
1 Review for Exam 2

Exercise 1 Let $V = \text{span} \left(\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -2 \end{bmatrix} \right)$. Recall that the span of a set of vectors in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$. From the choices below select ALL sets that can serve as bases for V .

Select All Correct Answers:

1. $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -2 \end{bmatrix} \right\}$

2. $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

3. $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -2 \end{bmatrix} \right\}$

4. $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -2 \end{bmatrix} \right\}$

5. $\left\{ \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

Exercise 2 Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$ be an ordered basis for $\mathbb{R}^3$. (Mentally verify that $\mathcal{B}$ is a basis of $\mathbb{R}^3$.)

Let $\mathbf{v} = \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$ be a vector of $\mathbb{R}^3$ (written with respect to the standard basis of $\mathbb{R}^3$). Find the coordinate vector for $\mathbf{v}$ with respect to $\mathcal{B}$.

$$\begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix}$$

Exercise 3 Consider matrix A and $\text{rref}(A)$ shown below.

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 & -1 \\ 1 & 2 & 2 & 3 & 0 \\ 0 & 0 & 1 & 3 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 0 & -3 & -2 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \text{rref}(A)$$

Find each of the following:

$$\text{rank}(A) = \dots$$

$$\dim(\text{row}(A)) = \dots$$

$$\dim(\text{col}(A)) = \dots$$

$$\dim(\text{null}(A)) = \dots$$

Select an appropriate basis for $\text{row}(A)$.

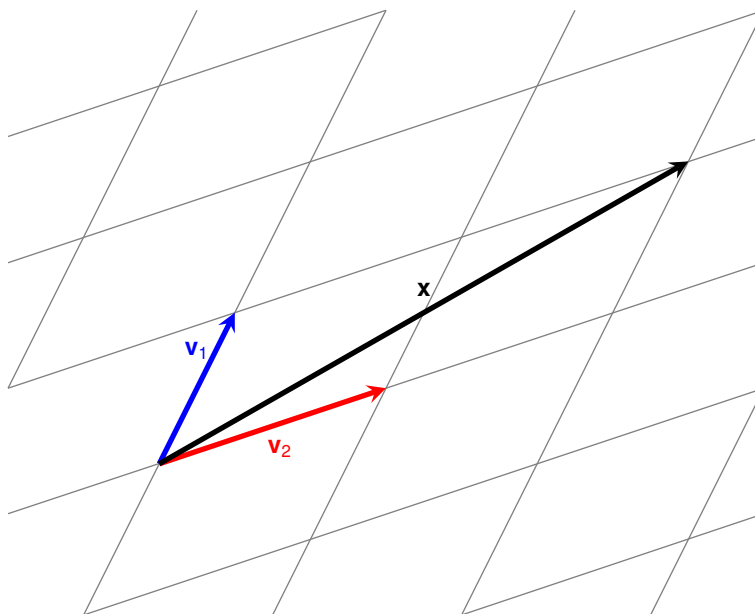
Multiple Choice:

1. $\left\{ \begin{bmatrix} 1 & 2 & 1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 2 & 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 3 & 1 \end{bmatrix} \right\}$
2. $\left\{ \begin{bmatrix} 1 & 2 & 0 & -3 & -2 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 3 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\}$
3. $\left\{ \begin{bmatrix} 1 & 2 & 0 & -3 & -2 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 3 & 1 \end{bmatrix} \right\}$

Using the algorithm presented in this text, find a basis for $\text{col}(A)$.

$$\left\{ \begin{bmatrix} \dots \\ \dots \\ \dots \\ \dots \end{bmatrix}, \begin{bmatrix} \dots \\ \dots \\ \dots \\ \dots \end{bmatrix} \right\}$$

Exercise 4 Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ be a basis for the plane depicted below. Find the coordinate vector for $\mathbf{x}$ with respect to $\mathcal{B}$.



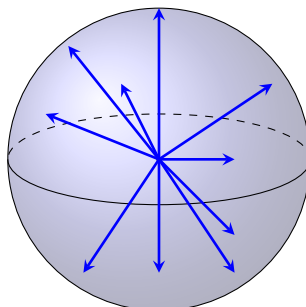
$$\begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix}$$

Exercise 5 Suppose that $\mathcal{B}_1 = \{\mathbf{v}_1, \mathbf{v}_2\}$ is an ordered basis for some subspace V of $\mathbb{R}^n$. Let $\mathcal{B}_2 = \{-\mathbf{v}_2, -2\mathbf{v}_1\}$. Verify that $\mathcal{B}_2$ is also an ordered basis for V .

Let $\mathbf{w}$ be a vector in V . If the coordinate vector for $\mathbf{w}$ with respect to $\mathcal{B}_1$ is $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$, find the coordinate vector for $\mathbf{w}$ with respect to $\mathcal{B}_2$.

$$\begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix}$$

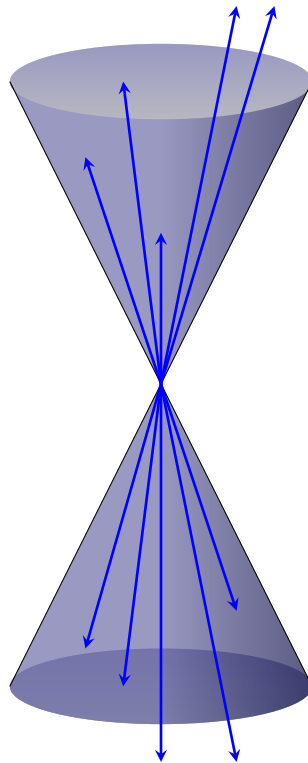
Exercise 6 Determine whether each set, S , of vectors is closed under vector addition, and scalar multiplication. Let S be the set of all vectors contained in a sphere of radius 1 centered at the origin.



Select All Correct Answers:

1. S is closed under scalar multiplication.
2. S is closed under vector addition.
3. None of the above.

Let S be the set of all vectors contained in an infinite double cone.



Select All Correct Answers:

1. S is closed under scalar multiplication.
2. S is closed under vector addition.
3. None of the above.

Exercise 7 Find the dimension of V if

$$V = \text{span} \left(\begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 1 \\ -1 \end{bmatrix} \right)$$
$$\dim(V) = \dots \dots$$

Exercise 8 True or False? If False, you should come up with a counterexample. If True, can you give a proof?

1. If V is a subspace of $\mathbb{R}^n$ and $\mathbf{x} + \mathbf{y}$ is in V , then $\mathbf{x}$ is in V or $\mathbf{y}$ is in V .

Multiple Choice:

- (i) True
- (ii) False

2. If V is a set in $\mathbb{R}^n$ such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ is in V whenever $\mathbf{v}_1$ and $\mathbf{v}_2$ are in V for any scalars c_1, c_2 , then V is a subspace.

Multiple Choice:

- (i) True
- (ii) False

3. Every set of four non-zero vectors in $\mathbb{R}^4$ is a basis.

Multiple Choice:

- (i) True
- (ii) False

4. $\mathbb{R}^3$ has a basis of the form $\{\mathbf{x}, \mathbf{x} + \mathbf{y}, \mathbf{y}\}$.

Multiple Choice:

- (i) True
- (ii) False

Exercise 9 Suppose a linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is such that

$$T \left(\begin{bmatrix} 2 \\ -3 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T \left(\begin{bmatrix} -1 \\ 7 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Then,

$$T \left(\begin{bmatrix} 1 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} \dots \dots \\ \dots \dots \end{bmatrix}$$

Exercise 10 Find the standard matrix M of a linear transformation $T_M : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ if

$$T \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}; \quad T \left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}; \quad T \left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$M = \begin{bmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \end{bmatrix}$$

Exercise 11 Suppose that an invertible linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is such that

$$T \left(\begin{bmatrix} 3 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 4 \\ -7 \end{bmatrix}; \quad T \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -5 \\ 4 \end{bmatrix}$$

Find the vector whose image under T is $\begin{bmatrix} 3 \\ -10 \end{bmatrix}$

$$T \left(\begin{bmatrix} \dots \\ \dots \end{bmatrix} \right) = \begin{bmatrix} 3 \\ -10 \end{bmatrix}$$

Exercise 12 True or False? If False, you should come up with a counterexample. If True, can you give a proof?

1. $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, given by $T(x, y) = (x, -y)$, is a linear transformation.

Multiple Choice:

- (i) True
- (ii) False

2. $T : \mathbb{R}^n \rightarrow \mathbb{R}$, given by $T(\mathbf{x}) = \mathbf{x} \cdot \mathbf{z}$ for some fixed vector $\mathbf{z} \in \mathbb{R}^n$, is a linear transformation.

Multiple Choice:

- (i) True
- (ii) False

3. $T : \mathbb{R} \rightarrow \mathbb{R}$, given by $T(x) = x^2$, is a linear transformation.

Multiple Choice:

- (i) True
- (ii) False

4. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation and let $\mathbf{v}_1, \dots, \mathbf{v}_k$ denote vectors in $\mathbb{R}^n$. If $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_k)\}$ is linearly independent, then $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is also linearly independent.

Multiple Choice:

- (i) *True*
- (ii) *False*

5. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation and suppose $\mathbf{v}_1, \mathbf{v}_2$ denote vectors in $\mathbb{R}^2$. If $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent, then $\{T(\mathbf{v}_1), T(\mathbf{v}_2)\}$ is also linearly independent.

Multiple Choice:

- (i) *True*
- (ii) *False*

Exercise 13 Let

$$A = \begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & 0 & 3 & 1 \\ 1 & 1 & -4 & 2 \end{bmatrix}$$

Use techniques discussed in [Image and Kernel of a Linear Transformation](#) to find the basis for the kernel and the image of the linear transformation, T_A , induced by A .

$$\text{Basis for } \text{im}(T_A): \left\{ \begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix}, \begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix} \right\}$$

$$\text{Basis for } \text{ker}(T_A): \left\{ \begin{bmatrix} \dots \\ \dots \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \dots \\ \dots \\ 0 \\ 1 \end{bmatrix} \right\}$$