
1 Bases and Dimension of Abstract Vector Spaces

When working with $\mathbb{R}^n$ and subspaces of $\mathbb{R}^n$ we developed several fundamental ideas including *span*, *linear independence*, *bases* and *dimension*. We will find that these concepts generalize easily to abstract vector spaces and that analogous results hold in these new settings.

Linear Independence

Definition 1 (Linear Independence). Let V be a vector space. Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ be vectors of V . We say that the set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ is *linearly independent* if the only solution to

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_p\mathbf{v}_p = \mathbf{0}$$

is the *trivial solution* $a_1 = a_2 = \dots = a_p = 0$.

If, in addition to the trivial solution, a *non-trivial solution* (not all $a_1, a_2, \dots, a_p$ are zero) exists, then we say that the set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ is *linearly dependent*.

Example 1. Show that $P = \{1 + x, 3x + x^2, 2 + x - x^2\}$ is linearly independent in $\mathbb{P}^2$.

Explanation. Consider the linear combination equation

$$\begin{aligned} a(1 + x) + b(3x + x^2) + c(2 + x - x^2) &= 0 \\ a + ax + 3bx + bx^2 + 2c + cx - cx^2 &= 0 \\ (a + 2c) + (a + 3b + c)x + (b - c)x^2 &= 0 \end{aligned}$$

The constant term, as well as the coefficients in front of x and x^2 , must be equal to 0. This gives us the following system of equations.

$$\begin{array}{rccccccc} a & + & & & + & 2c & = & 0 \\ a & + & 3b & + & & c & = & 0 \\ & & b & - & c & & = & 0 \end{array}$$

The only solution is $a = b = c = 0$. We conclude that P is linearly independent in $\mathbb{P}^2$.

Bases and Dimension

Recall that our motivation for defining a basis of a subspace of $\mathbb{R}^n$ was to have a collection of vectors such that every vector of the subspace can be expressed as a unique linear combination of the vectors in that collection. Definition of a basis (??) generalizes to abstract vector spaces as follows.

Definition 2. Let V be a vector space. A set $\mathcal{B}$ of vectors of V is called a *basis* of V provided that

1. $\text{span}(\mathcal{B}) = V$
2. $\mathcal{B}$ is linearly independent.

Theorem 1. Let V be a vector space, and let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a basis for V . Then every element $\mathbf{v}$ of V has a unique representation as linear combination of the elements of $\mathcal{B}$.

Proof By the definition of a basis, we know that $\mathbf{v}$ can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$. Suppose there are two such representations. Then,

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_n\mathbf{v}_n$$

$$\mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \dots + b_n\mathbf{v}_n$$

But then we have:

$$\begin{aligned} a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_n\mathbf{v}_n &= b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \dots + b_n\mathbf{v}_n \\ a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_n\mathbf{v}_n - (b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \dots + b_n\mathbf{v}_n) &= \mathbf{0} \\ (a_1 - b_1)\mathbf{v}_1 + (a_2 - b_2)\mathbf{v}_2 + \dots + (a_n - b_n)\mathbf{v}_n &= \mathbf{0} \end{aligned}$$

Because $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are linearly independent, we have $a_i - b_i = 0$ for $1 \leq i \leq n$. Consequently $a_i = b_i$ for $1 \leq i \leq n$. ■

In [Bases and Dimension](#) we defined the dimension of a subspace of $\mathbb{R}^n$ to be the number of elements in a basis. (Definition ??) We will adopt this definition for abstract vector spaces. As before, to ensure that *dimension* is well-defined we need to establish that this definition is independent of our choice of a basis. The proof of the following theorem is identical to the proof of its counterpart in $\mathbb{R}^n$. (Theorem ??)

Theorem 2. Let V be a vector space. Suppose $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_t\}$ and $\mathcal{C} = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_s\}$ be two bases of V . Then $s = t$.

Now we can state the definition.

Definition 3. Let V be a subspace of a vector space W . The *dimension* of V is the number, m , of elements in any basis of V . We write

$$\dim(V) = m$$

In our discussions up to this point, we have always assumed that a basis is nonempty and hence that the dimension of the space is at least 1. However, the zero space $\{\mathbf{0}\}$ has *no* basis. To accommodate for this, we will say that the zero vector space $\{\mathbf{0}\}$ is defined to have dimension 0:

$$\dim\{\mathbf{0}\} = 0$$

Our insistence that $\dim\{\mathbf{0}\} = 0$ amounts to saying that the empty set of vectors is a basis of $\{\mathbf{0}\}$. Thus the statement that “the dimension of a vector space is the number of vectors in any basis” holds even for the zero space.

Example 2. Recall that the vector space $\mathbb{M}_{m,n}$ consists of all $m \times n$ matrices. (See Example ?? of [Abstract Vector Spaces](#)). Find a basis and the dimension of $\mathbb{M}_{m,n}$.

Explanation. Let $\mathcal{B}$ consist of $m \times n$ matrices with exactly one entry equal to 1 and all other entries equal to 0. It is clear that every $m \times n$ matrix can be written as a linear combination of elements of $\mathcal{B}$. It is also easy to see that the elements of $\mathcal{B}$ are linearly independent. Thus $\mathcal{B}$ is a basis for $\mathbb{M}_{m,n}$. The set $\mathcal{B}$ contains mn elements, so $\dim(\mathbb{M}_{m,n}) = mn$.

Example 3. Recall that $\mathbb{P}^n$ is the set of all polynomials of degree n or less. (See Example ?? of [Abstract Vector Spaces](#)) Show that $\dim(\mathbb{P}^n) = n + 1$ and that $\{1, x, x^2, \dots, x^n\}$ is a basis of $\mathbb{P}^n$.

Explanation. Each polynomial $p(x) = a_0 + a_1x + \dots + a_nx^n$ in $\mathbb{P}^n$ is clearly a linear combination of $1, x, \dots, x^n$, so $\mathbb{P}^n = \text{span}\{1, x, \dots, x^n\}$.

Suppose $a_01 + a_1x + \dots + a_nx^n = 0$, then $a_0 = a_1 = \dots = a_n = 0$. So $\{1, x, \dots, x^n\}$ is linearly independent and hence is a basis containing $n + 1$ vectors. Thus, $\dim(\mathbb{P}^n) = n + 1$.

Example 4. Consider the subset

$$C_A = \{X \in \mathbb{M}_{2,2} : AX = XA\}$$

of $\mathbb{M}_{2,2}$.

It was shown in Example ?? of [Abstract Vector Spaces](#) that C_A is a subspace for any choice of the matrix A .

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Show that $\dim(C_A) = 2$ and find a basis of C_A .

Explanation. Suppose $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is in C_A . Then

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a+c & b+d \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & a \\ c & c \end{bmatrix}$$

This gives us two relationships:

$$b + d = a \quad \text{and} \quad c = 0$$

We can now express a generic element X of C_A as

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} b+d & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} b & b \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix} = b \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + d \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

The set $\mathcal{B}$ is linearly independent. (See Practice Problem ??) Every element X of C_A can be written as a linear combination of elements of $\mathcal{B}$. Thus $C_A = \text{span}\mathcal{B}$. Therefore $\mathcal{B}$ is a basis of C_A , and $\dim(C_A) = 2$.

Example 5. In Practice Problem ?? of [Abstract Vector Spaces](#) you demonstrated that the set of all symmetric $n \times n$ matrices is a subspace of $\mathbb{M}_{n,n}$.

Let V be a subspace of $\mathbb{M}_{2,2}$ consisting of all 2×2 symmetric matrices. Find the dimension of V .

Explanation. A matrix A is symmetric if $A^T = A$. In other words, a matrix A is symmetric when entries directly across the main diagonal are equal, so each 2×2 symmetric matrix has the form

$$\begin{bmatrix} a & c \\ c & b \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + c \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Hence the set $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ spans V . The reader can verify that $\mathcal{B}$ is linearly independent.

Thus $\mathcal{B}$ is a basis of V , so $\dim(V) = 3$.

Finite-Dimensional Vector Spaces

Our definition of dimension of a vector space depends on the vector space having a basis. In this section we will establish that any vector space spanned by finitely many vectors has a basis.

Definition 4. A vector space is said to be *finite-dimensional* if it is spanned by finitely many vectors.

Given a finite-dimensional vector space V we will find a basis for V by starting with a linearly independent subset of V and expanding it to a basis. The following results are more general versions of Lemmas ?? and ??, and Theorem ?? of [Bases and Dimension](#). The proofs are identical and we will omit them.

Lemma 1. Let V be a vector space spanned by n vectors. If a linearly independent subset S of V contains m vectors, then $m \leq n$.

Lemma 2. Let V be a vector space. Let $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ be a linearly independent subset of V . If $\mathbf{u}$ is not in $\text{span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$, then $\{\mathbf{u}, \mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly independent.

Theorem 3. Let V be a finite-dimensional vector space. Any linearly independent subset of V can be expanded to a basis of V .

Coordinate Vectors

Recall that in the context of $\mathbb{R}^n$ (and subspaces of $\mathbb{R}^n$) the requirement that elements of a basis be linearly independent guarantees that every element of the vector space has a *unique* representation in terms of the elements of the basis. (See Theorem ?? of [Introduction to Bases](#)) We proved the same property for abstract vector spaces in Theorem ??.

Uniqueness of representation in terms of the elements of a basis allows us to associate every element of a vector space V with a unique *coordinate vector* with respect to a given basis. Coordinate vectors were first introduced in [Introduction to Bases](#). We now give a formal definition.

Definition 5. Let V be a vector space, and let $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis for V . If $\mathbf{v} = a_1\mathbf{v}_1 + \dots + a_n\mathbf{v}_n$, then the vector in $\mathbb{R}^n$ whose components are the coefficients $a_1, \dots, a_n$ is said to be the *coordinate vector for $\mathbf{v}$ with respect to $\mathcal{B}$* . We denote the coordinate vector by $[\mathbf{v}]_{\mathcal{B}}$ and write:

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

Remark 1. The order of in which vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ appear in $\mathcal{B}$ of Definition ?? is important. Switching the order of these vectors would switch the order of the coordinate vector components. For this reason, we will often use the term *ordered basis* to describe $\mathcal{B}$ in the context of coordinate vectors.

Example 6. The coordinate vector for $p(x) = 4 - 3x^2 + 5x^3$ in $\mathbb{P}^4$ with respect to the ordered basis $\mathcal{B}_1 = \{1, x, x^2, x^3, x^4\}$ is

$$[p(x)]_{\mathcal{B}_1} = \begin{bmatrix} 4 \\ 0 \\ -3 \\ 5 \\ 0 \end{bmatrix}$$

Now let's change the order of the elements in $\mathcal{B}_1$. The coordinate vector for $p(x) = 4 - 3x^2 + 5x^3$ with respect to the ordered basis $\mathcal{B}_2 = \{x^4, x^3, x^2, x, 1\}$ is

$$[p(x)]_{\mathcal{B}_2} = \begin{bmatrix} 0 \\ 5 \\ -3 \\ 0 \\ 4 \end{bmatrix}$$

Example 7. Show that the set $\mathcal{B} = \{x, 1 + x, x + x^2\}$ is a basis for $\mathbb{P}^2$. Keep the order of elements in $\mathcal{B}$ and find the coordinate vector for $p(x) = 4 - x + 3x^2$ with respect to the ordered basis $\mathcal{B}$.

Explanation. We will begin by showing that the elements of $\mathcal{B}$ are linearly independent. Suppose

$$ax + b(1 + x) + c(x + x^2) = 0$$

then

$$b + (a + b + c)x + cx^2 = 0$$

This gives us the following system of equations:

$$\begin{array}{rcl} b & = & 0 \\ a + b + c & = & 0 \\ c & = & 0 \end{array}$$

The solution $a = b = c = 0$ is unique. We conclude that $\mathcal{B}$ is linearly independent.

Next, we need to show that $\mathcal{B}$ spans $\mathbb{P}^2$. To this end, we will consider a generic element $p(x) = \alpha + \beta x + \gamma x^2$ of $\mathbb{P}^2$ and attempt to express it as a linear combination of the elements of $\mathcal{B}$.

$$ax + b(1 + x) + c(x + x^2) = \alpha + \beta x + \gamma x^2$$

then

$$b + (a + b + c)x + cx^2 = \alpha + \beta x + \gamma x^2$$

Setting the coefficients of like terms equal to each other gives us

$$\begin{array}{rcl} b & = & \alpha \\ a + b + c & = & \beta \\ c & = & \gamma \end{array}$$

Solving this linear system of a , b and c gives us

$$a = \beta - \alpha - \gamma, \quad b = \alpha, \quad c = \gamma$$

(You should verify this.) This shows that every element of $\mathbb{P}^2$ can be written as a linear combination of elements of $\mathcal{B}$. Therefore $\mathcal{B}$ is a basis for $\mathbb{P}^2$.

To find the coordinate vector for $p(x) = 4 - x + 3x^2$ with respect to $\mathcal{B}$ we need to express $p(x)$ as a linear combination of the elements of $\mathcal{B}$. Fortunately, we have already done all the necessary work. For $p(x)$, $\alpha = 4$, $\beta = -1$ and $\gamma = 3$. This gives us the coefficients of the linear combination: $a = \beta - \alpha - \gamma = -8$, $b = \alpha = 4$, $c = \gamma = 3$. We now write $p(x)$ as a linear combination

$$p(x) = -8(x) + 4(1 + x) + 3(x + x^2)$$

The coordinate vector for $p(x)$ with respect to $\mathcal{B}$ is

$$[p(x)]_{\mathcal{B}} = \begin{bmatrix} -8 \\ 4 \\ 3 \end{bmatrix}$$

Example 8. Recall that the set V of all symmetric 2×2 matrices is a subspace of $\mathbb{M}_{2,2}$. In Example ?? we demonstrated that $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ is a basis for V . Let $A = \begin{bmatrix} 2 & -3 \\ -3 & 1 \end{bmatrix}$. Observe that A is an element of V .

1. Find the coordinate vector with respect to the ordered basis $\mathcal{B}$ for A .
2. Let $\mathcal{B}' = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ be another ordered basis for V . Find the coordinate vector for A with respect to $\mathcal{B}'$.

Explanation. We write A as a linear combination of the elements of $\mathcal{B}$.

$$A = \begin{bmatrix} 2 & -3 \\ -3 & 1 \end{bmatrix} = 2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - 3 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Thus, the coordinate vector with respect to $\mathcal{B}$ is

$$[A]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$$

The coordinate vector with respect to $\mathcal{B}'$ is

$$[A]_{\mathcal{B}'} = \begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix}$$

Coordinate vectors will play a vital role in establishing one of the most fundamental results in linear algebra, that all n -dimensional vector spaces have the same structure as $\mathbb{R}^n$. In Example ?? of [Isomorphic Vector Spaces](#), for instance, we will show that $\mathbb{P}^2$ is essentially the same as $\mathbb{R}^3$.

Practice Problems

Problem 1 Consider the subset

$$C_A = \{X \in \mathbb{M}_{2,2} : AX = XA\}$$

of $\mathbb{M}_{2,2}$. (See Example ??)

Prove that set $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a linearly independent subset of C_A .

Problem 2 Show that the following subset of $\mathbb{P}^2$ is linearly independent, and is a basis of $\mathbb{P}^2$.

$$\{1 + x, 1 - x, x + x^2\}$$

Problem 3 Show that the following subset of $\mathbb{P}^2$ is linearly independent, and is a basis of $\mathbb{P}^2$.

$$\{x^2, x + 1, 1 - x - x^2\}$$

Problem 4 Show that the following subset of $\mathbb{M}_{2,2}$ is linearly independent, and is a basis of $\mathbb{M}_{2,2}$.

$$\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$$

Problem 5 Find the coordinate vector for $p(x) = 6 - 2x + 4x^2$ in $\mathbb{P}^2$ with respect to

$$\mathcal{B} = \{1 + x, 1 - x, x + x^2\}$$

Answer:

$$[p(x)]_{\mathcal{B}} = \begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix}$$

Problem 6 Find the coordinate vector for $p(x) = 6 - 2x + 4x^2$ in $\mathbb{P}^2$ with respect to

$$\mathcal{B} = \{x^2, x + 1, 1 - x - x^2\}$$

Answer:

$$[p(x)]_{\mathcal{B}} = \begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix}$$

Problem 7 Find the coordinate vector for $A = \begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix}$ in $\mathbb{M}_{2,2}$ with respect to the ordered basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$$

Answer:

$$[A]_{\mathcal{B}} = \begin{bmatrix} \dots \\ \dots \\ \dots \\ \dots \end{bmatrix}$$

Text Source

The discussion of the zero space was adapted from Section 6.3 of Keith Nicholson's [Linear Algebra with Applications](#). (CC-BY-NC-SA)

W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2018, Open Edition, p. 349

Example Source

Examples ?? and ?? were adapted from Examples 6.3.1 and 6.3.10 of Keith Nicholson's *Linear Algebra with Applications*. (CC-BY-NC-SA)

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Exercise Source

Practice Problems ??, ?? and ?? are Exercises 6.3(a)(b)(c) from Keith Nicholson's *Linear Algebra with Applications*. (CC-BY-NC-SA)

W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2018, Open Edition, p. 351