
Chapter Reviews

June 5, 2026

Contents

1 Preliminaries

Learning Outcomes

After completing this chapter, students should be able to do the following.

- Perform basic operations, including addition, scalar multiplication, and dot product, on vectors in $\mathbb{R}^n$.
- Perform basic operations, including addition and scalar multiplication, on geometric representations of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.
- Transition seamlessly between algebraic and geometric representations of vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$.
- Determine the angle between vectors and test for orthogonality.
- Project one vector onto another.
- Compute and interpret the cross product of two vectors in $\mathbb{R}^3$.
- Transition seamlessly between parametric and non-parametric forms of the equation of a line in $\mathbb{R}^2$.
- Transition seamlessly between parametric and vector forms of the equation of a line in $\mathbb{R}^n$.
- Determine the equation of a plane in $\mathbb{R}^3$ meeting given characteristics.

Exercise 1 Select all pairs of parallel vectors.

Select All Correct Answers:

1. $\begin{bmatrix} -3 \\ -6 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 10 \\ -5 \end{bmatrix}$

2. $\begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$

3. $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

4. $\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -8 \\ 0 \\ 4 \end{bmatrix}$

Source

[Nicholson] W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2021, Open Edition, Problem 4.1.7.

Exercise 2 Find Q such that $\overrightarrow{PQ} = \mathbf{v}$ if

$$P(-1, 2, 2), \mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$$Q(\dots, \dots, \dots)$$

Source

[Nicholson] W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2021, Open Edition, Problem 4.1.10.

Exercise 3 Find a unit vector $\mathbf{u}$ in the direction of $\begin{bmatrix} 7 \\ -1 \\ 5 \end{bmatrix}$

$$\mathbf{u} = \dots \dots \begin{bmatrix} 7 \\ -1 \\ 5 \end{bmatrix}$$

Enter the exact answer. No decimal approximations.

Source

[Nicholson] W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2021, Open Edition, Problem 4.1.2(a).

Exercise 4 Select all pairs of orthogonal vectors.

Select All Correct Answers:

1. $\begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

2. $\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -6 \\ 4 \end{bmatrix}$

3. $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix}$

4. $\begin{bmatrix} -1 \\ 3 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \\ -1 \end{bmatrix}$

Exercise 5 Which set(s) of parametric equations describe the line containing $P(3, -2, 7)$, if the line is perpendicular to the plane $2x + y - 5z = 10$?

Select all that apply.

Select All Correct Answers:

1. $x = 2t + 3; y = t - 2; z = -5t + 7.$

2. $x = 3t + 2; y = -2t + 1; z = 7t - 5.$

3. $x = 10t + 3; y = 10t - 2; z = 10t + 7.$

4. $x = -2t + 3; y = -t - 2; z = 5t + 7.$

Exercise 6 Find the measure of the acute angle (in degrees) of intersection of lines L_1 and L_2 given by parametric equations

$$L_1 : \quad x = t, y = t, z = 0$$

$$L_2 : \quad x = 2t, y = t, z = t$$

... .. degrees

Exercise 7 Find $\mathbf{v} \times \mathbf{w}$ if

$$\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}; \quad \mathbf{w} = \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}$$

$$\mathbf{v} \times \mathbf{w} = \begin{bmatrix} \dots\dots \\ \dots\dots \\ \dots\dots \end{bmatrix}$$

Which of the following is true about $\mathbf{v} \times \mathbf{w}$?

Multiple Choice:

1. $\|\mathbf{v} \times \mathbf{w}\|$ is the product of $\|\mathbf{v}\|$ and $\|\mathbf{w}\|$.
2. $\mathbf{v} \times \mathbf{w}$ is parallel to ONE of the original vectors.
3. $\mathbf{v} \times \mathbf{w}$ is orthogonal to ONE of the original vectors.
4. $\mathbf{v} \times \mathbf{w}$ is orthogonal to BOTH of the original vectors.

Exercise 8 Which of the following planes is orthogonal to the line given by parametric equations below?

$$x = 2t - 10; \quad y = -t + 1; \quad z = 5t - 4$$

Multiple Choice:

1. $4x - 2y + 10z = 6$.
2. $-10x + y - 4z = 1$.
3. $5x - y + 2z = 0$
4. $3x + 6y - 2z = 4$

Exercise 9 Find the projection of $\mathbf{v} = \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix}$ onto a line with parametric equations

$$x = 2t; \quad y = 3t; \quad z = -t$$

Enter your answers in fraction form (no decimal approximations).

$$\begin{bmatrix} \dots\dots \\ \dots\dots \\ \dots\dots \end{bmatrix}$$

Exercise 10 True or False? If False, you should come up with a counterexample. If True, can you give a proof?

1. The zero vector is the only vector of length 0.

Multiple Choice:

- (i) True
- (ii) False

2. If $\|\mathbf{v} - \mathbf{w}\| = 0$, then $\mathbf{v} = \mathbf{w}$.

Multiple Choice:

- (i) True
- (ii) False

3. If $\mathbf{v} = -\mathbf{v}$, then $\mathbf{v} = \mathbf{0}$.

Multiple Choice:

- (i) True
- (ii) False

4. If $\|\mathbf{v}\| = \|\mathbf{w}\|$, then $\mathbf{v} = \mathbf{w}$.

Multiple Choice:

- (i) True
- (ii) False

5. If $\|\mathbf{v}\| = \|\mathbf{w}\|$, then $\mathbf{v} = \pm\mathbf{w}$.

Multiple Choice:

- (i) True
- (ii) False

6. If $\mathbf{v} = t\mathbf{w}$ for some scalar t , then $\mathbf{v}$ and $\mathbf{w}$ have the same direction.

Multiple Choice:

- (i) True
- (ii) False

7. If $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{v} + \mathbf{w}$ are non-zero, and $\mathbf{v}$ is parallel to $\mathbf{v} + \mathbf{w}$, then $\mathbf{v}$ and $\mathbf{w}$ are also parallel.

Multiple Choice:

- (i) True
- (ii) False

8. $\|\mathbf{v} + \mathbf{w}\| = \|\mathbf{v}\| + \|\mathbf{w}\|$

Multiple Choice:

- (i) True
- (ii) False

Source

[Nicholson] W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2021, Open Edition, Problem 4.1.21.

2 Systems of Linear Equations

Learning Outcomes

After completing this chapter, students should be able to do the following.

- Solve systems of linear equations using Gauss-Jordan elimination to reduce to echelon form.
- Interpret existence and uniqueness of solutions geometrically.
- Compute the rank of a matrix using Gaussian elimination.
- Implement Jacobi's method and the Gauss-Seidel method to solve linear systems.

2.1 Essential Vocabulary

Augmented matrix Every linear system

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ & & & & \vdots & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \end{array}$$

can be written in the *augmented matrix form* as follows:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array} \right]$$

The array to the left of the vertical bar is called the *coefficient matrix* of the linear system and is often given a capital letter name, like A . The vertical array to the right of the bar is called a *constant vector*.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

We will sometimes use the following notation to represent an augmented matrix.

$$\left[A \mid \mathbf{b} \right]$$

Back substitution When a matrix is in row-echelon form, we can compute the solution to the system by starting from the last equation and working backwards. This process is known as *back substitution*.

Basic variable See Leading variable

Coefficient matrix A *coefficient matrix* is a matrix whose entries are the coefficients of a system of linear equations. For the system

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ & & & & \vdots & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \end{array},$$

the *coefficient matrix* is $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$.

Consistent system A system of equations that has at least one solution.

Convergence when the iterates of an iterative method approach a solution

Divergence when the iterates of an iterative method fail to approach a solution

Elementary row operations The following three operations performed on a linear system are called *elementary row operations*.

1. Switching the order of equations (rows) i and j :

$$R_i \leftrightarrow R_j$$

2. Multiplying both sides of equation (row) i by the same non-zero constant, k , and replacing equation i with the result:

$$kR_i \rightarrow R_i$$

3. Adding k times equation (row) i to equation (row) j , and replacing equation j with the result:

$$R_j + kR_i \rightarrow R_j$$

Equivalent linear systems Linear systems are called *equivalent* if they have the same solution set.

Free variable When a linear system is in row-echelon form, the variables corresponding to columns that do not have any leading coefficients (if there are any) are known as *free variables*.

Gauss-Jordan elimination The process of using the elementary row operations on a matrix to transform it into reduced row-echelon form is called *Gauss-Jordan elimination*.

Gauss-Seidel Method An iterative method for solving linear systems that is a refinement of the Jacobi method, where we use computed values of variables alternately for quicker convergence.

Gaussian elimination The process of using the elementary row operations on a matrix to transform it into row-echelon form is called *Gaussian Elimination*.

Inconsistent system A system of equations that has no solution.

Iterative method A technique where we repeat the same procedure (called an *iteration*) many times (usually using a computer), and we obtain approximate solutions which we hope “converge to” the actual solution.

Jacobi’s method An iterative method for solving a system of equations where one variable is isolated in each equation in order to compute the coordinate of the next iterate.

Leading entry (leading 1) The first non-zero entry in a row of a matrix (when read from left to right) is called the *leading entry*. When the leading entry is 1, we refer to it as a *leading 1*.

Leading variable (or Basic variable) When a coefficient matrix is in row echelon form, a *leading variable* is a variable corresponding to a column of the matrix with at least one leading entry.

Linear equation A *linear equation* in variables $x_1, \dots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$$

where $a_1, \dots, a_n$ and b are constants.

Pivot In Gaussian elimination, an entry chosen to become a leading coefficient used to get zeros in the remaining rows.

Rank of a matrix The *rank* of matrix A , denoted by $\text{rank}(A)$, is the number of nonzero rows that remain after we reduce A to row-echelon form by elementary row operations.

Reduced row echelon form A matrix that is already in *row-echelon* form is said to be in *reduced row-echelon form* if:

1. Each leading entry is 1
2. All entries *above* and below each leading 1 are 0

Row echelon form A matrix is said to be in *row-echelon form* if:

1. All entries below each leading entry are 0.
2. Each leading entry is in a column to the right of the leading entries in the rows above it.
3. All rows of zeros, if there are any, are located below non-zero rows.

Row equivalent matrices Two matrices A and B are said to be row equivalent if there is a sequence of elementary row operations that converts A to B .

System of linear equations A finite set of linear (degree 1) equations each with the same variables.

Exercise 11 Find all solutions (if any) to the system.

$$\begin{aligned} 3x - y &= 0 \\ 2x - 3y &= 1 \end{aligned}$$

(... ..)

Enter your answers using fractions. If the system is inconsistent, type DNE in each answer cell.

Source

[Nicholson] W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2021, Open Edition, Problem 1.2.4 (b).

Exercise 12 Find all solutions (if any) to the system.

$$\begin{aligned} 3x - y &= 2 \\ -6x + 2y &= -4 \end{aligned}$$

(... .., t)

Enter your answers using fractions.

Source

[Nicholson] W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2021, Open Edition, Problem 1.2.4 (d).

Exercise 13 Find all solutions (if any) to the system.

$$\begin{aligned} 2x - 3y &= 5 \\ -2x + 3y &= 2 \end{aligned}$$

(... ..,)

Enter your answers using fractions. If the system is inconsistent, type DNE in each answer cell.

Source

[Nicholson] W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2021, Open Edition, Problem 1.2.4 (f).

Exercise 14 Find all solutions (if any) to the system.

$$\begin{aligned} -2x + 3y + 3z &= -9 \\ 3x - 4y + z &= 5 \\ -5x + 7y + 2z &= -14 \end{aligned}$$

$$(\dots, \dots, t)$$

If the system is inconsistent, type DNE in each answer cell.

Source

[Nicholson] W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2021, Open Edition, Problem 1.2.5 (b).

Exercise 15 Find all solutions (if any) to the system.

$$\begin{array}{rcrcrcrcrcl} x & + & 2y & - & z & = & 2 \\ 2x & + & 5y & - & 3z & = & 1 \\ x & + & 4y & - & 3z & = & 3 \end{array}$$

$$(\dots, \dots, \dots)$$

If the system is inconsistent, type DNE in each answer cell.

Source

[Nicholson] W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2021, Open Edition, Problem 1.2.5 (d).

Exercise 16 Find all solutions (if any) to the system.

$$\begin{array}{rcrcrcrcrcl} 3x & - & 2y & + & z & = & -2 \\ x & - & y & + & 3z & = & 5 \\ -x & + & y & + & z & = & -1 \end{array}$$

$$(\dots, \dots, \dots)$$

If the system is inconsistent, type DNE in each answer cell.

Source

[Nicholson] W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2021, Open Edition, Problem 1.2.5 (f).

Exercise 17 The augmented matrix of a system of linear equations has been carried to the following by row operations. Solve the system.

$$\left[\begin{array}{cccccc|c} 1 & -2 & 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 5 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 & 1 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$(\dots, r, \dots, s, \dots, t)$$

Source

[Nicholson] W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2021, Open Edition, Problem 1.2.3 (b).

Exercise 18 The augmented matrix of a system of linear equations has been carried to the following by row operations. Solve the system.

$$\left[\begin{array}{ccccc|c} 1 & -1 & 2 & 4 & 6 & 2 \\ 0 & 1 & 2 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

(.....,, s,, t)

Source

[Nicholson] W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2021, Open Edition, Problem 1.2.3 (d).

Exercise 19 Consider the following augmented matrix in which * denotes an arbitrary number and ■ denotes a nonzero number.

$$\left[\begin{array}{ccccc|c} \blacksquare & * & * & * & * & * \\ 0 & \blacksquare & * & * & * & * \\ 0 & 0 & \blacksquare & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & 0 \end{array} \right]$$

Which of the following is true? Select all that apply.

Select All Correct Answers:

1. The corresponding system of equations is consistent.
2. The corresponding system of equations is inconsistent.
3. The rank of the augmented matrix is 4.
4. The rank of the augmented matrix is 5.
5. The rank of the augmented matrix is 6.
6. There are no free variables.
7. There is one free variable.
8. There are two free variables.
9. The corresponding system has a unique solution.
10. The corresponding system has infinitely many solutions.

Bibliography

This problem is based on Ken Kuttler, *A First Course in Linear Algebra*, Lyryx 2017, Open Edition, pp. 42–49.

Exercise 20 Consider an augmented matrix of the form

$$\left[A \mid \mathbf{b} \right]$$

corresponding to a system of linear equations.

True or False?

1. If there is more than one solution, $\left[A \mid \mathbf{b} \right]$ has a row of zeros.

Multiple Choice:

- (i) True
- (ii) False

2. If $\left[A \mid \mathbf{b} \right]$ has a row of zeros, there is more than one solution.

Multiple Choice:

- (i) True
- (ii) False

3. If there is no solution, $\text{rref}(A)$ has a row of zeros.

Multiple Choice:

- (i) True
- (ii) False

4. If $\text{rref}(A)$ has a row of zeros, there is no solution.

Multiple Choice:

- (i) True
- (ii) False

5. There is NO system that is inconsistent for every $\mathbf{b}$.

Multiple Choice:

- (i) True
- (ii) False

6. If the system is consistent for some choice of $\mathbf{b}$, it is consistent for every choice of $\mathbf{b}$.

Multiple Choice:

- (i) True
- (ii) False

Source

[Nicholson] W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2021, Open Edition, Problem 1.2.12.

3 Big Ideas about Vectors

Learning Outcomes

After completing this chapter, students should be able to do the following.

- Express vectors as linear combinations of other vectors in $\mathbb{R}^n$ in both algebraic and geometric settings.
- Interpret a system of linear equations as an expression of one vector as a linear combination of other vectors in $\mathbb{R}^n$.
- Describe the span of a set of vectors in $\mathbb{R}^n$.
- Determine linear independence / dependence of a set of vectors in $\mathbb{R}^n$.

3.1 Essential Vocabulary

Linear combination of vectors A vector $\mathbf{v}$ is said to be a *linear combination* of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ if

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_n\mathbf{v}_n$$

for some scalars $a_1, a_2, \dots, a_n$.

Linearly dependent vectors Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ be vectors of $\mathbb{R}^n$. We say that the set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is *linearly independent* if the only solution to

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_k\mathbf{v}_k = \mathbf{0}$$

is the *trivial solution* $c_1 = c_2 = \dots = c_k = 0$.

If, in addition to the trivial solution, a *non-trivial solution* (not all $c_1, c_2, \dots, c_k$ are zero) exists, then we say that the set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is *linearly dependent*.

Linearly independent vectors Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ be vectors of $\mathbb{R}^n$. We say that the set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is *linearly independent* if the only solution to

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_k\mathbf{v}_k = \mathbf{0}$$

is the *trivial solution* $c_1 = c_2 = \dots = c_k = 0$.

If, in addition to the trivial solution, a *non-trivial solution* (not all $c_1, c_2, \dots, c_k$ are zero) exists, then we say that the set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is *linearly dependent*.

Redundant vectors Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ be a set of vectors in $\mathbb{R}^n$. If we can remove one vector without changing the span of this set, then that vector is *redundant*. In other words, if

$$\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k) = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \dots, \mathbf{v}_k)$$

we say that $\mathbf{v}_j$ is a redundant element of $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$, or simply redundant.

Span of a set of vectors Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ be vectors in $\mathbb{R}^n$. The set S of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ is called the *span* of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$. We write

$$S = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p)$$

and we say that vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ *span* S . Any vector in S is said to be *in the span* of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$. The set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ is called a *spanning set* for S .

Exercise 21 Express $\begin{bmatrix} 127 \\ -26 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 101 \\ 50 \end{bmatrix}$ and $\begin{bmatrix} 25 \\ 42 \end{bmatrix}$.

$$\dots\dots \begin{bmatrix} 101 \\ 50 \end{bmatrix} + \dots\dots \begin{bmatrix} 25 \\ 42 \end{bmatrix} = \begin{bmatrix} 127 \\ -26 \end{bmatrix}$$

Exercise 22 Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{w}$ are vectors in $\mathbb{R}^n$. What are the implications of the statement: $\mathbf{w}$ is in the span of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Select all that apply.

Select All Correct Answers:

1. Vector $\mathbf{w}$ a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$
2. The set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{w}\}$ contains redundant vectors.
3. Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{w}$ are linearly independent.
4. Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{w}$ are linearly dependent.
5. Vector $\mathbf{v}_1$ can be written as a linear combination of $\mathbf{v}_2, \mathbf{v}_3, \mathbf{w}$.
6. The equation $a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3 + d\mathbf{w} = \mathbf{0}$ has a non-trivial solution.

Exercise 23 Express $\begin{bmatrix} -1 \\ 4 \\ 5 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$.

If this is not possible, enter DNE in each answer cell.

$$\begin{bmatrix} -1 \\ 4 \\ 5 \end{bmatrix} = \dots\dots \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + \dots\dots \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} + \dots\dots \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

Exercise 24 Suppose three planes given by the equations below intersect at a single point.

$$\begin{aligned} a_1x + a_2y + a_3z &= 0 \\ b_1x + b_2y + b_3z &= 0 \\ c_1x + c_2y + c_3z &= 0 \end{aligned}$$

What can be said about the vectors $\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$, $\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$ and $\begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix}$?

Select all that apply.

Select All Correct Answers:

1. Nothing can be deduced from the information given.

2. The three vectors are linearly independent.
3. The three vectors are linearly dependent.
4. The three vectors are pairwise orthogonal.
5. The first vector can be written as a linear combination of the other two.
6. The equation $\alpha \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} + \beta \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} + \omega \begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix} = \mathbf{0}$ has only the trivial solution.

Exercise 25 Use the navigation bar arrows in the interactive below to advance the construction steps. Use the final result to express $\mathbf{w}$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

Access GeoGebra interactives through the online version of this text at

<https://ximera.osu.edu/linearalgebradv3/LinearAlgebraInteractiveIntro>.

$$\mathbf{w} = \dots \mathbf{u} + \dots \mathbf{v}$$

Exercise 26 True or False?

1. If $\{\mathbf{x}, \mathbf{y}\}$ is linearly independent, then $\{\mathbf{x}, \mathbf{y}, \mathbf{x} + \mathbf{y}\}$ is linearly independent.

Multiple Choice:

- (i) True
- (ii) False

2. If $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly independent, then $\{\mathbf{y}, \mathbf{z}\}$ is linearly independent.

Multiple Choice:

- (i) True
- (ii) False

3. If $\{\mathbf{y}, \mathbf{z}\}$ is linearly dependent, then $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly dependent.

Multiple Choice:

- (i) True
- (ii) False

4. If $a\mathbf{x} + b\mathbf{y} + c\mathbf{z} = \mathbf{0}$, then $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly independent.

Multiple Choice:

- (i) True
- (ii) False

Source

[Nicholson] W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2021, Open Edition, Problem 5.2.7.

4 Matrices

Learning Outcomes

After completing this chapter, students should be able to do the following.

- Perform addition and scalar multiplication of matrices.
- Give the dimensions of a matrix product, and compute the product.
- Discuss associativity and noncommutativity of matrix multiplication.
- Interpret a matrix-vector product as a linear combination of the columns of the matrix.
- Employ block multiplication of matrices, when advantageous.
- Compute the transpose of a matrix.
- Express a solution to a system of linear equations as a sum of a particular solution and a solution to the associated homogeneous system.
- Compute the inverse of a matrix using elementary row operations.
- Solve systems of linear equations using the inverse of the coefficient matrix, when possible.
- Create elementary matrices corresponding to each of the elementary row operations
- Solve a system of equations using LU factorization.

4.1 Essential Vocabulary

Column matrix (or column vector) A matrix with m rows and only 1 column.

Diagonal matrix A matrix $A = [a_{ij}]$ where $a_{ij} = 0$ whenever $i \neq j$

Dimensions of a matrix An $m \times n$ matrix is a matrix with m rows and n columns.

Elementary matrix A matrix formed by applying one elementary row operation to the identity matrix.

Homogeneous system A system of linear equations is called *homogeneous* if the system can be written in the form

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & 0 \\ & & & & \vdots & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & 0 \end{array},$$

or as a matrix equation as $A\mathbf{x} = \mathbf{0}$.

Identity matrix A square matrix with ones as diagonal entries and zeros for the remaining entries.

Inverse of a square matrix Let A be an $n \times n$ matrix. An $n \times n$ matrix B is called an *inverse* of A if

$$AB = BA = I$$

where I is an $n \times n$ identity matrix. If such an inverse matrix exists, we say that A is *invertible*. If an inverse does not exist, we say that A is not invertible. The inverse of A is denoted by A^{-1} .

LU factorization A factorization $A = LU$ where L is lower triangular and U is upper triangular with ones on the diagonal (called unit upper triangular). It is useful for solving $Ax = b$.

Matrix A rectangular array of numbers. It has m rows and n columns for some positive integers m and n .

Matrix addition Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two $m \times n$ matrices. Then the *sum of matrices* A and B , denoted by $A + B$, is an $m \times n$ matrix given by

$$A + B = [a_{ij} + b_{ij}]$$

Matrix factorization Representing a matrix as a product of two or more matrices.

Matrix multiplication Let A be an $m \times n$ matrix whose rows are vectors $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_n$. Let B be an $n \times p$ matrix with columns $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_p$. Then the matrix product AB is an $m \times p$ matrix with entries given by the dot products

$$AB = \begin{bmatrix} - & \mathbf{r}_1 & - \\ - & \mathbf{r}_2 & - \\ & \vdots & \\ - & \mathbf{r}_i & - \\ & \vdots & \\ - & \mathbf{r}_m & - \end{bmatrix} \begin{bmatrix} | & | & & | & | & | \\ \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_j & \dots & \mathbf{b}_p \\ | & | & & | & | & | \end{bmatrix} =$$

$$= \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{b}_1 & \mathbf{r}_1 \cdot \mathbf{b}_2 & \dots & \mathbf{r}_1 \cdot \mathbf{b}_j & \dots & \mathbf{r}_1 \cdot \mathbf{b}_p \\ \mathbf{r}_2 \cdot \mathbf{b}_1 & \mathbf{r}_2 \cdot \mathbf{b}_2 & \dots & \mathbf{r}_2 \cdot \mathbf{b}_j & \dots & \mathbf{r}_2 \cdot \mathbf{b}_p \\ \vdots & \vdots & & \vdots & & \vdots \\ \mathbf{r}_i \cdot \mathbf{b}_1 & \mathbf{r}_i \cdot \mathbf{b}_2 & \dots & \mathbf{r}_i \cdot \mathbf{b}_j & \dots & \mathbf{r}_i \cdot \mathbf{b}_p \\ \vdots & \vdots & & \vdots & & \vdots \\ \mathbf{r}_m \cdot \mathbf{b}_1 & \mathbf{r}_m \cdot \mathbf{b}_2 & \dots & \mathbf{r}_m \cdot \mathbf{b}_j & \dots & \mathbf{r}_m \cdot \mathbf{b}_p \end{bmatrix}$$

Matrix powers If A is a square matrix then we can define A^k to be the result of multiplying A by itself k times.

Negative of a matrix The additive inverse of a matrix, formed by multiplying the matrix by the scalar -1 .

Nonsingular matrix A square matrix A is said to be *nonsingular* provided that $\text{rref}(A) = I$. Otherwise we say that A is *singular*.

Partitioned matrices - block multiplication Subdividing a matrix into submatrices using imaginary horizontal and vertical lines - used to multiply matrices more efficiently.

Permutation matrix A matrix formed by permuting the rows of the identity matrix.

Properties of matrix algebra Addition and scalar multiplication of matrices have many nice properties. Matrix multiplication is not commutative, but the associative and distributive laws hold. The inverse and transpose operations of a product follow the "shoes and socks" rule.

Row matrix A matrix with only 1 row and n columns.

Scalar multiple of a matrix If $A = [a_{ij}]$ and k is a scalar, then $kA = [ka_{ij}]$.

Singular matrix A square matrix A is said to be *singular* provided that $\text{rref}(A)$ is NOT the identity matrix. If instead $\text{rref}(A) = I$, we say that A is *nonsingular*.

Square matrix A matrix with the same number of rows and columns.

Symmetric matrix An $n \times n$ matrix A is said to be *symmetric* if $A = A^T$. It is said to be *skew symmetric* if $A = -A^T$.

Transpose of a matrix Let $A = [a_{ij}]$ be an $m \times n$ matrix. Then the *transpose of A* , denoted by A^T , is the $n \times m$ matrix given by switching the rows and columns:

$$A^T = [a_{ij}]^T = [a_{ji}]$$

Zero matrix The $m \times n$ *zero matrix* is the $m \times n$ matrix having every entry equal to zero. The zero matrix is denoted by O .

Exercise 27 Find the inverse of the following matrix.

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -5 & 7 & -11 \\ -2 & 3 & -5 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{bmatrix}$$

Source

[Nicholson] W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2021, Open Edition, Problem 2.4.2 (d).

Exercise 28 Consider the system of equations:

$$\begin{array}{rcrcrcrcrcl} x & - & 3y & + & z & = & 2 \\ -3x & - & 4y & + & z & = & 0 \\ & & 2y & - & z & = & 1 \end{array}$$

Let A be the coefficient matrix corresponding to this system, and let $\mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

Suppose that we find that $(1, -2, -5)$ is a unique solution to this system. What does this tell us? Select ALL that apply.

Select All Correct Answers:

1. A is invertable.
2. Vector $\mathbf{b}$ is in the span of the columns of A .
3. Columns of A are linearly dependent.
4. Equation $A\mathbf{x} = \mathbf{b}$ has a unique solution.
5. Vector $\mathbf{b}$ can be written as a linear combination of the columns of A .

6. $A\mathbf{b} = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$

Exercise 29 Let $A = \begin{bmatrix} | & | & | \\ \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 \\ | & | & | \end{bmatrix}$ be a 3×3 matrix with columns $\mathbf{c}_1$, $\mathbf{c}_2$ and $\mathbf{c}_3$.

Suppose $A \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Which of the following can we conclude from the given information? Select ALL that apply.

Select All Correct Answers:

1. $\mathbf{c}_2 + \mathbf{c}_3 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$

2. $-\mathbf{c}_1 - \mathbf{c}_2 + 2\mathbf{c}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

3. A is non-singular.

4. $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is in $\text{span}(\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3)$.

5. Vector $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ can be written as a linear combination of the columns of A .

Exercise 30 Suppose A is a non-singular 3×3 matrix. Let $A^{-1} = \begin{bmatrix} | & | & | \\ \mathbf{d}_1 & \mathbf{d}_2 & \mathbf{d}_3 \\ | & | & | \end{bmatrix}$.

Suppose $A \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$

Express $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ as a linear combination of the columns of A^{-1} .

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \dots \mathbf{d}_1 + \dots \mathbf{d}_2 + \dots \mathbf{d}_3$$

Exercise 31 Let A be a 2×5 matrix. Compute the dimensions of each product.

$A^T A$ is a $\dots \times \dots$ matrix.

AA^T is a $\dots \times \dots$ matrix.

Exercise 32 Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$. Find elementary matrices E_1 and E_2 such that $C = E_2 E_1 A$.

$$E_1 = \begin{bmatrix} \dots & \dots \\ \dots & \dots \end{bmatrix}; \quad E_2 = \begin{bmatrix} \dots & \dots \\ \dots & \dots \end{bmatrix}$$

Source

[Nicholson] W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2021, Open Edition, Problem 2.5.3 (a).

Exercise 33 Find an LU-factorization of A .

$$A = \begin{bmatrix} 5 & -5 & 10 & 0 & 5 \\ -3 & 3 & 2 & 2 & 1 \\ -2 & 2 & 0 & -1 & 0 \\ 1 & -1 & 10 & 2 & 5 \end{bmatrix}$$

$$A = LU = \begin{bmatrix} \dots & 0 & 0 & 0 \\ \dots & \dots & 0 & 0 \\ \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots \end{bmatrix} \begin{bmatrix} \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Source

[Nicholson] W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2021, Open Edition, Example 2.7.3.

5 Subspaces of $\mathbb{R}^n$

Learning Outcomes

After completing this chapter, students should be able to do the following.

- Determine whether sets are closed under addition and/or scalar multiplication.
- Determine whether a set of vectors is a basis of a subspace of $\mathbb{R}^n$.
- Write the coordinates of a vector with respect to a given basis of $\mathbb{R}^n$.
- Identify the dimension of the span of a given set of vectors.
- Find a basis for each of the following: the row space, the column space, and the null space of a given matrix.
- Prove elementary theorems concerning rank of a matrix and the relationship between rank and nullity.

5.1 Essential Vocabulary

Basis A set S of vectors is called a *basis* of $\mathbb{R}^n$ (or a basis of a subspace V of $\mathbb{R}^n$) provided that

1. $\text{span}(S) = \mathbb{R}^n$ (or V)
2. S is linearly independent.

Closed under addition A set V is said to be *closed under addition* if for each element $\mathbf{u} \in V$ and $\mathbf{v} \in V$ the sum $\mathbf{u} + \mathbf{v}$ is also in V .

Closed under scalar multiplication A set V is said to be *closed under scalar multiplication* if for each element $\mathbf{v} \in V$ and for each scalar $k \in \mathbb{R}$ the product $k\mathbf{v}$ is also in V .

Column space of a matrix Let A be an $m \times n$ matrix. The *column space* of A , denoted by $\text{col}(A)$, is the subspace of $\mathbb{R}^m$ spanned by the columns of A .

Coordinate vector with respect to a basis Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ be an ordered basis. Then the *coordinate vector* $\mathbf{v}$

is the column vector $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$ such that $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_k\mathbf{v}_k$.

Dimension Let V be a subspace of $\mathbb{R}^n$. The *dimension* of V is the number, m , of elements in any basis of V . We write

$$\dim(V) = m$$

Null space of a matrix Let A be an $m \times n$ matrix. The *null space* of A , denoted by $\text{null}(A)$, is the set of all vectors $\mathbf{x}$ in $\mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{0}$. It is a subspace of $\mathbb{R}^n$.

Nullity of a matrix Let A be a matrix. The dimension of the null space of A is called the *nullity* of A .

$$\dim(\text{null}(A)) = \text{nullity}(A)$$

Ordered basis A basis in which the elements appear in a specific fixed order. Establishing an order is necessary because a coordinate vector with respect to a given basis relies on the order in which the basis elements appear.

Rank of a matrix Let A be a matrix. The dimension of the row space of A is called the *rank* of A .

$$\dim(\text{row}(A)) = \text{rank}(A)$$

Rank-Nullity theorem Let A be an $m \times n$ matrix. Then

$$\text{rank}(A) + \text{nullity}(A) = n$$

Row space of a matrix Let A be an $m \times n$ matrix. The *row space* of A , denoted by $\text{row}(A)$, is the subspace of $\mathbb{R}^n$ spanned by the rows of A .

Subspace Suppose that V is a nonempty subset of $\mathbb{R}^n$ that is closed under addition and closed under scalar multiplication. Then V is a *subspace* of $\mathbb{R}^n$.

Exercise 34 Let $V = \text{span} \left(\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -2 \end{bmatrix} \right)$. Recall that the span of a set of vectors in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$. From the choices below select ALL sets that can serve as bases for V .

Select All Correct Answers:

1. $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -2 \end{bmatrix} \right\}$

2. $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

3. $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -2 \end{bmatrix} \right\}$

4. $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -2 \end{bmatrix} \right\}$

5. $\left\{ \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

Exercise 35 Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$ be an ordered basis for $\mathbb{R}^3$. (Mentally verify that $\mathcal{B}$ is a basis of $\mathbb{R}^3$.)

Let $\mathbf{v} = \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$ be a vector of $\mathbb{R}^3$ (written with respect to the standard basis of $\mathbb{R}^3$). Find the coordinate vector for $\mathbf{v}$ with respect to $\mathcal{B}$.

$$\begin{bmatrix} \dots \dots \\ \dots \dots \\ \dots \dots \end{bmatrix}$$

Exercise 36 Consider matrix A and $\text{rref}(A)$ shown below.

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 & -1 \\ 1 & 2 & 2 & 3 & 0 \\ 0 & 0 & 1 & 3 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 0 & -3 & -2 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \text{rref}(A)$$

Find each of the following:

$$\text{rank}(A) = \dots$$

$$\dim(\text{row}(A)) = \dots$$

$$\dim(\text{col}(A)) = \dots$$

$$\dim(\text{null}(A)) = \dots$$

Select an appropriate basis for $\text{row}(A)$.

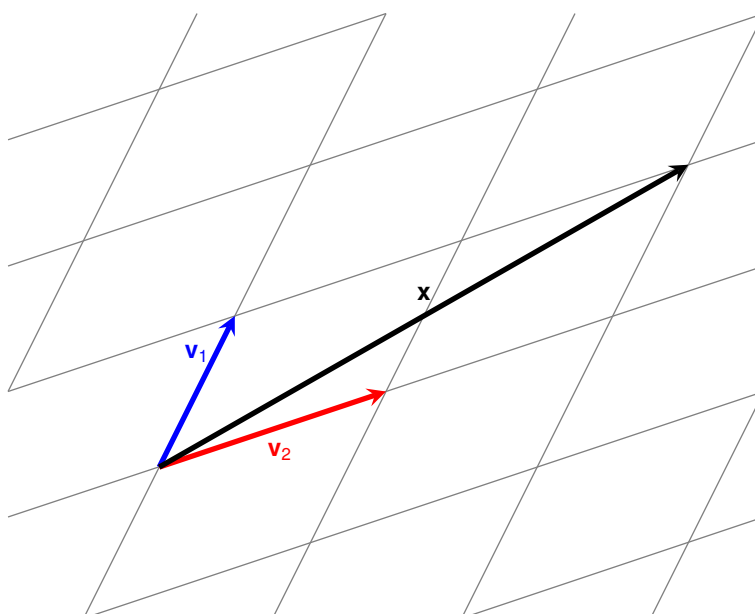
Multiple Choice:

1. $\left\{ \begin{bmatrix} 1 & 2 & 1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 2 & 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 3 & 1 \end{bmatrix} \right\}$
2. $\left\{ \begin{bmatrix} 1 & 2 & 0 & -3 & -2 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 3 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\}$
3. $\left\{ \begin{bmatrix} 1 & 2 & 0 & -3 & -2 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 3 & 1 \end{bmatrix} \right\}$

Using the algorithm in *Subspaces of $\mathbb{R}^n$ Associated with Matrices*, find a basis for $\text{col}(A)$.

$$\left\{ \begin{bmatrix} \dots \\ \dots \\ \dots \\ \dots \end{bmatrix}, \begin{bmatrix} \dots \\ \dots \\ \dots \\ \dots \end{bmatrix} \right\}$$

Exercise 37 Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ be a basis for the plane depicted below. Find the coordinate vector for $\mathbf{x}$ with respect to $\mathcal{B}$.



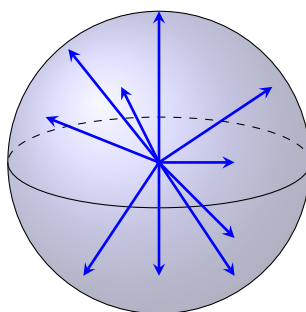
$$\begin{bmatrix} \dots \\ \dots \end{bmatrix}$$

Exercise 38 Suppose that $\mathcal{B}_1 = \{\mathbf{v}_1, \mathbf{v}_2\}$ is an ordered basis for some subspace V of $\mathbb{R}^n$. Let $\mathcal{B}_2 = \{-\mathbf{v}_2, -2\mathbf{v}_1\}$. Verify that $\mathcal{B}_2$ is also an ordered basis for V .

Let $\mathbf{w}$ be a vector in V . If the coordinate vector for $\mathbf{w}$ with respect to $\mathcal{B}_1$ is $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$, find the coordinate vector for $\mathbf{w}$ with respect to $\mathcal{B}_2$.

$$\begin{bmatrix} \dots \\ \dots \end{bmatrix}$$

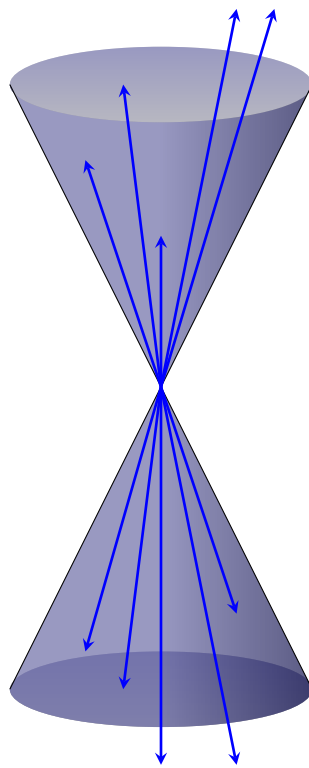
Exercise 39 Determine whether each set, S , of vectors is closed under vector addition, and scalar multiplication. Let S be the set of all vectors contained in a sphere of radius 1 centered at the origin.



Select All Correct Answers:

1. S is closed under scalar multiplication.
2. S is closed under vector addition.
3. None of the above.

Let S be the set of all vectors contained in an infinite double cone.



Select All Correct Answers:

1. S is closed under scalar multiplication.
2. S is closed under vector addition.
3. None of the above.

Exercise 40 Find the dimension of V if

$$V = \text{span} \left(\begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 1 \\ -1 \end{bmatrix} \right)$$

$\dim(V) = \dots \dots$

Exercise 41 True or False? If False, you should come up with a counterexample. If True, can you give a proof?

1. If V is a subspace of $\mathbb{R}^n$ and $\mathbf{x} + \mathbf{y}$ is in V , then $\mathbf{x}$ is in V or $\mathbf{y}$ is in V .

Multiple Choice:

- (i) True

(ii) *False*

2. If V is a set in $\mathbb{R}^n$ such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ is in V whenever $\mathbf{v}_1$ and $\mathbf{v}_2$ are in V for any scalars c_1, c_2 , then V is a subspace.

Multiple Choice:

(i) *True*

(ii) *False*

3. Every set of four non-zero vectors in $\mathbb{R}^4$ is a basis.

Multiple Choice:

(i) *True*

(ii) *False*

4. $\mathbb{R}^3$ has a basis of the form $\{\mathbf{x}, \mathbf{x} + \mathbf{y}, \mathbf{y}\}$.

Multiple Choice:

(i) *True*

(ii) *False*

Source

[Nicholson] W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2021, Open Edition, Exercise 5.1.

6 Linear Transformations

Learning Outcomes

After completing this chapter, students should be able to do the following.

- Interpret a matrix as a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.
- Produce a matrix representation of a linear transformation given its action on the standard unit vectors.
- Identify some basic geometric transformations of the plane.
- Discuss the kernel and image of the transformation geometrically and in terms of nullity and rank of the matrix.
- Give examples of linear transformations that are invertible or not invertible.
- Interpret a matrix product as a composition of linear transformations.

6.1 Essential Vocabulary

Composition of linear transformations Let U , V and W be vector spaces, and let $T : U \rightarrow V$ and $S : V \rightarrow W$ be linear transformations. The *composition* of S and T is the transformation $S \circ T : U \rightarrow W$ given by

$$(S \circ T)(\mathbf{u}) = S(T(\mathbf{u}))$$

The matrix of a composition is the product of the matrices corresponding to the transformations in the composition, in the same order.

Image of a linear transformation Let V and W be vector spaces, and let $T : V \rightarrow W$ be a linear transformation. The *image* of T , denoted by $\text{im}(T)$, is the set

$$\text{im}(T) = \{T(\mathbf{v}) : \mathbf{v} \in V\}$$

In other words, the image of T consists of individual images of all vectors of V .

Inverse of a linear transformation Let V and W be vector spaces, and let $T : V \rightarrow W$ be a linear transformation. A transformation $S : W \rightarrow V$ that satisfies $S \circ T = \text{id}_V$ and $T \circ S = \text{id}_W$ is called an *inverse* of T . If T has an inverse, T is called *invertible*.

Kernel of a linear transformation Let V and W be vector spaces, and let $T : V \rightarrow W$ be a linear transformation. The *kernel* of T , denoted by $\ker(T)$, is the set

$$\ker(T) = \{\mathbf{v} : T(\mathbf{v}) = \mathbf{0}\}$$

In other words, the kernel of T consists of all vectors of V that map to $\mathbf{0}$ in W .

Linear transformation A transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is called a *linear transformation* if the following are true for all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, and scalars k .

$$T(k\mathbf{u}) = kT(\mathbf{u})$$

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$

Standard matrix of a linear transformation Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Then the matrix

$$A = \begin{bmatrix} | & | & & | \\ T(\mathbf{e}_1) & T(\mathbf{e}_2) & \dots & T(\mathbf{e}_n) \\ | & | & & | \end{bmatrix}$$

is known as the *standard matrix of the linear transformation* T .

Exercise 42 Suppose a linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is such that

$$T \left(\begin{bmatrix} 2 \\ -3 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T \left(\begin{bmatrix} -1 \\ 7 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

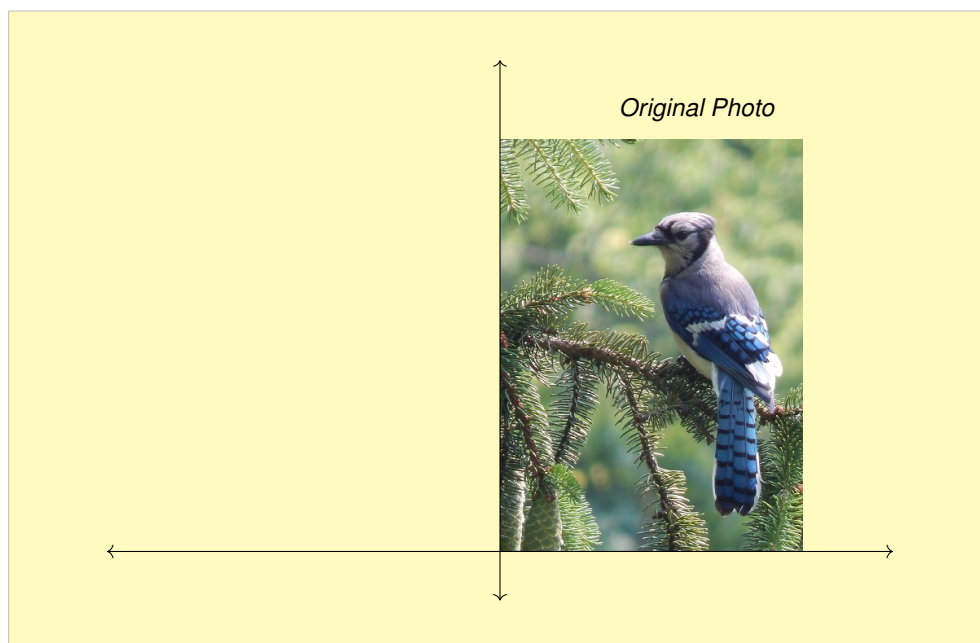
Then,

$$T \left(\begin{bmatrix} 1 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} \dots\dots \\ \dots\dots \end{bmatrix}$$

Exercise 43 Let

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Consider the linear transformation $T_M : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ induced by M . What is the image of the photograph below under T_M ?



Click on the picture to select the correct image from the four choices below.

Multiple Choice:

Image A



1.

Image B



2.

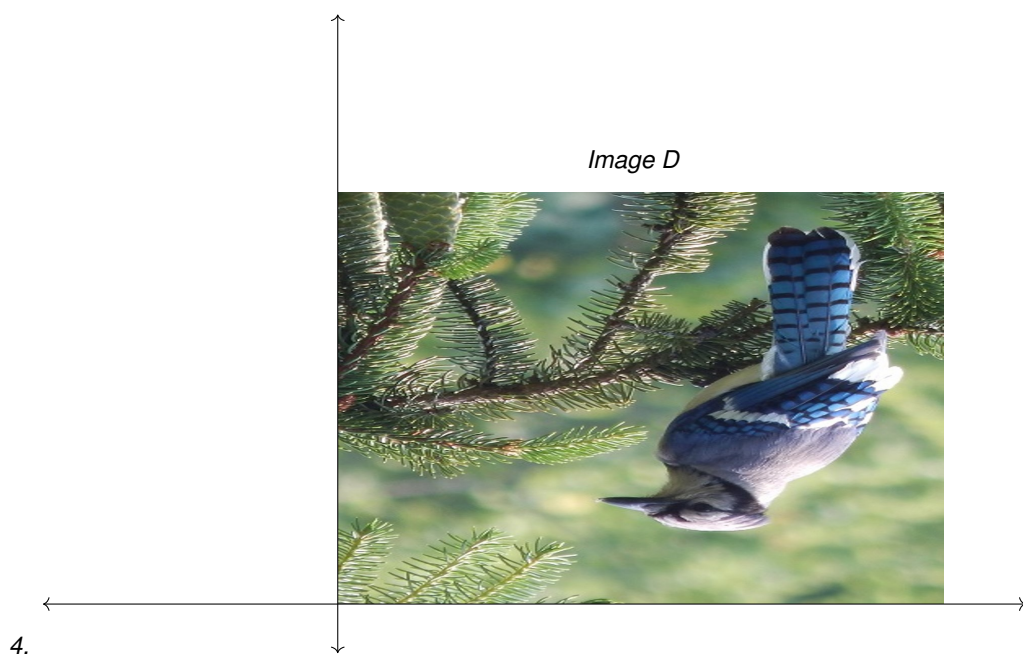


Photo Credit

[Davis] Erik Davis, CC-BY 4.0.

Exercise 44 Find the standard matrix M of a linear transformation $T_M : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ if

$$T \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}; \quad T \left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}; \quad T \left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$M = \begin{bmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \end{bmatrix}$$

Exercise 45 Suppose that an invertible linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is such that

$$T\left(\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ -7 \end{bmatrix}; \quad T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -5 \\ 4 \end{bmatrix}$$

Find the vector whose image under T is $\begin{bmatrix} 3 \\ -10 \end{bmatrix}$

$$T\left(\begin{bmatrix} \dots \\ \dots \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -10 \end{bmatrix}$$

Exercise 46 True or False? If False, you should come up with a counterexample. If True, can you give a proof?

1. $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, given by $T(x, y) = (x, -y)$, is a linear transformation.

Multiple Choice:

- (i) True
- (ii) False

2. $T : \mathbb{R}^n \rightarrow \mathbb{R}$, given by $T(\mathbf{x}) = \mathbf{x} \cdot \mathbf{z}$ for some fixed vector $\mathbf{z} \in \mathbb{R}^n$, is a linear transformation.

Multiple Choice:

- (i) True
- (ii) False

3. $T : \mathbb{R} \rightarrow \mathbb{R}$, given by $T(x) = x^2$, is a linear transformation.

Multiple Choice:

- (i) True
- (ii) False

4. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation and let $\mathbf{v}_1, \dots, \mathbf{v}_k$ denote vectors in $\mathbb{R}^n$. If $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_k)\}$ is linearly independent, then $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is also linearly independent.

Multiple Choice:

- (i) True
- (ii) False

5. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation and suppose $\mathbf{v}_1, \mathbf{v}_2$ denote vectors in $\mathbb{R}^2$. If $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent, then $\{T(\mathbf{v}_1), T(\mathbf{v}_2)\}$ is also linearly independent.

Multiple Choice:

- (i) True
- (ii) False

Source

[Nicholson] W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2021, Open Edition, Exercises 7.1.1, 7.1.2, and 7.1.17,

Exercise 47 Let

$$A = \begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & 0 & 3 & 1 \\ 1 & 1 & -4 & 2 \end{bmatrix}$$

Use techniques discussed in *Image and Kernel of a Linear Transformation* to find the basis for the kernel and the image of the linear transformation, T_A , induced by A .

$$\text{Basis for } \text{im}(T_A): \left\{ \begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix}, \begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix} \right\}$$

$$\text{Basis for } \text{ker}(T_A): \left\{ \begin{bmatrix} \dots \\ \dots \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \dots \\ \dots \\ 0 \\ 1 \end{bmatrix} \right\}$$

Source

[Nicholson] W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2021, Open Edition, Exercises 7.2.1 (b).

7 The Determinant

Learning Outcomes

After completing this chapter, students should be able to do the following.

- Find determinants using cofactor expansion along a row or column.
- Interpret determinants as areas and volumes.
- Describe how row operations affect the determinant.
- Describe the significance when the determinant is zero.
- Analyze the determinant of a product algebraically and geometrically.
- Solve a system of equations using Cramer's rule.

7.1 Essential Vocabulary

Adjugate of a matrix (the term **Adjoint** is also sometimes used) The transpose of the matrix of cofactors of a matrix - it is part of a formula for the inverse of a matrix.

Cofactor expansion A method to compute $\det A$ using determinants of minor matrices associated with one row or one column.

Cramer's rule A method of solving systems of equations that uses determinants.

Determinant A function that assigns a scalar output to each square matrix A , denoted $\det A$ - it is nonzero if and only if A is invertible. Geometrically speaking, the determinant of a linear transformation of a square matrix is the factor by which area (or volume or hypervolume) is scaled by the transformation.

Laplace Expansion Theorem The determinant of a matrix can be computed using cofactor expansion along ANY row or ANY column.

Properties of determinants

1. The determinant of a triangular matrix is the product of its diagonal entries.
 2. The determinant of a matrix is equal to the determinant of its transpose.
 3. The determinant of the inverse of a matrix is the reciprocal of the determinant of the matrix.
 4. A matrix with a zero row has determinant zero.
 5. Interchanging two rows of a matrix changes the sign of its determinant.
 6. A matrix with two identical rows has determinant zero.
 7. Multiplying a row of a matrix by k multiplies the determinant by a factor of k .
 8. Multiplying a matrix by k multiplies the determinant by a factor of k^n .
 9. Adding a multiple of one row of a matrix to another row does not change the determinant.
 10. A matrix is singular if and only if its determinant is zero.
 11. The determinant of a product is equal to the product of the determinants.
-

Exercise 48 Find the determinant of A if

$$A = \begin{bmatrix} 1 & 3 & 4 & 1 \\ k & 0 & 0 & 2 \\ 0 & 1 & -2 & 1 \\ 1 & -1 & 0 & 1 \end{bmatrix}$$

where k is a constant.

$$\det(A) = \dots\dots$$

For what value of k is A singular?

$$k = \dots\dots$$

Exercise 49 Find the determinant of A by first adding all other rows to the first row.

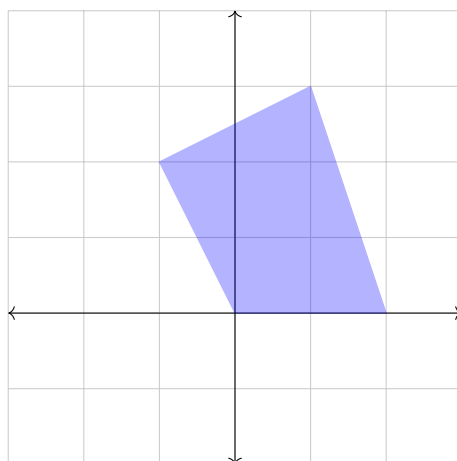
$$A = \begin{bmatrix} x-1 & 2 & 3 \\ 2 & -3 & x-2 \\ -2 & x & -2 \end{bmatrix}$$

$$\det(A) = \dots\dots$$

Source

[Nicholson] W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2021, Open Edition, Exercises 3.1.14 (a).

Exercise 50 Find the area of the polygon shown below.



$$\text{Area} = \dots\dots \text{ square units}$$

Exercise 51 Let B be a parallelepiped determined by vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^3$. Suppose the volume of B is 10 cubic units. Let B' be a parallelepiped determined by $-\mathbf{u}$, $2\mathbf{v}$, and $\mathbf{w} + \mathbf{u}$. Find the volume of B' .

volume of $B' = \dots \dots$ cubic units

Exercise 52 Suppose A is an invertible 2×2 matrix such that $\det A = 4$. Suppose $A^{-1} = \begin{bmatrix} 1 & x \\ 1/2 & 3/2 \end{bmatrix}$. Find x .

$x = \dots \dots$

Exercise 53 True or False? If False, you should come up with a counterexample.

Suppose A and B are $n \times n$ matrices ($n \geq 2$).

1. $\det(AB) = \det A \det B$

Multiple Choice:

- (i) True
- (ii) False

2. If $\det A = \frac{1}{\det B}$, then $A = B^{-1}$.

Multiple Choice:

- (i) True
- (ii) False

3. $\det(A + B) = \det A + \det B$.

Multiple Choice:

- (i) True
- (ii) False

4. If $A = 2B$, then $\det A = 2 \det B$.

Multiple Choice:

- (i) True
- (ii) False

5. If $A = -B$, then $\det A = -\det B$ when n is odd, and $\det A = \det B$ when n is even.

Multiple Choice:

- (i) True
- (ii) False

8 Eigenvalues and Eigenvectors

Learning Outcomes

After completing this chapter, students should be able to do the following.

- Define eigenvalues and eigenvectors geometrically.
- Use characteristic polynomials to compute eigenvalues and eigenvectors.
- Define algebraic multiplicity and geometric multiplicity.
- Utilize properties to show that two matrices are not similar.
- Use eigenvectors of matrices, when possible, to diagonalize a matrix.
- Plot the Gershgorin region of a matrix.
- Implement the Power Method and its variants.

8.1 Essential Vocabulary

Algebraic multiplicity of an eigenvalue The multiplicity of an eigenvalue as a root of the characteristic equation.

Characteristic equation The equation

$$\det(A - \lambda I) = 0$$

is called the *characteristic equation* of A .

Characteristic polynomial The polynomial

$$\det(A - \lambda I)$$

is called the *characteristic polynomial* of A .

Diagonalizable matrix Let A be an $n \times n$ matrix. Then A is said to be *diagonalizable* if there exists an invertible matrix P such that

$$P^{-1}AP = D$$

where D is a diagonal matrix. In other words, a matrix A is diagonalizable if it is similar to a diagonal matrix, $A \sim D$.

Eigenspace If λ is an eigenvalue of an $n \times n$ matrix, the set of all eigenvectors associated to λ along with the zero vector is the *eigenspace* associated to λ . The eigenspace is a subspace of $\mathbb{R}^n$.

Eigenvalue Let A be an $n \times n$ matrix. We say that a scalar λ is an *eigenvalue* of A if

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some nonzero vector $\mathbf{x}$. We say that $\mathbf{x}$ is an *eigenvector* of A associated with the eigenvalue λ .

Eigenvalue decomposition If λ is an eigenvalue of an $n \times n$ matrix, the set of all eigenvectors associated to λ along with the zero vector is the *eigenspace* associated to λ . The eigenspace is a subspace of $\mathbb{R}^n$.

Eigenvector Let A be an $n \times n$ matrix. We say that a non-zero vector $\mathbf{x}$ is an *eigenvector* of A if

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some scalar λ . We say that λ is an *eigenvalue* of A associated with the eigenvector $\mathbf{x}$.

Geometric multiplicity of an eigenvalue The *geometric multiplicity* of an eigenvalue λ is the dimension of the corresponding eigenspace $\mathcal{S}_\lambda$.

Gershgorin disk A circle in the complex plane which has a diagonal entry of a matrix as its center and the sum of the absolute values of the other entries in that row (or column) as its radius.

Gershgorin's Theorem *Gershgorin's theorem* says that the n eigenvalues of an $n \times n$ matrix can be found in the region in the complex plane consisting of the n Gershgorin disks.

Power method (and its variants) The *power method* is an iterative method for computing the dominant eigenvalue of a matrix. Its variants can compute the smallest eigenvalue or the eigenvalue closest to some target.

Properties of similar matrices Similar matrices must have the same...

1. determinant,
 2. rank,
 3. trace,
 4. characteristic polynomial,
and
 5. eigenvalues.
-

Similar matrices If A and B are $n \times n$ matrices, we say that A and B are *similar*, if $B = P^{-1}AP$ for some invertible matrix P . In this case we write $A \sim B$.

Exercise 54 Find eigenvalues of $A = \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix}$.

Eigenvalues (in increasing order): $\lambda_1 = \dots$, $\lambda_2 = \dots$

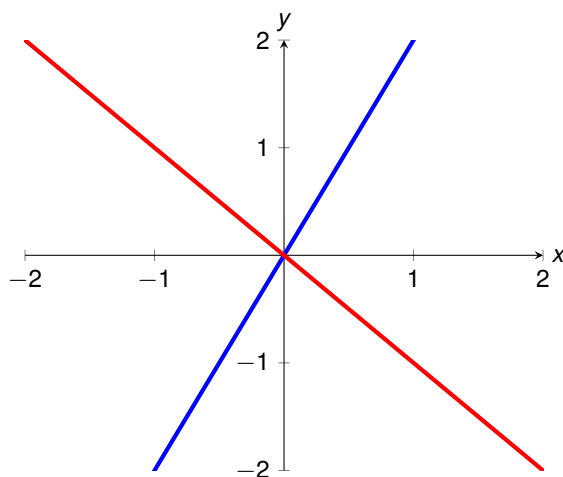
Compute a basis for the eigenspace associated with each of these eigenvalues.

A basis for S_{λ_1} : $\left\{ \begin{bmatrix} 1 \\ \dots \end{bmatrix} \right\}$

A basis for S_{λ_2} : $\left\{ \begin{bmatrix} 4 \\ \dots \end{bmatrix} \right\}$

Exercise 55 Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation induced by some matrix A . The sketch below shows two lines. Suppose the following:

- when transformation T is applied to vectors along the line $y = 2x$, the vectors reverse direction, but keep their lengths.
- when transformation T is applied to vectors along the line $y = -x$, the vectors keep their direction, but triple in length.



Find the eigenvalues of A .

Eigenvalues of A (in increasing order): $\lambda_1 = \dots$, $\lambda_2 = \dots$

Find a basis for the eigenspace associated with each of these eigenvalues.

A basis for S_{λ_1} : $\left\{ \begin{bmatrix} 1 \\ \dots \end{bmatrix} \right\}$

A basis for S_{λ_2} : $\left\{ \begin{bmatrix} 1 \\ \dots \end{bmatrix} \right\}$

Exercise 56 Let $A = \begin{bmatrix} 2 & 3 & -3 \\ 1 & 0 & -1 \\ 1 & 1 & -2 \end{bmatrix}$ and let $B = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix}$.

Each of these matrices has the same characteristic polynomial, $(2 - \lambda)(1 + \lambda)^2$. Which of the following is true? (Justify your answer.)

Multiple Choice:

1. A is diagonalizable but B is not diagonalizable.
2. A is not diagonalizable but B is diagonalizable.
3. Both A and B are diagonalizable.
4. Neither A nor B is diagonalizable.

Source

[Nicholson] W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2021, Open Edition, Problem 3.3.26.

Exercise 57 For square matrices A and B . True or False? If False, you should come up with a counterexample. If True, can you give a proof?

1. If λ is an eigenvalue of A , then $-\lambda$ is an eigenvalue of A^{-1} .

Multiple Choice:

- (i) True
- (ii) False

2. If λ is an eigenvalue of A , then $\lambda - c$ is an eigenvalue of $A - cI$.

Multiple Choice:

- (i) True
- (ii) False

3. If λ is an eigenvalue of A , then 2λ is an eigenvalue of $2A$.

Multiple Choice:

- (i) True
- (ii) False

Exercise 58 For matrices A , B and C . True or False? If False, you should come up with a counterexample. If True, can you give a proof?

1. If A is similar to B , then A^2 is similar to B^2 .

Multiple Choice:

- (i) True
- (ii) False

2. If A^2 is similar to B^2 , then A is similar to B .

Multiple Choice:

- (i) True
- (ii) False

3. If A^T is similar to B , then A is similar to B^T .

Multiple Choice:

- (i) True
- (ii) False

4. If A is similar to B , and B is similar to C , then A is similar to C .

Multiple Choice:

- (i) True
- (ii) False

Exercise 59 Let $A = \begin{bmatrix} -6 & -76 & -16 \\ 2 & 21 & 4 \\ -2 & -64 & -17 \end{bmatrix}$.

1. The vector $\mathbf{v}_1 = \begin{bmatrix} -4 \\ 1 \\ -3 \end{bmatrix}$ is an eigenvector of A because $A\mathbf{v}_1 = \lambda_1\mathbf{v}_1$ where $\lambda_1 =$

Multiple Choice:

- (i) -3
- (ii) -1
- (iii) 2

2. The vector $\mathbf{v}_2 = \begin{bmatrix} 8 \\ -2 \\ 7 \end{bmatrix}$ is an eigenvector of A because $A\mathbf{v}_2 = \lambda_2\mathbf{v}_2$ where $\lambda_2 =$

Multiple Choice:

- (i) -3
- (ii) -1
- (iii) 2

3. The vector $\mathbf{v}_3 = \begin{bmatrix} 4 \\ -1 \\ 4 \end{bmatrix}$ is an eigenvector of A because $A\mathbf{v}_3 = \lambda_3\mathbf{v}_3$ where $\lambda_3 =$

Multiple Choice:

- (i) -3
- (ii) -1
- (iii) 2

4. Since the Power Method converges to the dominant eigenvector, it can be used to approximate which eigenvector of A ?

Multiple Choice:

- (i) $\mathbf{v}_1$
- (ii) $\mathbf{v}_2$
- (iii) $\mathbf{v}_3$

9 Orthogonality

Learning Outcomes

After completing this chapter, students should be able to do the following.

- Discuss orthogonal and orthonormal bases, Gram-Schmidt orthogonalization, orthogonal complements and projections.
- Explain how orthogonal projections relate to least square approximations.
- Discuss general inner product spaces and symmetric matrices, and associated norms.
- Discuss the singular value decomposition of a matrix.

9.1 Essential Vocabulary

Fundamental subspaces of a matrix $\text{null}(A)$ is the orthogonal complement of $\text{row}(A)$, and $\text{null}(A^T)$ is the orthogonal complement of $\text{col}(A)$.

Gram-Schmidt process An iterative process which constructs an orthogonal basis for a subspace. The idea is to build the orthogonal set one vector at a time, by taking a vector not in the span of the vectors in the current iteration of the set, and subtracting its orthogonal projection onto each of those vectors.

Orthogonal Basis A set of orthogonal vectors that spans a subspace. (Any orthogonal set of vectors must be linearly independent.)

Orthogonal complement of a subspace If W is a subspace, we define the *orthogonal complement* $W^\perp$ as the set of all vectors orthogonal to every vector in W , i.e.,

$$W^\perp = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \cdot \mathbf{y} = 0 \text{ for all } \mathbf{y} \in W\}$$

Orthogonal Decomposition Theorem Let W be a subspace of $\mathbb{R}^n$ and let $\mathbf{x} \in \mathbb{R}^n$. Then there exist unique vectors $\mathbf{w} \in W$ and $\mathbf{w}^\perp \in W^\perp$ such that $\mathbf{x} = \mathbf{w} + \mathbf{w}^\perp$.

Orthogonal matrices An $n \times n$ matrix Q is called an *orthogonal matrix* if its columns form an orthonormal set. This will happen if and only if its rows form an orthonormal set. Note also that Q is an orthogonal matrix if and only if it is an invertible matrix such that $Q^{-1} = Q^T$.

Orthogonal projection onto a subspace Let W be a subspace of $\mathbb{R}^n$ with orthogonal basis $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m\}$. If $\mathbf{x}$ is in $\mathbb{R}^n$, the vector

$$\mathbf{w} = \text{proj}_W(\mathbf{x}) = \text{proj}_{\mathbf{f}_1} \mathbf{x} + \text{proj}_{\mathbf{f}_2} \mathbf{x} + \dots + \text{proj}_{\mathbf{f}_m} \mathbf{x}$$

is called the *orthogonal projection* of $\mathbf{x}$ onto W .

Orthogonal set of vectors Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ be a set of nonzero vectors in $\mathbb{R}^n$. Then this set is called an *orthogonal set* if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for all $i \neq j$. Moreover, if $\|\mathbf{v}_i\| = 1$ for $i = 1, \dots, m$ (i.e. each vector in the set is a unit vector), we say the set of vectors is an *orthonormal set*.

Orthogonally diagonalizable matrix An $n \times n$ matrix A is said to be *orthogonally diagonalizable* if an orthogonal matrix Q can be found such that $Q^{-1}AQ = Q^T A Q$ is diagonal.

Orthonormal basis A set of orthonormal vectors that spans a subspace. (Any orthogonal set of vectors must be linearly independent by Theorem ??.)

Orthonormal set of vectors Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ be a set of nonzero vectors in $\mathbb{R}^n$. Then this set is called an *orthogonal set* if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for all $i \neq j$. Moreover, if $\|\mathbf{v}_i\| = 1$ for $i = 1, \dots, m$ (i.e. each vector in the set is a unit vector), we say the set of vectors is an *orthonormal set*.

Properties of orthogonal matrices If Q is an orthogonal matrix, then...

1. $Q^{-1} = Q^T$ is orthogonal,
2. $\det Q = \pm 1$,
3. if λ is an eigenvalue of Q , then $|\lambda| = 1$,
4. the product of Q with any other orthogonal matrix will be an orthogonal matrix (i.e. orthogonal matrices are closed under matrix multiplication),
and
5. Q is a length-preserving and angle-preserving linear transformation.

QR factorization Let A be an $m \times n$ matrix with independent columns. A *QR-factorization* of A expresses it as $A = QR$ where Q is $m \times n$ with orthonormal columns and R is an invertible and upper triangular matrix with positive diagonal entries.

Rank-Nullity Theorem for matrices (??) Let A be an $m \times n$ matrix. Then

$$\text{rank}(A) + \text{nullity}(A) = n.$$

This implies that the dimension of a subspace of $\mathbb{R}^n$ plus the dimension of its orthogonal complement equals n .

Spectral Theorem If A is a real $n \times n$ matrix, then A is symmetric if and only if A is orthogonally diagonalizable.

Exercise 60 Vectors $\mathbf{q}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{q}_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ form an orthonormal basis for the plane $x - y - 2z = 0$.

(You should know how to verify that $\mathbf{q}_1, \mathbf{q}_2$ are orthonormal.) Use orthogonal projections to express $\mathbf{x} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$ as a linear combination of $\mathbf{q}_1$ and $\mathbf{q}_2$.

Enter exact coefficients below. Do not round.

$$\mathbf{x} = \dots \dots \mathbf{q}_1 + \dots \dots \mathbf{q}_2$$

Exercise 61 Vectors $\mathbf{q}_1 = \frac{1}{7} \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix}$ and $\mathbf{q}_2 = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$ form an orthonormal basis for the plane $3x + 20y + 9z = 0$.

(You should know how to verify that $\mathbf{q}_1, \mathbf{q}_2$ are orthonormal.) Express the orthogonal projection of $\mathbf{x} = \begin{bmatrix} 14 \\ 7 \\ 21 \end{bmatrix}$ onto the plane as a linear combination of $\mathbf{q}_1$ and $\mathbf{q}_2$.

Enter exact coefficients below. Do not round.

$$\mathbf{x} = \dots \dots \mathbf{q}_1 + \dots \dots \mathbf{q}_2$$

Exercise 62 Use technology to find the line of best fit.

Desmos link: <https://www.desmos.com/calculator/xfoeppxjbn>

Enter your computed coefficients to three decimal places.

$$y = \dots \dots x + \dots \dots$$

Plot your your line of best fit in the Desmos interactive above.

xfoeppxjbn

Exercise 63 Let W be the span of the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}$ $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 4 \\ -4 \end{bmatrix}$.

In this exercise we will compute a basis for $W^\perp$.

1. Form a matrix A by using $\mathbf{v}_1$ and $\mathbf{v}_2$ as the rows of A . Then A is a 2×4 matrix, and $W^\perp$ is the

Multiple Choice:

(i) Row

(ii) *Column*(iii) *Null*space of A .

$$2. \text{ A basis of } \text{null}(W^\perp) : \left\{ \begin{bmatrix} 5 \\ \dots \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ \dots \\ 0 \\ 1 \end{bmatrix} \right\}$$

Exercise 64 If $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 7 \\ 0 & 7 & 3 \end{bmatrix}$, find an orthogonal matrix Q such that $Q^{-1}AQ$ is diagonal.

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ \dots & 1/\sqrt{2} & 1/\sqrt{2} \\ \dots & 1/\sqrt{2} & \dots \end{bmatrix}$$

Exercise 65 True or False? If False, you should come up with a counterexample. If True, can you give a proof?

1. If A is symmetric, then A has orthonormal columns.

Multiple Choice:

- (i) *True*
- (ii) *False*

2. The product of two orthogonal matrices is an orthogonal matrix.

Multiple Choice:

- (i) *True*
- (ii) *False*

3. The orthogonal complement of the row space of a matrix is the null space of that matrix.

Multiple Choice:

- (i) *True*
- (ii) *False*

4. For any proper subspace of $\mathbb{R}^n$, it is possible to produce an orthogonal basis.

Multiple Choice:

- (i) *True*
- (ii) *False*

10 Vector Spaces

Learning Outcomes

After completing this chapter, students should be able to do the following.

- Use axioms for abstract vector spaces (over the real or complex fields) to discuss examples (and non-examples) of abstract vector spaces such as subspaces of the space of all polynomials.
- Discuss the existence of a basis of an abstract vector space.
- Describe coordinates of a vector relative to a given basis.
- For a linear transformation between vector spaces, discuss its matrix relative to given bases.
- Discuss how the matrix of a linear transformation with respect to a basis changes when the basis is changed.
- Discuss the advantages of a change of basis that leads to a simplified matrix and simplified description of a linear map.

10.1 Essential Vocabulary

Coordinate vector

Let V be a vector space, and let $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis for V . If $\mathbf{v} = a_1\mathbf{v}_1 + \dots + a_n\mathbf{v}_n$, then the vector in $\mathbb{R}^n$ whose components are the coefficients $a_1, \dots, a_n$ is said to be the *coordinate vector for $\mathbf{v}$ with respect to $\mathcal{B}$* . We denote the coordinate vector by $[\mathbf{v}]_{\mathcal{B}}$ and write:

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

Finite-dimensional vector space

A vector space is said to be *finite-dimensional* if it is spanned by finitely many vectors.

Isomorphism

Let V and W be vector spaces. If there exists an invertible linear transformation $T : V \rightarrow W$ we say that V and W are *isomorphic* and write $V \cong W$. The invertible linear transformation T is called an *isomorphism*.

Matrix of a linear transformation

Let V and W be finite-dimensional vector spaces with ordered bases $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ and $\mathcal{C}$, respectively. Suppose $T : V \rightarrow W$ is a linear transformation.

$$\text{Let } A = \begin{bmatrix} | & | & & | \\ [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \dots & [T(\mathbf{v}_n)]_{\mathcal{C}} \\ | & | & & | \end{bmatrix}$$

Then $A[\mathbf{v}]_{\mathcal{B}} = [T(\mathbf{v})]_{\mathcal{C}}$ for all vectors $\mathbf{v}$ in V .

Matrix A is called the matrix of T with respect to ordered bases $\mathcal{B}$ and $\mathcal{C}$.

One-to-one linear transformation

A linear transformation $T : V \rightarrow W$ is *one-to-one* if

$$T(\mathbf{v}_1) = T(\mathbf{v}_2) \text{ implies that } \mathbf{v}_1 = \mathbf{v}_2$$

Onto linear transformation

A linear transformation $T : V \rightarrow W$ is *onto* if for every element $\mathbf{w}$ of W , there exists an element $\mathbf{v}$ of V such that $T(\mathbf{v}) = \mathbf{w}$.

Subspace

A nonempty subset U of a vector space V is called a *subspace* of V , provided that U is itself a vector space when given the same addition and scalar multiplication as V .

Vector Space

Let V be a nonempty set. Suppose that elements of V can be added together and multiplied by scalars. The set V , together with operations of addition and scalar multiplication, is called a *vector space* provided that

V is closed under addition

V is closed under scalar multiplication

and the following properties hold for $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in V and scalars k and p :

1. Commutative Property of Addition: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. Associative Property of Addition: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. Existence of Additive Identity: $\mathbf{u} + \mathbf{0} = \mathbf{u}$
4. Existence of Additive Inverse: $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5. Distributive Property over Vector Addition: $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
6. Distributive Property over Scalar Addition: $(k + p)\mathbf{u} = k\mathbf{u} + p\mathbf{u}$
7. Associative Property for Scalar Multiplication: $k(p\mathbf{u}) = (kp)\mathbf{u}$
8. Multiplication by 1: $1\mathbf{u} = \mathbf{u}$

We will refer to elements of V as *vectors*.

Note that definitions of span, linear independence, basis, and dimension are analogous to those for subspaces of $\mathbb{R}^n$.

Exercise 66 Are the following vector spaces with the indicated operations?

1. The set V of all polynomials of degree greater than or equal to 3, together with 0; operations of $\mathbb{P}$.

Multiple Choice:

- (i) V is a vector space.
- (ii) V is not a vector space.

2. The set V of all polynomials of degree less than or equal to 3; operations of $\mathbb{P}$.

Multiple Choice:

- (i) V is a vector space.
- (ii) V is not a vector space.

3. The set V of 2×2 matrices with zero determinant; usual matrix operations.

Multiple Choice:

- (i) V is a vector space.
- (ii) V is not a vector space.

4. The set V of all 2×2 matrices whose entries sum to 0; usual matrix operations.

Multiple Choice:

- (i) V is a vector space.
- (ii) V is not a vector space.

Source

[Nicholson] W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2021, Open Edition, Exercises 6.1.2.

Exercise 67 Describe the following subspace of $\mathbb{M}_{3,3}$.

$$\text{span} \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\}$$

The subspace consists of:

Multiple Choice:

- 1. Matrices A of $\mathbb{M}_{3,3}$ such that $A^T = A$.
- 2. All invertible matrices of $\mathbb{M}_{3,3}$.

3. Upper and lower triangular matrices of $\mathbb{M}_{3,3}$.

4. All of $\mathbb{M}_{3,3}$.

Exercise 68 Let V denote the set of ordered triples (x, y, z) and define addition in V as in $\mathbb{R}^3$. For each of the following definitions of scalar multiplication, select all properties of scalar multiplication for vector spaces that hold and decide whether V is a vector space.

1.

$$a(x, y, z) = (ax, y, az)$$

Select all properties of scalar multiplication that hold true.

Select All Correct Answers:

- (i) Closure under scalar multiplication
- (ii) Distributive Property over Vector Addition: $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
- (iii) Distributive Property over Scalar Addition: $(k + p)\mathbf{u} = k\mathbf{u} + p\mathbf{u}$
- (iv) Associative Property for Scalar Multiplication: $k(p\mathbf{u}) = (kp)\mathbf{u}$
- (v) Multiplication by 1: $1\mathbf{u} = \mathbf{u}$

Is V a vector space?

Multiple Choice:

- (i) V is a vector space.
- (ii) V is not a vector space.

2.

$$a(x, y, z) = (ax, 0, az)$$

Select all properties of scalar multiplication that hold true.

Select All Correct Answers:

- (i) Closure under scalar multiplication
- (ii) Distributive Property over Vector Addition: $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
- (iii) Distributive Property over Scalar Addition: $(k + p)\mathbf{u} = k\mathbf{u} + p\mathbf{u}$
- (iv) Associative Property for Scalar Multiplication: $k(p\mathbf{u}) = (kp)\mathbf{u}$
- (v) Multiplication by 1: $1\mathbf{u} = \mathbf{u}$

Is V a vector space?

Multiple Choice:

- (i) V is a vector space.
- (ii) V is not a vector space.

3.

$$a(x, y, z) = (0, 0, 0)$$

Select all properties of scalar multiplication that hold true.

Select All Correct Answers:

- (i) Closure under scalar multiplication
- (ii) Distributive Property over Vector Addition: $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
- (iii) Distributive Property over Scalar Addition: $(k + p)\mathbf{u} = k\mathbf{u} + p\mathbf{u}$
- (iv) Associative Property for Scalar Multiplication: $k(p\mathbf{u}) = (kp)\mathbf{u}$
- (v) Multiplication by 1: $1\mathbf{u} = \mathbf{u}$

Is V a vector space?

Multiple Choice:

- (i) V is a vector space.
- (ii) V is not a vector space.

4.

$$a(x, y, z) = (2ax, 2ay, 2az)$$

Select all properties of scalar multiplication that hold true.

Select All Correct Answers:

- (i) Closure under scalar multiplication
- (ii) Distributive Property over Vector Addition: $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
- (iii) Distributive Property over Scalar Addition: $(k + p)\mathbf{u} = k\mathbf{u} + p\mathbf{u}$
- (iv) Associative Property for Scalar Multiplication: $k(p\mathbf{u}) = (kp)\mathbf{u}$
- (v) Multiplication by 1: $1\mathbf{u} = \mathbf{u}$

Is V a vector space?

Multiple Choice:

- (i) V is a vector space.
- (ii) V is not a vector space.

Source

[Nicholson] W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2021, Open Edition, Exercises 6.1.1.

Exercise 69 Find the coordinates of $\mathbf{v} = ax^2 + bx + c$ of $\mathbb{P}^2$ with respect to the ordered basis $\mathcal{B} = \{x^2, x + 1, x + 2\}$.

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} \dots \\ \dots \\ \dots \\ \dots \end{bmatrix}$$

Source

[Nicholson] W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2021, Open Edition, Exercises 9.1.1 (b).

Exercise 70 Suppose $T : \mathbb{P}^2 \rightarrow \mathbb{P}^3$ is a linear transformation such that

$$T(x^2) = x^3, \quad T(x + 1) = 0, \quad T(x - 1) = x$$

Find $T(x^2 + x + 1)$.

$$T(x^2 + x + 1) = \dots$$

Source

[Nicholson] W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2021, Open Edition, Exercises 7.1.4 (c).

Exercise 71 Suppose $T : \mathbb{M}_{2,2} \rightarrow \mathbb{R}$ is a linear transformation such that

$$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = 3$$

$$T\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = -1$$

$$T\left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\right) = 0$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Find $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \dots$$

Source

[Nicholson] W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2021, Open Edition, Exercises 7.1.4 (d).

Exercise 72 Find the matrix of the linear transformation $T : \mathbb{M}_{2,2} \rightarrow \mathbb{M}_{2,2}$, given by $T(A) = A^T$, if the basis of the domain ($\mathcal{B}$) and the basis of the codomain ($\mathcal{D}$) are given by

$$\mathcal{B} = \mathcal{D} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$\begin{bmatrix} \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

Source

[Nicholson] W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2021, Open Edition, Exercises 9.1.3 (b).