
1 Inner Product Spaces

We have used the dot product in $\mathbb{R}^n$ to compute the length of vectors (Corollary ??) and also the [angle between vectors](#). The goal of this section is to define an *inner product* on an arbitrary vector space V over the real numbers. The dot product is an example an inner product for $\mathbb{R}^n$.

Definition 1. An *inner product* on a real vector space V is a function that assigns a real number $\langle \mathbf{v}, \mathbf{w} \rangle$ to every pair $\mathbf{v}, \mathbf{w}$ of vectors in V in such a way that the following properties are satisfied.

1. $\langle \mathbf{v}, \mathbf{w} \rangle$ is a real number for all $\mathbf{v}$ and $\mathbf{w}$ in V .
2. $\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle$ for all $\mathbf{v}$ and $\mathbf{w}$ in V .
3. $\langle \mathbf{v} + \mathbf{w}, \mathbf{u} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{w}, \mathbf{u} \rangle$ for all $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ in V .
4. $\langle r\mathbf{v}, \mathbf{w} \rangle = r\langle \mathbf{v}, \mathbf{w} \rangle$ for all $\mathbf{v}$ and $\mathbf{w}$ in V and all r in $\mathbb{R}$.
5. $\langle \mathbf{v}, \mathbf{v} \rangle > 0$ for all $\mathbf{v} \neq \mathbf{0}$ in V .

A real vector space V with an inner product $\langle \cdot, \cdot \rangle$ will be called an *inner product space*. Note that every subspace of an inner product space is again an inner product space using the same inner product.

Example 1. $\mathbb{R}^n$ is an inner product space with the dot product as inner product:

$$\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v} \cdot \mathbf{w} \quad \text{for all } \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$$

See Theorem ?. This is also called the *euclidean* inner product, and $\mathbb{R}^n$, equipped with the dot product, is called *euclidean n -space*.

Example 2. If A and B are $m \times n$ matrices, define $\langle A, B \rangle = \text{tr}(AB^T)$ where $\text{tr}(X)$ is the trace of the square matrix X . Show that $\langle \cdot, \cdot \rangle$ is an inner product in $\mathbb{M}_{mn}$.

Explanation. Property ?? is clear. Since $\text{tr}(P) = \text{tr}(P^T)$ for every square matrix P , we have Property ??:

$$\langle A, B \rangle = \text{tr}(AB^T) = \text{tr}[(AB^T)^T] = \text{tr}(BA^T) = \langle B, A \rangle$$

Next, Property ?? and Property ?? follow because trace is a linear transformation $\mathbb{M}_{mn} \rightarrow \mathbb{R}$ (Practice Problem ??). Turning to Property ??, let $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_m$ denote the rows of the matrix A . Then the (i, j) -entry of AA^T is $\mathbf{r}_i \cdot \mathbf{r}_j$, so

$$\langle A, A \rangle = \text{tr}(AA^T) = \mathbf{r}_1 \cdot \mathbf{r}_1 + \mathbf{r}_2 \cdot \mathbf{r}_2 + \dots + \mathbf{r}_m \cdot \mathbf{r}_m$$

But $\mathbf{r}_j \cdot \mathbf{r}_j$ is the sum of the squares of the entries of $\mathbf{r}_j$, so this shows that $\langle A, A \rangle$ is the sum of the squares of all nm entries of A . Property ?? follows.

The next example is important in analysis.

Example 3. Let $\mathcal{F}[a, b]$ be the set of all functions $f : [a, b] \rightarrow \mathbb{R}$. Observe that $\mathcal{F}[a, b]$ is a vector space. Let $\mathcal{C}[a, b]$ be a subset of $\mathcal{F}[a, b]$ consisting of all continuous functions. Why is $\mathcal{C}[a, b]$ a subspace of $\mathcal{F}[a, b]$? Show that

$$\langle f, g \rangle = \int_a^b f(x)g(x)dx$$

defines an inner product on $\mathcal{C}[a, b]$.

Remark 1. This example (and others later that refer to it) can be omitted with no loss of continuity by students with no calculus background.

Explanation. Property ?? and Property ?? are clear. As to Property ??,

$$\langle rf, g \rangle = \int_a^b rf(x)g(x)dx = r \int_a^b f(x)g(x)dx = r\langle f, g \rangle$$

Property ?? is similar. Finally, theorems of calculus show that $\langle f, f \rangle = \int_a^b f(x)^2 dx \geq 0$ and, if f is continuous, that this is zero if and only if f is the zero function. This gives Property ??.

If $\mathbf{v}$ is any vector, then, using Property ?? of Definition ??, we get

$$\langle \mathbf{0}, \mathbf{v} \rangle = \langle \mathbf{0} + \mathbf{0}, \mathbf{v} \rangle = \langle \mathbf{0}, \mathbf{v} \rangle + \langle \mathbf{0}, \mathbf{v} \rangle$$

and it follows that the number $\langle \mathbf{0}, \mathbf{v} \rangle$ must be zero. This observation is recorded for reference in the following theorem, along with several other properties of inner products. The other proofs are left as Practice Problem ??.

Theorem 1. Let $\langle \cdot, \cdot \rangle$ be an inner product on a space V ; let $\mathbf{v}$, $\mathbf{u}$, and $\mathbf{w}$ denote vectors in V ; and let r denote a real number.

1. $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$
2. $\langle \mathbf{v}, r\mathbf{w} \rangle = r\langle \mathbf{v}, \mathbf{w} \rangle = \langle r\mathbf{v}, \mathbf{w} \rangle$
3. $\langle \mathbf{v}, \mathbf{0} \rangle = 0 = \langle \mathbf{0}, \mathbf{v} \rangle$
4. $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$

Remark 2. If r is a complex number, then ?? must be modified. See Theorem ?? in [Complex Matrices](#).

If $\langle \cdot, \cdot \rangle$ is an inner product on a space V , then, given $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V ,

$$\langle r\mathbf{u} + s\mathbf{v}, \mathbf{w} \rangle = \langle r\mathbf{u}, \mathbf{w} \rangle + \langle s\mathbf{v}, \mathbf{w} \rangle = r\langle \mathbf{u}, \mathbf{w} \rangle + s\langle \mathbf{v}, \mathbf{w} \rangle$$

for all r and s in $\mathbb{R}$ by Property ?? and Property ?? of Definition ?. Moreover, there is nothing special about the fact that there are two terms in the linear combination or that it is in the first component:

$$\langle r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_n\mathbf{v}_n, \mathbf{w} \rangle = r_1\langle \mathbf{v}_1, \mathbf{w} \rangle + r_2\langle \mathbf{v}_2, \mathbf{w} \rangle + \cdots + r_n\langle \mathbf{v}_n, \mathbf{w} \rangle$$

and

$$\langle \mathbf{v}, s_1\mathbf{w}_1 + s_2\mathbf{w}_2 + \cdots + s_m\mathbf{w}_m \rangle = s_1\langle \mathbf{v}, \mathbf{w}_1 \rangle + s_2\langle \mathbf{v}, \mathbf{w}_2 \rangle + \cdots + s_m\langle \mathbf{v}, \mathbf{w}_m \rangle$$

hold for all r_i and s_i in $\mathbb{R}$ and all $\mathbf{v}$, $\mathbf{w}$, $\mathbf{v}_i$, and $\mathbf{w}_j$ in V . These results are described by saying that inner products “preserve” linear combinations. For example,

$$\begin{aligned} \langle 2\mathbf{u} - \mathbf{v}, 3\mathbf{u} + 2\mathbf{v} \rangle &= \langle 2\mathbf{u}, 3\mathbf{u} \rangle + \langle 2\mathbf{u}, 2\mathbf{v} \rangle + \langle -\mathbf{v}, 3\mathbf{u} \rangle + \langle -\mathbf{v}, 2\mathbf{v} \rangle \\ &= 6\langle \mathbf{u}, \mathbf{u} \rangle + 4\langle \mathbf{u}, \mathbf{v} \rangle - 3\langle \mathbf{v}, \mathbf{u} \rangle - 2\langle \mathbf{v}, \mathbf{v} \rangle \\ &= 6\langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle - 2\langle \mathbf{v}, \mathbf{v} \rangle \end{aligned}$$

If A is a symmetric $n \times n$ matrix and $\mathbf{x}$ and $\mathbf{y}$ are columns in $\mathbb{R}^n$, we regard the 1×1 matrix $\mathbf{x}^T A \mathbf{y}$ as a number. If we write

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T A \mathbf{y} \quad \text{for all columns } \mathbf{x}, \mathbf{y} \text{ in } \mathbb{R}^n$$

then Properties ??-?? of Definition ?? follow from matrix arithmetic (only Property ?? of Definition ?? requires that A is symmetric). Property ?? of Definition ?? reads

$$\mathbf{x}^T A \mathbf{x} > 0 \quad \text{for all columns } \mathbf{x} \neq \mathbf{0} \text{ in } \mathbb{R}^n$$

and this condition characterizes the positive definite matrices (Theorem ??). This proves the first assertion in the next theorem.

Theorem 2. If A is any $n \times n$ positive definite matrix, then

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T A \mathbf{y} \text{ for all columns } \mathbf{x}, \mathbf{y} \text{ in } \mathbb{R}^n$$

defines an inner product on $\mathbb{R}^n$, and every inner product on $\mathbb{R}^n$ arises in this way.

Proof Given an inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$, let $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ be the standard basis of $\mathbb{R}^n$. If $\mathbf{x} = \sum_{i=1}^n x_i \mathbf{e}_i$ and $\mathbf{y} = \sum_{j=1}^n y_j \mathbf{e}_j$ are two vectors in $\mathbb{R}^n$, compute $\langle \mathbf{x}, \mathbf{y} \rangle$ by adding the inner product of each term $x_i \mathbf{e}_i$ to each term $y_j \mathbf{e}_j$. The result is a double sum.

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^n \sum_{j=1}^n \langle x_i \mathbf{e}_i, y_j \mathbf{e}_j \rangle = \sum_{i=1}^n \sum_{j=1}^n x_i \langle \mathbf{e}_i, \mathbf{e}_j \rangle y_j$$

As the reader can verify, this is a matrix product:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} \langle \mathbf{e}_1, \mathbf{e}_1 \rangle & \langle \mathbf{e}_1, \mathbf{e}_2 \rangle & \cdots & \langle \mathbf{e}_1, \mathbf{e}_n \rangle \\ \langle \mathbf{e}_2, \mathbf{e}_1 \rangle & \langle \mathbf{e}_2, \mathbf{e}_2 \rangle & \cdots & \langle \mathbf{e}_2, \mathbf{e}_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mathbf{e}_n, \mathbf{e}_1 \rangle & \langle \mathbf{e}_n, \mathbf{e}_2 \rangle & \cdots & \langle \mathbf{e}_n, \mathbf{e}_n \rangle \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Hence $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T A \mathbf{y}$, where A is the $n \times n$ matrix whose (i, j) -entry is $\langle \mathbf{e}_i, \mathbf{e}_j \rangle$. The fact that

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \langle \mathbf{e}_j, \mathbf{e}_i \rangle$$

shows that A is symmetric. Finally, A is positive definite by Theorem ??. ■

Thus, just as every linear operator $\mathbb{R}^n \rightarrow \mathbb{R}^n$ corresponds to an $n \times n$ matrix, every inner product on $\mathbb{R}^n$ corresponds to a positive definite $n \times n$ matrix. In particular, the dot product corresponds to the identity matrix I_n .

Remark 3. If we refer to the inner product space $\mathbb{R}^n$ without specifying the inner product, we mean that the dot product is to be used.

Example 4. Let the inner product $\langle \cdot, \cdot \rangle$ be defined on $\mathbb{R}^2$ by

$$\left\langle \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right\rangle = 2v_1 w_1 - v_1 w_2 - v_2 w_1 + v_2 w_2$$

Find a symmetric 2×2 matrix A such that $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T A \mathbf{y}$ for all $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^2$.

Explanation. The (i, j) -entry of the matrix A is the coefficient of $v_i w_j$ in the expression, so $A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$.

Incidentally, if $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, then

$$\langle \mathbf{x}, \mathbf{x} \rangle = 2x^2 - 2xy + y^2 = x^2 + (x - y)^2 \geq 0$$

for all $\mathbf{x}$, so $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ implies $\mathbf{x} = \mathbf{0}$. Hence $\langle \cdot, \cdot \rangle$ is indeed an inner product, so A is positive definite.

Let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathbb{R}^n$ given as in Theorem ?? by a positive definite matrix A . If $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T$, then $\langle \mathbf{x}, \mathbf{x} \rangle = \mathbf{x}^T A \mathbf{x}$ is an expression in the variables $x_1, x_2, \dots, x_n$ called a *quadratic form*. For more on quadratic forms, see Section 8.8 of [Nicholson], pp. 472–482.

Norm and Distance

Definition 2. As in $\mathbb{R}^n$, if $\langle \cdot, \cdot \rangle$ is an inner product on a space V , the *norm* $\|\mathbf{v}\|$ of a vector $\mathbf{v}$ in V is defined by

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

We define the *distance* between vectors $\mathbf{v}$ and $\mathbf{w}$ in an inner product space V to be

$$d(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|$$

Remark 4. If the dot product is used in $\mathbb{R}^n$, the norm $\|\mathbf{x}\|$ of a vector $\mathbf{x}$ is usually called the *length* of $\mathbf{x}$.

Note that Property ?? of Definition ?? guarantees that $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$, so $\|\mathbf{v}\|$ is a real number.

Example 5. The norm of a continuous function $f = f(x)$ in $\mathcal{C}[a, b]$ (with the inner product from Example ??) is given by

$$\|f\| = \sqrt{\int_a^b f(x)^2 dx}$$

Hence $\|f\|^2$ is the area beneath the graph of $y = f(x)^2$ between $x = a$ and $x = b$.

Example 6. Show that $\langle \mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2$ in any inner product space.

Explanation.

$$\begin{aligned} \langle \mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle &= \langle \mathbf{u}, \mathbf{u} \rangle - \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle - \langle \mathbf{v}, \mathbf{v} \rangle \\ &= \|\mathbf{u}\|^2 - \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle - \|\mathbf{v}\|^2 \\ &= \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2 \end{aligned}$$

A vector $\mathbf{v}$ in an inner product space V is called a *unit vector* if $\|\mathbf{v}\| = 1$. The set of all unit vectors in V is called the *unit ball* in V . For example, if $V = \mathbb{R}^2$ (with the dot product) and $\mathbf{v} = (x, y)$, then

$$\|\mathbf{v}\|^2 = 1 \quad \text{if and only if} \quad x^2 + y^2 = 1$$

Hence the unit ball in $\mathbb{R}^2$ is the *unit circle* $x^2 + y^2 = 1$ with centre at the origin and radius 1. However, the shape of the unit ball varies with the choice of inner product.

Example 7. Let $a > 0$ and $b > 0$. If $\mathbf{v} = (x, y)$ and $\mathbf{w} = (x_1, y_1)$, define an inner product on $\mathbb{R}^2$ by

$$\langle \mathbf{v}, \mathbf{w} \rangle = \frac{xx_1}{a^2} + \frac{yy_1}{b^2}$$

The reader can verify (Practice Problem ??) that this is indeed an inner product. In this case

$$\|\mathbf{v}\|^2 = 1 \quad \text{if and only if} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

so the unit ball is the ellipse shown in the diagram.

Example ?? graphically illustrates the fact that norms and distances in an inner product space V vary with the choice of inner product in V .

Theorem 3. If $\mathbf{v} \neq \mathbf{0}$ is any vector in an inner product space V , then $\frac{1}{\|\mathbf{v}\|}\mathbf{v}$ is the unique unit vector that is a positive multiple of $\mathbf{v}$.

The next theorem reveals an important and useful fact about the relationship between norms and inner products, extending the Cauchy inequality for $\mathbb{R}^n$ (Theorem ??).

Theorem 4 (Cauchy-Schwarz Inequality). If $\mathbf{v}$ and $\mathbf{w}$ are two vectors in an inner product space V , then

$$\langle \mathbf{v}, \mathbf{w} \rangle^2 \leq \|\mathbf{v}\|^2 \|\mathbf{w}\|^2$$

Moreover, equality occurs if and only if one of $\mathbf{v}$ and $\mathbf{w}$ is a scalar multiple of the other.

Remark 5. Hermann Amandus Schwarz (1843–1921) was a German mathematician at the University of Berlin. He had strong geometric intuition, which he applied with great ingenuity to particular problems. A version of the inequality appeared in 1885.

Proof of Cauchy-Schwarz Inequality Write $\|\mathbf{v}\| = a$ and $\|\mathbf{w}\| = b$. Using Theorem ?? we compute:

$$\begin{aligned} \|b\mathbf{v} - a\mathbf{w}\|^2 &= b^2 \|\mathbf{v}\|^2 - 2ab\langle \mathbf{v}, \mathbf{w} \rangle + a^2 \|\mathbf{w}\|^2 = 2ab(ab - \langle \mathbf{v}, \mathbf{w} \rangle) \\ \|b\mathbf{v} + a\mathbf{w}\|^2 &= b^2 \|\mathbf{v}\|^2 + 2ab\langle \mathbf{v}, \mathbf{w} \rangle + a^2 \|\mathbf{w}\|^2 = 2ab(ab + \langle \mathbf{v}, \mathbf{w} \rangle) \end{aligned} \tag{1}$$

It follows that $ab - \langle \mathbf{v}, \mathbf{w} \rangle \geq 0$ and $ab + \langle \mathbf{v}, \mathbf{w} \rangle \geq 0$, and hence that $-ab \leq \langle \mathbf{v}, \mathbf{w} \rangle \leq ab$. But then $|\langle \mathbf{v}, \mathbf{w} \rangle| \leq ab = \|\mathbf{v}\| \|\mathbf{w}\|$, as desired.

Conversely, if $|\langle \mathbf{v}, \mathbf{w} \rangle| = \|\mathbf{v}\| \|\mathbf{w}\| = ab$ then $\langle \mathbf{v}, \mathbf{w} \rangle = \pm ab$. Hence (??) shows that $b\mathbf{v} - a\mathbf{w} = \mathbf{0}$ or $b\mathbf{v} + a\mathbf{w} = \mathbf{0}$. It follows that one of $\mathbf{v}$ and $\mathbf{w}$ is a scalar multiple of the other, even if $a = 0$ or $b = 0$. ■

Example 8. If f and g are continuous functions on the interval $[a, b]$, then (see Example ??)

$$\left(\int_a^b f(x)g(x)dx \right)^2 \leq \int_a^b f(x)^2 dx \int_a^b g(x)^2 dx$$

Another famous inequality, the so-called *triangle inequality* (See [Triangle Inequality](#) in the Appendix), also comes from the Cauchy-Schwarz inequality. It is included in the following list of basic properties of the norm of a vector.

Theorem 5. If V is an inner product space, the norm $\|\cdot\|$ has the following properties.

1. $\|\mathbf{v}\| \geq 0$ for every vector $\mathbf{v}$ in V .
2. $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = \mathbf{0}$.
3. $\|r\mathbf{v}\| = |r| \|\mathbf{v}\|$ for every $\mathbf{v}$ in V and every r in $\mathbb{R}$.
4. $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$ for all $\mathbf{v}$ and $\mathbf{w}$ in V (*triangle inequality*).

Proof Because $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$, properties ?? and ?? follow immediately from ?? and ?? of Theorem ??. As to ??, compute

$$\|r\mathbf{v}\|^2 = \langle r\mathbf{v}, r\mathbf{v} \rangle = r^2 \langle \mathbf{v}, \mathbf{v} \rangle = r^2 \|\mathbf{v}\|^2$$

Hence ?? follows by taking positive square roots. Finally, the fact that $\langle \mathbf{v}, \mathbf{w} \rangle \leq \|\mathbf{v}\| \|\mathbf{w}\|$ by the Cauchy-Schwarz inequality gives

$$\begin{aligned} \|\mathbf{v} + \mathbf{w}\|^2 &= \langle \mathbf{v} + \mathbf{w}, \mathbf{v} + \mathbf{w} \rangle = \|\mathbf{v}\|^2 + 2\langle \mathbf{v}, \mathbf{w} \rangle + \|\mathbf{w}\|^2 \\ &\leq \|\mathbf{v}\|^2 + 2\|\mathbf{v}\| \|\mathbf{w}\| + \|\mathbf{w}\|^2 \\ &= (\|\mathbf{v}\| + \|\mathbf{w}\|)^2 \end{aligned}$$

Hence ?? follows by taking positive square roots. ■

It is worth noting that the usual triangle inequality for absolute values,

$$|r + s| \leq |r| + |s| \text{ for all real numbers } r \text{ and } s$$

is a special case of ?? where $V = \mathbb{R} = \mathbb{R}^1$ and the dot product $\langle r, s \rangle = rs$ is used.

In many calculations in an inner product space, it is required to show that some vector $\mathbf{v}$ is zero. This is often accomplished most easily by showing that its norm $\|\mathbf{v}\|$ is zero. Here is an example.

Example 9. Let $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a spanning set for an inner product space V . If $\mathbf{v}$ in V satisfies $\langle \mathbf{v}, \mathbf{v}_i \rangle = 0$ for each $i = 1, 2, \dots, n$, show that $\mathbf{v} = \mathbf{0}$.

Explanation. Write $\mathbf{v} = r_1 \mathbf{v}_1 + \dots + r_n \mathbf{v}_n$, r_i in $\mathbb{R}$. To show that $\mathbf{v} = \mathbf{0}$, we show that $\|\mathbf{v}\|^2 = \langle \mathbf{v}, \mathbf{v} \rangle = 0$. Compute:

$$\langle \mathbf{v}, \mathbf{v} \rangle = \langle \mathbf{v}, r_1 \mathbf{v}_1 + \dots + r_n \mathbf{v}_n \rangle = r_1 \langle \mathbf{v}, \mathbf{v}_1 \rangle + \dots + r_n \langle \mathbf{v}, \mathbf{v}_n \rangle = 0$$

by hypothesis, and the result follows.

The norm properties in Theorem ?? translate to the following properties of distance familiar from geometry. The proof is Practice Problem ??.

Theorem 6. Let V be an inner product space.

1. $d(\mathbf{v}, \mathbf{w}) \geq 0$ for all $\mathbf{v}, \mathbf{w}$ in V .
2. $d(\mathbf{v}, \mathbf{w}) = 0$ if and only if $\mathbf{v} = \mathbf{w}$.
3. $d(\mathbf{v}, \mathbf{w}) = d(\mathbf{w}, \mathbf{v})$ for all $\mathbf{v}$ and $\mathbf{w}$ in V .
4. $d(\mathbf{v}, \mathbf{w}) \leq d(\mathbf{v}, \mathbf{u}) + d(\mathbf{u}, \mathbf{w})$ for all $\mathbf{v}, \mathbf{u}$, and $\mathbf{w}$ in V .

Practice Problems

Problem 1 In each case, determine which of Property ??–Property ?? in Definition ?? fail to hold.

$$1. V = \mathbb{R}^2, \left\langle \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \right\rangle = x_1 y_1 x_2 y_2$$

$$2. V = \mathbb{R}^3, \left\langle \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \right\rangle = x_1 y_1 - x_2 y_2 + x_3 y_3$$

Click the arrow to see the answer. Property ?? fails.

$$3. V = \mathbb{C}, \langle z, w \rangle = z \bar{w}, \text{ where } \bar{w} \text{ is complex conjugation}$$

Click the arrow to see the answer. Property ?? fails, as sometimes we get a complex number. However, we will return to this definition of $\langle \cdot, \cdot \rangle$ in [Complex Matrices](#) – see Definition ??

$$4. V = \mathbb{P}^3, \langle p(x), q(x) \rangle = p(1)q(1)$$

Click the arrow to see the answer. Property ?? fails.

$$5. V = \mathbb{M}_{22}, \langle A, B \rangle = \det(AB)$$

$$6. V = \mathcal{F}[0, 1], \langle f, g \rangle = f(1)g(0) + f(0)g(1)$$

Click the arrow to see the answer. Property ?? fails.

Problem 2 Let V be an inner product space. If $U \subseteq V$ is a subspace, show that U is an inner product space using the same inner product.

Problem 3 In each case, find a scalar multiple of $\mathbf{v}$ that is a unit vector.

$$1. \mathbf{v} = f \text{ in } \mathcal{C}[0, 1] \text{ where } f(x) = x^2$$

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx$$

$$2. \mathbf{v} = f \text{ in } \mathcal{C}[-\pi, \pi] \text{ where } f(x) = \cos x$$

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx$$

Click the arrow to see the answer. $\frac{1}{\sqrt{\pi}}f$

$$3. \mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \text{ in } \mathbb{R}^2 \text{ where } \langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{w}$$

$$4. \mathbf{v} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \text{ in } \mathbb{R}^2, \langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \mathbf{w}$$

Click the arrow to see the answer. $\frac{1}{\sqrt{17}} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

Problem 4 In each case, find the distance between $\mathbf{u}$ and $\mathbf{v}$.

1. $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}; \langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v}$

2. $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 3 \end{bmatrix}; \langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v}$

Click the arrow to see the answer. $\sqrt{3}$

3. $\mathbf{u} = f, \mathbf{v} = g$ in $C[0, 1]$ where $f(x) = x^2$ and $g(x) = 1 - x; \langle f, g \rangle = \int_0^1 f(x)g(x)dx$

4. $\mathbf{u} = f, \mathbf{v} = g$ in $C[-\pi, \pi]$ where $f(x) = 1$ and $g(x) = \cos x; \langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx$

Click the arrow to see the answer. $\sqrt{3\pi}$

Problem 5 Let $a_1, a_2, \dots, a_n$ be positive numbers. Given $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$, define $\langle \mathbf{v}, \mathbf{w} \rangle = a_1 v_1 w_1 + \dots +$

$a_n v_n w_n$. Show that this is an inner product on $\mathbb{R}^n$.

Problem 6 If $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is a basis of V and if $\mathbf{v} = v_1 \mathbf{b}_1 + \dots + v_n \mathbf{b}_n$ and $\mathbf{w} = w_1 \mathbf{b}_1 + \dots + w_n \mathbf{b}_n$ are vectors in V , define

$$\langle \mathbf{v}, \mathbf{w} \rangle = v_1 w_1 + \dots + v_n w_n.$$

Show that this is an inner product on V .

Problem 7 Let $\operatorname{re}(z)$ denote the real part of the complex number z . Show that $\langle \cdot, \cdot \rangle$ is an inner product on $\mathbb{C}$ if $\langle \mathbf{z}, \mathbf{w} \rangle = \operatorname{re}(z\bar{w})$.

Problem 8 If $T : V \rightarrow V$ is an isomorphism of the inner product space V , show that

$$\langle \mathbf{v}, \mathbf{w} \rangle_1 = \langle T(\mathbf{v}), T(\mathbf{w}) \rangle$$

defines a new inner product $\langle \cdot, \cdot \rangle_1$ on V .

Problem 9 Show that every inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$ has the form $\langle \mathbf{x}, \mathbf{y} \rangle = (\mathbf{U}\mathbf{x}) \cdot (\mathbf{U}\mathbf{y})$ for some upper triangular matrix $\mathbf{U}$ with positive diagonal entries.

Problem 10 In each case, show that $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T \mathbf{A} \mathbf{w}$ defines an inner product on $\mathbb{R}^2$ and hence show that $\mathbf{A}$ is positive definite.

$$1. A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$2. A = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$$

Click the arrow to see the answer. $\langle \mathbf{v}, \mathbf{v} \rangle = 5v_1^2 - 6v_1v_2 + 2v_2^2 = \frac{1}{5}[(5v_1 - 3v_2)^2 + v_2^2]$

$$3. A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

$$4. A = \begin{bmatrix} 3 & 4 \\ 4 & 6 \end{bmatrix}$$

Click the arrow to see the answer. $\langle \mathbf{v}, \mathbf{v} \rangle = 3v_1^2 + 8v_1v_2 + 6v_2^2 = \frac{1}{3}[(3v_1 + 4v_2)^2 + 2v_2^2]$

Problem 11 In each case, find a symmetric matrix A such that $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T A \mathbf{w}$.

$$1. \left\langle \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right\rangle = v_1 w_1 + 2v_1 w_2 + 2v_2 w_1 + 5v_2 w_2$$

$$2. \left\langle \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right\rangle = v_1 w_1 - v_1 w_2 - v_2 w_1 + 2v_2 w_2$$

Click the arrow to see the answer. $\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$

$$3. \left\langle \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \right\rangle = 2v_1 w_1 + v_2 w_2 + v_3 w_3 - v_1 w_2 - v_2 w_1 + v_2 w_3 + v_3 w_2$$

$$4. \left\langle \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \right\rangle = v_1 w_1 + 2v_2 w_2 + 5v_3 w_3 - 2v_1 w_3 - 2v_3 w_1$$

Click the arrow to see the answer. $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ -2 & 0 & 5 \end{bmatrix}$

Problem 12 If A is symmetric and $\mathbf{x}^T A \mathbf{x} = 0$ for all columns $\mathbf{x}$ in $\mathbb{R}^n$, show that $A = 0$.

Click the arrow to see the answer. By the condition, $\langle \mathbf{x}, \mathbf{y} \rangle = \frac{1}{2} \langle \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y} \rangle = 0$ for all $\mathbf{x}, \mathbf{y}$. Let $\mathbf{e}_i$ denote column i of I . If $A = [a_{ij}]$, then $a_{ij} = \mathbf{e}_i^T A \mathbf{e}_j = \langle \mathbf{e}_i, \mathbf{e}_j \rangle = 0$ for all i and j .

Problem 13 Show that the sum of two inner products on V is again an inner product.

Problem 14 Let $\|\mathbf{u}\| = 1$, $\|\mathbf{v}\| = 2$, $\|\mathbf{w}\| = \sqrt{3}$, $\langle \mathbf{u}, \mathbf{v} \rangle = -1$, $\langle \mathbf{u}, \mathbf{w} \rangle = 0$ and $\langle \mathbf{v}, \mathbf{w} \rangle = 3$. Compute:

1. $\langle \mathbf{v} + \mathbf{w}, 2\mathbf{u} - \mathbf{v} \rangle$

2. $\langle \mathbf{u} - 2\mathbf{v} - \mathbf{w}, 3\mathbf{w} - \mathbf{v} \rangle$

....

Problem 15 Given the data in Practice Problem ??, show that $\mathbf{u} + \mathbf{v} = \mathbf{w}$.

Problem 16 Show that no vectors exist such that $\|\mathbf{u}\| = 1$, $\|\mathbf{v}\| = 2$, and $\langle \mathbf{u}, \mathbf{v} \rangle = -3$.

Problem 17 Complete Example ??.

Problem 18 Prove Theorem ??.

Problem 19 Prove Theorem ??.

Problem 20 Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in an inner product space V .

1. Expand $\langle 2\mathbf{u} - 7\mathbf{v}, 3\mathbf{u} + 5\mathbf{v} \rangle$.

2. Expand $\langle 3\mathbf{u} - 4\mathbf{v}, 5\mathbf{u} + \mathbf{v} \rangle$.

Click the arrow to see the answer. $15\|\mathbf{u}\|^2 - 17\langle \mathbf{u}, \mathbf{v} \rangle - 4\|\mathbf{v}\|^2$

3. Show that $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + 2\langle \mathbf{u}, \mathbf{v} \rangle + \|\mathbf{v}\|^2$.

4. Show that $\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 - 2\langle \mathbf{u}, \mathbf{v} \rangle + \|\mathbf{v}\|^2$.

Click the arrow to see the answer. $\|\mathbf{u} + \mathbf{v}\|^2 = \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle = \|\mathbf{u}\|^2 + 2\langle \mathbf{u}, \mathbf{v} \rangle + \|\mathbf{v}\|^2$

Problem 21 Show that

$$\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 = \frac{1}{2} \{ \|\mathbf{v} + \mathbf{w}\|^2 + \|\mathbf{v} - \mathbf{w}\|^2 \}$$

for any $\mathbf{v}$ and $\mathbf{w}$ in an inner product space.

Problem 22 Let $\langle \cdot, \cdot \rangle$ be an inner product on a vector space V . Show that the corresponding distance function is translation invariant. That is, show that $d(\mathbf{v}, \mathbf{w}) = d(\mathbf{v} + \mathbf{u}, \mathbf{w} + \mathbf{u})$ for all $\mathbf{v}, \mathbf{w}$, and $\mathbf{u}$ in V .

Problem 23 1. Show that $\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{4} [\|\mathbf{u} + \mathbf{v}\|^2 - \|\mathbf{u} - \mathbf{v}\|^2]$ for all $\mathbf{u}, \mathbf{v}$ in an inner product space V .

2. If $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle'$ are two inner products on V that have equal associated norm functions, show that $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle'$ holds for all $\mathbf{u}$ and $\mathbf{v}$.

Problem 24 Let $\mathbf{v}$ denote a vector in an inner product space V .

1. Show that $W = \{\mathbf{w} \mid \mathbf{w} \text{ in } V, \langle \mathbf{v}, \mathbf{w} \rangle = 0\}$ is a subspace of V .

2. Let W be as in (a). If $V = \mathbb{R}^3$ with the dot product, and if $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, find a basis for W .

Click the arrow to see the answer. $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$

Problem 25 Given vectors $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n$ and $\mathbf{v}$, assume that $\langle \mathbf{v}, \mathbf{w}_i \rangle = 0$ for each i . Show that $\langle \mathbf{v}, \mathbf{w} \rangle = 0$ for all $\mathbf{w}$ in $\text{span}\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$.

Problem 26 If $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ and $\langle \mathbf{v}, \mathbf{v}_i \rangle = \langle \mathbf{w}, \mathbf{v}_i \rangle$ holds for each i . Show that $\mathbf{v} = \mathbf{w}$.

Problem 27 Use the Cauchy-Schwarz inequality in an inner product space to show that:

1. If $\|\mathbf{u}\| \leq 1$, then $\langle \mathbf{u}, \mathbf{v} \rangle^2 \leq \|\mathbf{v}\|^2$ for all $\mathbf{v}$ in V .
2. $(x \cos \theta + y \sin \theta)^2 \leq x^2 + y^2$ for all real x, y , and θ .
3. $\|r_1 \mathbf{v}_1 + \dots + r_n \mathbf{v}_n\|^2 \leq [r_1 \|\mathbf{v}_1\| + \dots + r_n \|\mathbf{v}_n\|]^2$ for all vectors $\mathbf{v}_i$, and all $r_i > 0$ in $\mathbb{R}$.

Problem 28 If A is a $2 \times n$ matrix, let $\mathbf{u}$ and $\mathbf{v}$ denote the rows of A .

1. Show that $AA^T = \begin{bmatrix} \|\mathbf{u}\|^2 & \mathbf{u} \cdot \mathbf{v} \\ \mathbf{u} \cdot \mathbf{v} & \|\mathbf{v}\|^2 \end{bmatrix}$.

2. Show that $\det(AA^T) \geq 0$.

Problem 29 1. If $\mathbf{v}$ and $\mathbf{w}$ are nonzero vectors in an inner product space V , show that $-1 \leq \frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\|\mathbf{v}\| \|\mathbf{w}\|} \leq 1$,

and hence that a unique angle θ exists such that

$\frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\|\mathbf{v}\| \|\mathbf{w}\|} = \cos \theta$ and $0 \leq \theta \leq \pi$. This angle θ is called the angle between $\mathbf{v}$ and $\mathbf{w}$.

2. Find the angle between $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \\ 3 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$ in $\mathbb{R}^5$ with the dot product.

3. If θ is the angle between $\mathbf{v}$ and $\mathbf{w}$, show that the law of cosines is valid:

$$\|\mathbf{v} - \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2 \|\mathbf{v}\| \|\mathbf{w}\| \cos \theta.$$

Problem 30 If $V = \mathbb{R}^2$, define $\left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\| = |x| + |y|$.

-
1. Show that $\|\cdot\|$ satisfies the conditions in Theorem ??.
 2. Show that $\|\cdot\|$ does not arise from an inner product on $\mathbb{R}^2$ given by a matrix A .

Text Source

This section was adapted from Section 10.1 of Keith Nicholson's *Linear Algebra with Applications*. (CC-BY-NC-SA)
W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2018, Open Edition, pp. 521–530.