
Additional Problems

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Solved Problems for Chapter 2

Problem 1 You have a system of k equations in two variables, $k \geq 2$. Explain the geometric significance of

1. No solution.
The k lines do not have a point common to all of them.
2. A unique solution.
All k lines intersect at a single point.
3. An infinite number of solutions.
The k lines coincide.

Click the arrow to see answer.

Problem 2 Consider the following augmented matrix in which $*$ denotes an arbitrary number and $\blacksquare$ denotes a nonzero number. Determine whether the given augmented matrix corresponds to a consistent system. If consistent, is the solution unique?

$$\left[\begin{array}{ccccc|c} \blacksquare & * & * & * & * & * \\ 0 & \blacksquare & * & * & 0 & * \\ 0 & 0 & 0 & 0 & \blacksquare & 0 \\ 0 & 0 & 0 & 0 & \blacksquare & \blacksquare \end{array} \right]$$

Click the arrow to see answer. The third equation implies that $x_5 = 0$. The fourth equation implies that $x_5 \neq 0$. We conclude that the system is inconsistent.

Problem 3 Suppose a system of equations has fewer equations than variables. Will such a system necessarily be consistent? If so, explain why and if not, give an example which is not consistent.

Click the arrow to see answer. No. Consider $x + y + z = 2$ and $x + y + z = 1$.

Problem 4 If a system of equations has more equations than variables, can it have a solution? If so, give an example and if not, explain why not.

Click the arrow to see answer. These can have a solution. For example, $x + y = 1$, $2x + 2y = 2$, $3x + 3y = 3$ even has an infinite set of solutions.

Problem 5 Find h such that

$$\left[\begin{array}{cc|c} 2 & h & 4 \\ 3 & 6 & 7 \end{array} \right]$$

is the augmented matrix of an inconsistent system.

$$\left[\begin{array}{cc|c} 2 & h & 4 \\ 3 & 6 & 7 \end{array} \right] \rightsquigarrow \left[\begin{array}{cc|c} 3 & (3/2)h & 6 \\ 3 & 6 & 7 \end{array} \right]. \text{ The system is inconsistent for } h = 4.$$

Problem 6 Find h such that

$$\left[\begin{array}{cc|c} 1 & h & 3 \\ 2 & 4 & 6 \end{array} \right]$$

is the augmented matrix of a consistent system.

Click the arrow to see answer. The reduced row-echelon form will never have a row of the form $[0 \ 0 \ | \ 1]$. The system is consistent for all h .

Problem 7 Choose h and k such that the augmented matrix shown has each of the following:

1. one solution
2. no solution
3. infinitely many solutions

$$\left[\begin{array}{cc|c} 1 & 2 & 2 \\ 2 & h & k \end{array} \right]$$

Click the arrow to see answer. If $h \neq 4$, then there is exactly one solution. If $h = 4$ and $k \neq 4$, then there are no solutions. If $h = 4$ and $k = 4$, then there are infinitely many solutions.

Problem 8 Determine if the system is consistent. If so, is the solution unique?

$$\begin{array}{rrrrrr} x & + & 2y & + & z & - & w & = & 2 \\ x & - & y & + & z & + & w & = & 1 \\ 2x & + & y & - & z & & & = & 1 \\ 4x & + & 2y & + & z & & & = & 5 \end{array}$$

Click the arrow to see answer. There is no solution because the rref of the augmented matrix contains a row of the form $[0 \ 0 \ 0 \ 0 \ | \ 1]$

$$\text{rref} \left(\left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 2 \\ 1 & -1 & 1 & 1 & 1 \\ 2 & 1 & -1 & 0 & 1 \\ 4 & 2 & 1 & 0 & 5 \end{array} \right] \right) = \left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & 0 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right].$$

Problem 9 Determine if the system is consistent. If so, is the solution unique?

$$\begin{array}{rrcrcl} x & + & 2y & + & z & - & w & = & 2 \\ x & - & y & + & z & + & w & = & 0 \\ 2x & + & y & - & z & & & = & 1 \\ 4x & + & 2y & + & z & & & = & 3 \end{array}$$

Click the arrow to see answer.

$$\text{rref} \left(\left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 2 \\ 1 & -1 & 1 & 1 & 0 \\ 2 & 1 & -1 & 0 & 1 \\ 4 & 2 & 1 & 0 & 3 \end{array} \right] \right) = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1/3 & 1/3 \\ 0 & 1 & 0 & -2/3 & 2/3 \\ 0 & 0 & 1 & 0 & 1/3 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

There are infinitely many solutions: $w = t$, $z = \frac{1}{3}$, $y = \frac{2}{3} + \frac{2}{3}t$, $x = \frac{1}{3} - \frac{1}{3}t$.

Problem 10 Find the solution of the system whose augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 1 & 3 & 4 & 2 \\ 1 & 0 & 2 & 1 \end{array} \right]$$

Click the arrow to see answer.

$$\text{rref} \left(\left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 1 & 3 & 4 & 2 \\ 1 & 0 & 2 & 1 \end{array} \right] \right) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6/5 \\ 0 & 1 & 0 & 2/5 \\ 0 & 0 & 1 & -1/10 \end{array} \right]$$

$$x = \frac{6}{5}, y = \frac{2}{5}, z = -\frac{1}{10}$$

Problem 11 Solve the system whose augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 0 & 4 & 2 \end{array} \right]$$

Click the arrow to see answer. The reduced row echelon form is $\left[\begin{array}{ccc|c} 1 & 0 & 4 & 2 \\ 0 & 1 & -4 & -1 \end{array} \right]$ and so the solution is $z = t$, $y = 4t$, $x = 2 - 4t$.

Problem 12 Solve the system if the rref of its augmented matrix is

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 9 & 3 \\ 0 & 1 & 0 & 0 & -4 & 0 \\ 0 & 0 & 1 & 0 & -7 & -1 \\ 0 & 0 & 0 & 1 & 6 & 1 \end{array} \right]$$

Click the arrow to see answer. $x_5 = t$, $x_4 = 1 - 6t$, $x_3 = -1 + 7t$, $x_2 = 4t$, $x_1 = 3 - 9t$.

Problem 13 Solve the system if the rref of its augmented matrix is

$$\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -1/2 & 5/2 \\ 0 & 1 & 0 & 0 & 1/2 & 3/2 \\ 0 & 0 & 0 & 1 & 3/2 & -1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Click the arrow to see answer. The free variables are $x_5 = t$, $x_3 = s$. The other variables are given by $x_4 = -\frac{1}{2} - \frac{3}{2}t$, $x_2 = \frac{3}{2} - \frac{1}{2}t$, $x_1 = \frac{5}{2} + \frac{1}{2}t - 2s$.

Problem 14 Suppose a system of equations has fewer equations than variables and you have found a solution to this system of equations. Is it possible that your solution is the only one? Explain.

Click the arrow to see answer. No. The rank of the coefficient matrix in this case is smaller than the number of columns (variables). So, there has to be a free variable. The parameter (t is a typical choice) assigned to the free variable will guarantee infinitely many solutions.

Problem 15 Suppose A is an $m \times n$ matrix. Explain why the rank of A is always no larger than $\min(m, n)$.

Click the arrow to see answer. It is because you cannot have more leading 1's than columns and you cannot have more leading 1's than rows.

Problem 16 Find the rank of the following matrix.

$$\begin{bmatrix} 4 & 4 & 20 & -1 & 17 \\ 1 & 1 & 5 & 0 & 5 \\ 1 & 1 & 5 & -1 & 2 \\ 3 & 3 & 15 & -3 & 6 \end{bmatrix}$$

Click the arrow to see answer.

$$\text{rref} \left(\begin{bmatrix} 4 & 4 & 20 & -1 & 17 \\ 1 & 1 & 5 & 0 & 5 \\ 1 & 1 & 5 & -1 & 2 \\ 3 & 3 & 15 & -3 & 6 \end{bmatrix} \right) = \begin{bmatrix} 1 & 1 & 5 & 0 & 5 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank is the number of leading 1's. The rank of the given matrix is 2.

Bibliography

These problems come from the end of Chapter 1 of Ken Kuttler's *A First Course in Linear Algebra*. (CC-BY)

Ken Kuttler, *A First Course in Linear Algebra*, Lyryx 2017, Open Edition, pp. 42–49.

3 Challenge Problems

Challenge Problems for Chapter 2

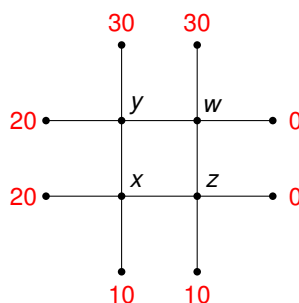
Problem 17 A man is ordered by his doctor to take 5 units of vitamin A, 13 units of vitamin B, and 23 units of vitamin C each day. Three brands of vitamin pills are available, and the number of units of each vitamin per pill are shown in the accompanying table.

Brand	A	B	C
1	1	2	4
2	1	1	3
3	0	1	1

- Find all combinations of pills that provide exactly the required amount of vitamins (no partial pills allowed).
- If brands 1, 2, and 3 cost 3 cents, 2 cents, and 5 cents per pill, respectively, find the least expensive treatment.
 of brand 1, of brand 2, of brand 3

Problem 18 A restaurant owner plans to use x tables seating 4, y tables seating 6, and z tables seating 8, for a total of 20 tables. When fully occupied, the tables seat 108 customers. If only half of the x tables, half of the y tables, and one-fourth of the z tables are used, each fully occupied, then 46 customers will be seated. Find x , y , and z .

Problem 19 The steady state temperature, u , of a plate solves Laplace's equation, $\Delta u = 0$. One way to approximate the solution is to divide the plate into a square mesh and require the temperature at each node to equal the average of the temperature at the four adjacent nodes. In the following picture, the numbers represent the observed temperature at the indicated nodes. Find the temperature at the interior nodes, indicated by x , y , z , and w . One of the equations is $z = \frac{1}{4}(10 + 0 + w + x)$.



$$w = \dots$$

$$x = \dots$$

$$y = \dots$$

$$z = \dots$$

Problem 20 Solve the following system of equations for x and y .

$$\begin{aligned} x^2 + xy - y^2 &= 1 \\ 2x^2 - xy + 3y^2 &= 13 \\ x^2 + 3xy + 2y^2 &= 0 \end{aligned}$$

Problem 21 Find coefficients a , b and c such that the graph of $y = ax^2 + bx + c$ passes through the points $(-2, 1)$, $(2, 5)$, $(4, 1)$. Graph the equation you found to check your answer. Given any three points in the plane, does a solution always exist? What are the possibilities when only two points are given? What about more than three points? Illustrate your answers with examples.

Problem 22 Use the concepts from Problem ?? to establish the fact that two distinct points determine a line.

Problem 23 A circle with radius r , centered at the origin is a graph of $x^2 + y^2 - r^2 = 0$. How many points does it take to uniquely determine a circle with this equation? In general, a circle is a graph of $x^2 + y^2 + ax + by + c = 0$. How many points determine a circle? Can a circle be drawn through any three points? Support your answers geometrically and algebraically.

Bibliography

These problems come from the end of Chapter 1 of Ken Kuttler's [A First Course in Linear Algebra](#). (CC-BY)

Ken Kuttler, *A First Course in Linear Algebra*, Lyryx 2017, Open Edition, pp. 42–49.

4 Chapter 3 Problems

5 Solved Problems

Solved Problems for Chapter 3

Problem 24 If possible, express the vector

$$\mathbf{v} = \begin{bmatrix} 4 \\ 4 \\ -3 \end{bmatrix}$$

as a linear combination of the vectors

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \text{ and } \mathbf{u}_2 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

Click the arrow to see answer. We start with a linear combination equation $a \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + b \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ -3 \end{bmatrix}$. This gives us the following:

$$\left[\begin{array}{cc|c} 3 & 2 & 4 \\ 1 & -2 & 4 \\ -1 & 1 & -3 \end{array} \right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array} \right]$$

This gives us $\mathbf{v} = 2\mathbf{u}_1 - \mathbf{u}_2$.

Problem 25 Decide whether

$$\mathbf{v} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$

is a linear combination of the vectors

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \text{ and } \mathbf{u}_2 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

Click the arrow to see answer.

$$\left[\begin{array}{cc|c} 3 & 2 & 4 \\ 1 & -2 & 4 \\ -1 & 1 & 4 \end{array} \right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]$$

The last row indicates that there is no solution to the linear combination equation $a\mathbf{u}_1 + b\mathbf{u}_2 = \mathbf{v}$. Therefore, $\mathbf{v}$ is not a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$.

Problem 26 Determine whether the following set of vectors is linearly independent.

$$\left\{ \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\}$$

Click the arrow for answer.

The linear combination equation $a \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = \mathbf{0}$ has only the trivial solution. We can see this as follows.

$$\text{rref} \left(\begin{bmatrix} 2 & 1 & 1 \\ -3 & 0 & 1 \\ 0 & 1 & 4 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore the given set of vectors is linearly independent.

Problem 27 Determine whether the following set of vectors is linearly independent.

$$\left\{ \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\}$$

Click the arrow for answer.

The linear combination equation $a \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix} + b \begin{bmatrix} 4 \\ -1 \\ 8 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = \mathbf{0}$ has a non-trivial solution. We can see this as follows.

$$\text{rref} \left(\begin{bmatrix} 2 & 4 & 1 \\ -3 & -1 & 1 \\ 0 & 8 & 4 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore the given set of vectors is linearly dependent.

Problem 28 Suppose $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ is a set of vectors from $\mathbb{R}^n$. Show that $\mathbf{0}$ is in $\text{span}\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$.

Click the arrow to see answer. Let all of the coefficients in the linear combination be zero: $\sum_{i=1}^k 0\mathbf{x}_i = \mathbf{0}$

Problem 29 True or false?

Click the arrow to see each answer.

1. The span of any one vector in $\mathbb{R}^n$ is a line.

This is technically false because the span of the zero vector is a point. For all non-zero vectors this statement is true.

2. The span of two non-zero vectors in $\mathbb{R}^n$ is a plane.

This is false. If two vectors are scalar multiples of each other (i.e. they are linearly dependent), then their span is a line, not a plane.

3. Given two linearly independent vectors, $\mathbf{v}$ and $\mathbf{w}$, let $\mathbf{n}$ be a normal vector to the plane spanned by $\mathbf{v}$ and $\mathbf{w}$. The set $\{\mathbf{v}, \mathbf{w}, \mathbf{n}\}$ is linearly independent.

This is true. A normal to the plane is a vector perpendicular to the plane. Only vectors that lie in the plane spanned by $\mathbf{v}$ and $\mathbf{w}$ can be expressed as a linear combination of $\mathbf{v}$ and $\mathbf{w}$. Since $\mathbf{n}$ is not in the plane, it cannot be expressed as a linear combination of $\mathbf{v}$ and $\mathbf{w}$. Therefore the set $\{\mathbf{v}, \mathbf{w}, \mathbf{n}\}$ is linearly independent.

Bibliography

Some of the Review Exercises come from the end of Chapter 4 of Ken Kuttler's [A First Course in Linear Algebra](#). (CC-BY)

Ken Kuttler, *A First Course in Linear Algebra*, Lyryx 2017, Open Edition, pp. 151–152.

6 Challenge Problems

Challenge Problems for Chapter 3

Problem 30 Suppose $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ is a linearly independent set in $\mathbb{R}^n$, and that $\mathbf{w}$ is not in $\text{span}(\mathbf{v}_1, \dots, \mathbf{v}_m)$.

1. Is $\mathbf{w}$ in $\text{span}(\mathbf{v}_1 + \mathbf{w}, \dots, \mathbf{v}_m + \mathbf{w})$ (YES, / NO)
2. Is $\{\mathbf{v}_1 + \mathbf{w}, \dots, \mathbf{v}_m + \mathbf{w}\}$ linearly independent?
(YES, / NO)

Problem 31 Suppose $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$ are the rows of the 3×3 matrix A . Then we can interpret the solution to the system of equations $[A|\mathbf{0}]$ as the intersection of three planes containing the origin. Discuss what this intersection would look like geometrically if the reduced row echelon form of $[A|\mathbf{0}]$ is of the form:

1.

$$\left[\begin{array}{ccc|c} 1 & 0 & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

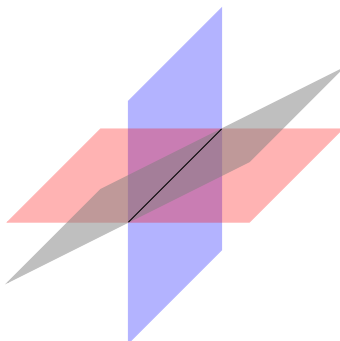
2.

$$\left[\begin{array}{ccc|c} 1 & * & * & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

3. Are there any other possibilities?

Problem 32 Show that for any matrix A , $\text{rref}(A)$ has a row of zeros if and only if one of the rows of A can be expressed as a linear combination of the others.

Problem 33 Recall that a plane in $\mathbb{R}^3$ has an equation of the form $ax + by + cz = d$, where a , b , c are the components of a normal vector to the plane. Suppose three planes intersect in a line, as shown below.



We can find the line of intersection by solving a system of equations. Suppose the augmented matrix $[A|\mathbf{d}]$ corresponds to this system.

1. Explain why $\text{rref}[A|\mathbf{d}]$ will have a row of zeros.
2. Consider the normal vectors to the three planes. What geometric property of these particular three normal vectors explains the row of zeros in the reduced row-echelon form?

Bibliography

The first problem came from the end of Chapter 1 of Keith Nicholson's *Linear Algebra with Applications*. (CC-BY-NC-SA)

W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2018, Open Edition, pp. 33–34.

7 Chapter 4 Problems

8 Solved Problems

Solved Problems for Chapter 4

Problem 34 Using only the properties given in Theorem ?? and Theorem ??, show that the additive inverse of A , $-A$, is unique.

Click on the arrow to see answer. Suppose B is also an additive inverse of A . Then

$$-A = -A + (A + B) = (-A + A) + B = 0 + B = B$$

Problem 35 Using only the properties given in Theorem ?? and Theorem ??, show that the $n \times m$ zero matrix, O , is unique.

Click on the arrow to see answer. Suppose O' is also an $n \times m$ additive identity. Then $O' = O' + O = O$.

Problem 36 Using only the properties given in Theorem ?? and Theorem ??, show that $0A = O$.

Click on the arrow to see answer. $0A = (0 + 0)A = 0A + 0A$. Now add $-(0A)$ to both sides. Then $O = 0A$.

Problem 37 Using only the properties given in Theorem ?? and Theorem ??, as well as previous problems, show $(-1)A = -A$.

Click on the arrow to see answer. $A + (-1)A = (1 + (-1))A = 0A = O$. Therefore, from the uniqueness of the additive inverse proved in the above Problem ??, it follows that $-A = (-1)A$.

Problem 38 Consider the matrices $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 & 2 \\ -3 & 2 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$,

$$D = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}, E = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Find the following if possible. If it is not possible explain why.

1. $-3A$

Click on the arrow to see answer. $\begin{bmatrix} -3 & -6 & -9 \\ -6 & -3 & -21 \end{bmatrix}$

2. $3B - A$

Click on the arrow to see answer. $\begin{bmatrix} 8 & -5 & 3 \\ -11 & 5 & -4 \end{bmatrix}$

3. AC

Click on the arrow to see answer. Not possible

4. CB

Click on the arrow to see answer. $\begin{bmatrix} -3 & 3 & 4 \\ 6 & -1 & 7 \end{bmatrix}$

5. AE

Click on the arrow to see answer. Not possible

6. EA

Click on the arrow to see answer. Not possible

Problem 39 Let $A = \begin{bmatrix} -1 & -1 \\ 3 & 3 \end{bmatrix}$. Find all 2×2 matrices, B such that $AB = O$.

Click on the arrow to see answer.

$$\begin{bmatrix} -1 & -1 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} -x - z & -w - y \\ 3x + 3z & 3w + 3y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Solution is: $w = -y, x = -z$ so the matrices are of the form $\begin{bmatrix} x & y \\ -x & -y \end{bmatrix}$.

Problem 40 Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 3 & k \end{bmatrix}$. Is it possible to choose k such that $AB = BA$? If so, what should k equal?

Click on the arrow to see answer.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & k \end{bmatrix} = \begin{bmatrix} 7 & 2k+2 \\ 15 & 4k+6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & k \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 3k+3 & 4k+6 \end{bmatrix}$$

Thus you must have $\begin{matrix} 3k+3=15 \\ 2k+2=10 \end{matrix}$. Therefore $k = 4$.

Problem 41 Find 2×2 matrices A and B such that $A \neq O$ and $B \neq O$ but $AB = O$.

Click on the arrow to see answer. Let $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Problem 42 Suppose $AB = AC$ and A is an invertible $n \times n$ matrix. Does it follow that $B = C$? Explain why or why not.

Click on the arrow to see answer. Yes $B = C$. Multiply $AB = AC$ on the left by A^{-1} .

Problem 43 Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 1 & 0 & 2 \end{bmatrix}$$

Find A^{-1} if possible. If A^{-1} does not exist, explain why.

Click on the arrow to see answer. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 1 & 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 4 & -5 \\ 0 & 1 & -2 \\ 1 & -2 & 3 \end{bmatrix}$

Problem 44 Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 4 & 5 & 10 \end{bmatrix}$$

Find A^{-1} if possible. If A^{-1} does not exist, explain why.

Click on the arrow to see answer. The reduced row echelon form is $\begin{bmatrix} 1 & 0 & \frac{5}{3} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix}$. There is no inverse.

Problem 45 Show that if A^{-1} exists for an $n \times n$ matrix, then it is unique. That is, if $BA = I$ and $AB = I$, then $B = A^{-1}$.

Click on the arrow to see answer. $A^{-1} = A^{-1}I = A^{-1}(AB) = (A^{-1}A)B = IB = B$.

Problem 46 Show that if A is an invertible $n \times n$ matrix, then so is A^T and $(A^T)^{-1} = (A^{-1})^T$.

Click the arrow to see answer.

You need to show that $(A^{-1})^T$ acts like the inverse of A^T because from uniqueness in the above problem, this will imply it is the inverse. From properties of the transpose,

$$A^T (A^{-1})^T = (A^{-1}A)^T = I^T = I$$

$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$$

Hence $(A^{-1})^T = (A^T)^{-1}$ and this last matrix exists.

Problem 47 Show $(AB)^{-1} = B^{-1}A^{-1}$ by verifying that

$$AB(B^{-1}A^{-1}) = I$$

and

$$B^{-1}A^{-1}(AB) = I$$

Click on the arrow to see answer. $(AB)B^{-1}A^{-1} = A(BB^{-1})A^{-1} = AA^{-1} = I$ $B^{-1}A^{-1}(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$

Problem 48 Find an LU factorization of the coefficient matrix and use it to solve the system of equations.

$$x + 2y = 5$$

$$2x + 3y = 6$$

Click on the arrow to see answer. An LU factorization of the coefficient matrix is

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$$

First solve

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

which gives $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$. Then solve

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$$

which says that $y = 4$ and $x = -3$.

Problem 49 Find an LU factorization of the coefficient matrix and use it to solve the system of equations.

$$x + 2y + z = 1$$

$$y + 3z = 2$$

$$2x + 3y = 6$$

Click on the arrow to see answer. An LU factorization of the coefficient matrix is

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 2 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

First solve

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$$

which yields $u = 1, v = 2, w = 6$. Next solve

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$$

This yields $z = 6, y = -16, x = 27$.

Bibliography

These problems come from the end of Chapter 2 of Ken Kuttler's *A First Course in Linear Algebra*. (CC-BY)

Ken Kuttler, *A First Course in Linear Algebra*, Lyryx 2017, Open Edition, pp. 90–98, 104–106.

9 Challenge Problems

Challenge Problems for Chapter 4

Problem 50 Solve for the matrix X if:

1. $PXQ = R$;

2. $XP = S$;

where $P = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 3 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \end{bmatrix}$,
 $R = \begin{bmatrix} -1 & 1 & -4 \\ -4 & 0 & -6 \\ 6 & 6 & -6 \end{bmatrix}$, $S = \begin{bmatrix} 1 & 6 \\ 3 & 1 \end{bmatrix}$

Problem 51 Consider

$$p(X) = X^3 - 5X^2 + 11X - 4I.$$

1. If $p(U) = \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}$ compute $p(U^T)$.

2. If $p(U) = 0$ where U is $n \times n$, find U^{-1} in terms of U .

Click on the arrow to see answer. $U^{-1} = \frac{1}{4}(U^2 - 5U + 11I)$.

Problem 52 Assume that a system $A\mathbf{x} = \mathbf{b}$ of linear equations has at least two distinct solutions $\mathbf{y}$ and $\mathbf{z}$.

1. Show that $\mathbf{x}_k = \mathbf{y} + k(\mathbf{y} - \mathbf{z})$ is a solution for every k .

2. Show that $\mathbf{x}_k = \mathbf{x}_m$ implies $k = m$.

3. Deduce that $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

Click on the arrow to see answer.

1. If $\mathbf{x}_k = \mathbf{x}_m$, then $\mathbf{y} + k(\mathbf{y} - \mathbf{z}) = \mathbf{y} + m(\mathbf{y} - \mathbf{z})$. So $(k - m)(\mathbf{y} - \mathbf{z}) = \mathbf{0}$. But $\mathbf{y} - \mathbf{z}$ is not zero (because $\mathbf{y}$ and $\mathbf{z}$ are distinct), so $k - m = 0$.

Problem 53 Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

1. show that $A^2 = I$.
2. What is wrong with the following argument? If $A^2 = I$, then $A^2 - I = 0$, so $(A - I)(A + I) = 0$, whence $A = I$ or $A = -I$.

Problem 54 Let E and F be elementary matrices obtained from the identity matrix by adding multiples of row k to rows p and q . If $k \neq p$ and $k \neq q$, show that $EF = FE$.

Bibliography

These exercises come from the end of Chapter 2 of Keith Nicholson's *Linear Algebra with Applications*. (CC-BY-NC-SA)

W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2018, Open Edition, pp. 143–144.

10 Chapter 5 Problems

11 Solved Problems

Solved Problems for Chapter 5

Problem 55 Let $M = \left\{ \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \in \mathbb{R}^4 : |u_1| \leq 4 \right\}$. Is M a subspace? Explain.

Click the arrow to see answer. No. $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is in M but $10 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is not in M .

Problem 56 Let $M = \left\{ \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \in \mathbb{R}^4 : u_i \geq 0 \text{ for each } i = 1, 2, 3, 4 \right\}$. Is M a subspace? Explain.

Click the arrow to see answer. This is not a subspace. $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is in M . However, $(-1) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is not in M .

Problem 57 Let $\mathbf{w} \in \mathbb{R}^4$ and let $M = \left\{ \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \in \mathbb{R}^4 : \mathbf{w} \cdot \mathbf{u} = 0 \right\}$. Is M a subspace? Explain.

Click the arrow to see answer. Yes, this is a subspace. First, M is not empty because the zero vector is in M . You can verify closure under vector addition and scalar multiplication using the properties of the dot product as follows. If $\mathbf{u}$ is in M , then $\mathbf{w} \cdot \mathbf{u} = 0$ and $\mathbf{w} \cdot k\mathbf{u} = 0$ for all constants k . Finally, if $\mathbf{u}_1$ and $\mathbf{u}_2$ are in M , then $\mathbf{w} \cdot (\mathbf{u}_1 + \mathbf{u}_2) = \mathbf{w} \cdot \mathbf{u}_1 + \mathbf{w} \cdot \mathbf{u}_2 = 0$.

Problem 58 Let $M = \left\{ \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \in \mathbb{R}^4 : u_3 = u_1 = 0 \right\}$. Is M a subspace? Explain.

Click the arrow to see answer. This is a subspace. M is not empty because the zero vector is in it. M is closed under vector addition and scalar multiplication.

Problem 59 If you have 5 vectors in $\mathbb{R}^5$ and the vectors are linearly independent, can it always be concluded they span $\mathbb{R}^5$? Explain.

Click the arrow to see answer. Yes. If not, there would exist a vector not in the span. But then you could add in this vector and obtain a linearly independent set of vectors with more vectors than a basis.

Problem 60 If you have 6 vectors in $\mathbb{R}^5$, is it possible they are linearly independent? Explain.

Click the arrow to see answer. A matrix A with five rows and six columns will contain a non-pivot column, giving rise to a non-trivial solution (infinitely many of them) to the equation $A\mathbf{x} = \mathbf{0}$. This shows that the columns of A are linearly dependent.

Problem 61 Suppose V, W are subspaces of $\mathbb{R}^n$. Let $V \cap W$ be all vectors which are in both V and W . Show that $V \cap W$ is a subspace also.

Click the arrow to see answer. If $\mathbf{x}, \mathbf{y} \in V \cap W$, then for scalars α, β , the linear combination $\alpha\mathbf{x} + \beta\mathbf{y}$ must be in both V and W since they are both subspaces.

Problem 62 Find the rank of the following matrix, and find a basis for the column spaces.

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 3 & 1 & 10 & 0 \\ -1 & 1 & -2 & 1 \\ 1 & -1 & 2 & -2 \end{bmatrix}$$

Click the arrow to see answer.

$$\text{rref} \left(\begin{bmatrix} 1 & 0 & 3 & 0 \\ 3 & 1 & 10 & 0 \\ -1 & 1 & -2 & 1 \\ 1 & -1 & 2 & -2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank of the matrix is 3. We can use columns 1, 2, and 4 as a basis for the column space.

Problem 63 Find a basis for $\text{null}(A)$.

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

Click the arrow to see answer.

$$\text{rref} \left(\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

A basis for the null space is $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$.

Problem 64 Find a basis for $\text{null}(A)$

$$A = \begin{bmatrix} 2 & -1 & 3 & 5 \\ 2 & 0 & 1 & 2 \\ 6 & 4 & -5 & -6 \\ 0 & 2 & -4 & -6 \end{bmatrix}$$

Click the arrow to see answer.

$$\text{rref} \left(\begin{bmatrix} 2 & -1 & 3 & 5 \\ 2 & 0 & 1 & 2 \\ 6 & 4 & -5 & -6 \\ 0 & 2 & -4 & -6 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 1/2 & 1 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A basis for the null space is $\left\{ \begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Problem 65 Suppose matrix A has linearly independent columns. What is the null space of A ?

Click the arrow to see answer.

The only solution to $A\mathbf{x} = \mathbf{0}$ is the trivial one. Therefore, the only vector in $\text{null}(A)$ is the zero vector.

Problem 66 Let $V = \text{span} \left(\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ -2 \end{bmatrix} \right)$. Find a basis for V . What is the dimension of V ?

Click the arrow to see answer.

Form a matrix A using the given vectors as columns. $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$. The leading 1's are in the first two columns. We can use the first two of the given vectors to form a basis of V . Therefore $\dim(V) = 2$.

Bibliography

Some of the problems come from the end of Chapter 4 of Ken Kuttler's *A First Course in Linear Algebra*. (CC-BY) Ken Kuttler, *A First Course in Linear Algebra*, Lyryx 2017, Open Edition, pp. 227–232.

12 Challenge Problems

Challenge Problems for Chapter 5

Problem 67 Suppose V has dimension p and W has dimension q and they are each contained in a subspace, U which has dimension equal to n where $n > \max(p, q)$. What are the possibilities for the dimension of $V \cap W$?

Click the arrow to see the answer. Let $\{x_1, \dots, x_k\}$ be a basis for $V \cap W$. Then there is a basis for V and W which are respectively

$$\{x_1, \dots, x_k, y_{k+1}, \dots, y_p\}, \{x_1, \dots, x_k, z_{k+1}, \dots, z_q\}$$

It follows that you must have $k + p - k + q - k \leq n$ and so you must have

$$p + q - n \leq k$$

Problem 68 Let $\{v_1, v_2, \dots, v_n\}$ be a basis of $\mathbb{R}^n$. Let A be an $n \times n$ matrix.

1. If A is invertible, show that $\{Av_1, Av_2, \dots, Av_n\}$ is a basis of $\mathbb{R}^n$.
2. If $\{Av_1, Av_2, \dots, Av_n\}$ is a basis of $\mathbb{R}^n$, show that A is invertible.

Problem 69 Show that $\text{null}(A) = \text{null}(A^T A)$ for any real matrix A .

Problem 70 Let A be an $m \times n$ matrix of rank r . Show that $\dim(\text{null}(A)) = n - r$.

Problem 71 Let U and W be subspaces of $\mathbb{R}^3$. If $\dim(U) = \dim(W) = 2$, and $U \neq W$, show that $\dim(U \cap W) = 1$.

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These problems came from Chapter 6 of Keith Nicholson's *Linear Algebra with Applications*. (CC-BY-NC-SA)
 W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2018, Open Edition, pp. 358–359, and p. 370.

13 Chapter 6 Problems

14 Solved Problems

Solved Problems for Chapter 6

Problem 72 Show that the function T_u defined by $T_u(\mathbf{v}) = \mathbf{v} - \text{proj}_u(\mathbf{v})$ is also a linear transformation.

Click on the arrow to see answer.

$$\begin{aligned} T_u(a\mathbf{v} + b\mathbf{w}) &= a\mathbf{v} + b\mathbf{w} - \frac{(a\mathbf{v} + b\mathbf{w}) \cdot \mathbf{u}}{\|\mathbf{u}\|^2} \mathbf{u} = \\ &= a\mathbf{v} - a \frac{(\mathbf{v} \cdot \mathbf{u})}{\|\mathbf{u}\|^2} \mathbf{u} + b\mathbf{w} - b \frac{(\mathbf{w} \cdot \mathbf{u})}{\|\mathbf{u}\|^2} \mathbf{u} = aT_u(\mathbf{v}) + bT_u(\mathbf{w}) \end{aligned}$$

Problem 73 Let $\mathbf{u}$ be a fixed vector. The function T_u defined by $T_u\mathbf{v} = \mathbf{u} + \mathbf{v}$ has the effect of translating all vectors by adding $\mathbf{u} \neq \mathbf{0}$. Show that T_u is not a linear transformation.

Click the arrow to see answer.

Linear transformations take $\mathbf{0}$ to $\mathbf{0}$ which T does not. Also $T_a(\mathbf{u} + \mathbf{v}) \neq T_a\mathbf{u} + T_a\mathbf{v}$.

Problem 74 Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\pi/3$.

Click on the arrow to see answer.

$$\begin{bmatrix} \cos\left(\frac{\pi}{3}\right) & -\sin\left(\frac{\pi}{3}\right) \\ \sin\left(\frac{\pi}{3}\right) & \cos\left(\frac{\pi}{3}\right) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{bmatrix}$$

Problem 75 Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $2\pi/3$ and then reflects across the x axis.

Click on the arrow to see answer.

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\left(\frac{2\pi}{3}\right) & -\sin\left(\frac{2\pi}{3}\right) \\ \sin\left(\frac{2\pi}{3}\right) & \cos\left(\frac{2\pi}{3}\right) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & \frac{1}{2} \end{bmatrix}$$

Problem 76 Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\pi/6$ and then reflects across the x axis followed by a reflection across the y axis.

Click on the arrow to see answer.

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\left(\frac{\pi}{6}\right) & -\sin\left(\frac{\pi}{6}\right) \\ \sin\left(\frac{\pi}{6}\right) & \cos\left(\frac{\pi}{6}\right) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\sqrt{3} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} \end{bmatrix}$$

Problem 77 Let T be a linear transformation induced by the matrix $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ and let S be a linear transformation induced by $B = \begin{bmatrix} 0 & -2 \\ 4 & 2 \end{bmatrix}$. Find matrix of $S \circ T$ and find $(S \circ T)(\mathbf{x})$ for $\mathbf{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Click on the arrow to see answer. The matrix of $S \circ T$ is given by BA .

$$\begin{bmatrix} 0 & -2 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 10 & 8 \end{bmatrix}$$

Now, $(S \circ T)(\mathbf{x}) = (BA)\mathbf{x}$.

$$\begin{bmatrix} 2 & -4 \\ 10 & 8 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \end{bmatrix}$$

Problem 78 Let T be a linear transformation and suppose $T\left(\begin{bmatrix} 1 \\ -4 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$. Suppose S is a linear transformation induced by the matrix $B = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$. Find $(S \circ T)(\mathbf{x})$ for $\mathbf{x} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$.

Click on the arrow to see answer. To find $(S \circ T)(\mathbf{x})$ we compute $S(T(\mathbf{x}))$.

$$\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -4 \\ -11 \end{bmatrix}$$

Problem 79 Let T be a linear transformation given by

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Find a basis for $\ker(T)$ and a basis for $\text{im}(T)$. Find the dimension of the kernel and the image of T .

Click on the arrow to see answer. A basis for $\ker(T)$ is $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ and a basis for $\text{im}(T)$ is $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$. There are many other possibilities for the specific bases. $\dim(\ker(T)) = 1$ and $\dim(\text{im}(T)) = 1$.

Problem 80 Let T be a linear transformation given by

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Find a basis for $\ker(T)$ and a basis for $\operatorname{im}(T)$.

Click on the arrow to see answer.

In this case $\ker(T) = \{0\}$ and $\operatorname{im}(T) = \mathbb{R}^2$ (pick any basis of $\mathbb{R}^2$).

Problem 81 Let T be a linear transformation given by

$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

What is $\dim(\ker(T))$?

Click on the arrow to see answer. We can easily see that $\dim(\operatorname{im}(T)) = 1$, and thus $\dim(\ker(T)) = 3 - \dim(\operatorname{im}(T)) = 3 - 1 = 2$.

Problem 82 Suppose $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is a linear transformation such that $T \left(\begin{bmatrix} 2 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 7 \\ -6 \\ 1 \end{bmatrix}$, and $T \left(\begin{bmatrix} 4 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -2 \\ 5 \\ 0 \end{bmatrix}$. Find the image of $\begin{bmatrix} -2 \\ -2 \end{bmatrix}$ under T .

Click the arrow to see answer.

First, observe that $\begin{bmatrix} -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 4 \\ 1 \end{bmatrix}$. Therefore,

$$T \left(\begin{bmatrix} -2 \\ -2 \end{bmatrix} \right) = T \left(\begin{bmatrix} 2 \\ -1 \end{bmatrix} \right) - T \left(\begin{bmatrix} 4 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 7 \\ -6 \\ 1 \end{bmatrix} - \begin{bmatrix} -2 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ -11 \\ 1 \end{bmatrix}$$

Problem 83 Suppose $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear transformation that maps $\mathbf{i}$ to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\mathbf{j}$ to $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$. Find vector $\mathbf{v}$ such that $T(\mathbf{v}) = \begin{bmatrix} 20 \\ -30 \end{bmatrix}$.

Click the arrow to see answer.

Observe that T is induced by the matrix $A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$. We are looking for $\mathbf{v}$ such that $\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix} \mathbf{v} = \begin{bmatrix} 20 \\ -30 \end{bmatrix}$. Multiplying both sides by A^{-1} gives us

$$\mathbf{v} = \begin{bmatrix} 0.6 & 0.4 \\ -0.2 & 0.2 \end{bmatrix} \begin{bmatrix} 20 \\ -30 \end{bmatrix} = \begin{bmatrix} 0 \\ -10 \end{bmatrix}$$

Bibliography

Some of the problems come from Chapter 5 of Ken Kuttler's *A First Course in Linear Algebra*. (CC-BY)

Ken Kuttler, *A First Course in Linear Algebra*, Lyryx 2017, Open Edition, pp. 272–315.

15 Challenge Problems

Challenge Problems for Chapter 6

Problem 84 Argue geometrically to prove that the following transformations are linear:

1. Rotation of the plane about the origin through angle θ .
2. Reflection of the plane about the line $y = mx$.

Problem 85 Find the matrix of the linear transformation which rotates every vector in $\mathbb{R}^3$ counter clockwise about the z axis when viewed from the positive z axis through an angle of 30° and then reflects through the xy plane.

Click on the arrow to see answer.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\left(\frac{\pi}{6}\right) & -\sin\left(\frac{\pi}{6}\right) & 0 \\ \sin\left(\frac{\pi}{6}\right) & \cos\left(\frac{\pi}{6}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Problem 86 Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation such that $\ker(T) = \mathbb{R}^n$. Find $\text{im}(T)$.

Problem 87 Let $\mathbf{v}$ be a non-zero vector in $\mathbb{R}^n$. Given any vector $\mathbf{w}$ in $\mathbb{R}^m$, show there exists a linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ with $T(\mathbf{v}) = \mathbf{w}$.

Problem 88 Given $\mathbf{y}$ in $\mathbb{R}^n$, define $S_{\mathbf{y}} : \mathbb{R}^n \rightarrow \mathbb{R}$ by $S_{\mathbf{y}}(\mathbf{x}) = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

1. Show that $S_{\mathbf{y}}$ is a linear transformation.
2. Show that every linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}$ arises in this way (i.e. $T = S_{\mathbf{y}}$ for some $\mathbf{y}$ in $\mathbb{R}^n$.)

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Some of these problems come from Chapter 5 of Ken Kuttler's *A First Course in Linear Algebra*. (CC-BY)

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W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2018, Open Edition, pp. 376–386.

16 Chapter 7 Problems

17 Solved Problems

Solved Problems for Chapter 7

Problem 89 Let $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ -2 & 5 & 1 \end{bmatrix}$. When doing cofactor expansion along the top row, we encounter three minor matrices. What are they?

Click the arrow to see answer.

$$\begin{bmatrix} 1 & 3 \\ 5 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 3 \\ -2 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ -2 & 5 \end{bmatrix}$$

Problem 90 Find the following determinant by (a) expanding along the first row, (b) second column.

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 2 & 1 & 1 \end{vmatrix}$$

Click the arrow to see answer.

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 2 & 1 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} - 2 \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} + 1 \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} = 6$$

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 2 & 1 & 1 \end{vmatrix} = -2 \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} + 1 \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 6$$

Problem 91 Compute the determinant by co-factor expansion. Pick the easiest row or column to use.

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 2 & 1 & 3 & 1 \end{vmatrix}$$

Click the arrow to see answer. Expand along the third row.

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 2 & 1 & 3 & 1 \end{vmatrix} = -2 \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 2 & 1 & 3 \end{vmatrix} = -2 \left(1 \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} \right) = -2(2) = -4$$

Problem 92 An operation is done to get from the first matrix to the second. Identify what was done and tell how it will affect the value of the determinant.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Click the arrow to see answer. It does not change the determinant. This was just taking the transpose.

Problem 93 An operation is done to get from the first matrix to the second. Identify what was done and tell how it will affect the value of the determinant.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

Click the arrow to see answer. In this case two rows were switched and so the resulting determinant is -1 times the first.

Problem 94 An operation is done to get from the first matrix to the second. Identify what was done and tell how it will affect the value of the determinant.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} a & b \\ a+c & b+d \end{bmatrix}$$

Click the arrow to see answer. The determinant is unchanged. It was just the first row added to the second.

Problem 95 An operation is done to get from the first matrix to the second. Identify what was done and tell how it will affect the value of the determinant.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} a & b \\ 2c & 2d \end{bmatrix}$$

Click the arrow to see answer. The second row was multiplied by 2 so the determinant of the result is 2 times the original determinant.

Problem 96 An operation is done to get from the first matrix to the second. Identify what was done and tell how it will affect the value of the determinant.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

Click the arrow to see answer. In this case the two columns were switched so the determinant of the second is -1 times the determinant of the first.

Problem 97 Let A be an $n \times n$ matrix and suppose there are $n - 1$ rows such that all rows are linear combinations of these $n - 1$ rows. Show $\det(A) = 0$.

Click the arrow to see answer. If the determinant is nonzero, then it will remain nonzero with row operations applied to the matrix. In this case, you can obtain a row of zeros by doing row operations. Thus the determinant must be zero.

Problem 98 Construct 2×2 matrices A and B to illustrate the property that $\det A \det B = \det(AB)$.

Click the arrow to see answer.

$$\det \left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -5 & 6 \end{bmatrix} \right) = -8$$

$$\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \det \begin{bmatrix} -1 & 2 \\ -5 & 6 \end{bmatrix} = -2 \times 4 = -8$$

Problem 99 An $n \times n$ matrix is called nilpotent if for some positive integer, k it follows $A^k = O$. If A is a nilpotent matrix and k is the smallest possible integer such that $A^k = O$, what are the possible values of $\det(A)$?

Click the arrow to see answer. $\det A = 0$ because $0 = \det(O) = \det(A^k) = (\det(A))^k$.

Problem 100 A matrix is said to be orthogonal if $A^T A = I$. Thus the inverse of an orthogonal matrix is just its transpose. What are the possible values of $\det(A)$ if A is an orthogonal matrix?

Click the arrow to see answer. You would need $\det(AA^T) = \det(A) \det(A^T) = \det(A)^2 = 1$ and so $\det(A) = 1$, or -1 .

Problem 101 Let A and B be two $n \times n$ matrices. We say that A is **similar** to B and write $A \sim B$ provided that there exists an invertible matrix P such that $A = P^{-1}BP$. Show that if $A \sim B$, then $\det(A) = \det(B)$.

Click the arrow to see answer. $\det(A) = \det(S^{-1}BS) = \det(S^{-1}) \det(B) \det(S) = \det(B) \det(S^{-1}S) = \det(B)$.

Problem 102 Determine whether each statement is true or false. If true, provide a proof. If false, provide a counter example.

1. If any two columns of a square matrix are equal, then the determinant of the matrix equals zero.
2. If A^{-1} exists then $\det(A^{-1}) = \det(A)^{-1}$.
3. If A is a real $n \times n$ matrix, then $\det(A^T A) \geq 0$.
4. If $AX = 0$ for some $X \neq 0$, then $\det(A) = 0$.

Click the arrow to see answer. All of the statements are true.

1. $\det(A) = \det(A^T)$. If two columns of A are equal, then two rows of A^T are equal. Applying row operations to A^T will produce a matrix with a row of zeros.
2. Suppose A is invertible. Then

$$1 = \det(I) = \det(AA^{-1}) = \det(A) \det(A^{-1})$$

Divide both right and left side by $\det(A)$ to obtain the result.

3. This follows from the fact that $\det(A) = \det(A^T)$.
4. If $AX = 0$ for some $X \neq \mathbf{0}$ then there is a non-trivial linear combination of the columns of A that is equal to $\mathbf{0}$. This means that the columns of A are linearly dependent. This implies that the rows are also linearly dependent (why?). Applying elementary row operations will lead us to a row of zeros.

Problem 103 Let

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}$$

Determine whether the matrix A has an inverse by finding whether the determinant is non-zero.

Click the arrow to see answer.

$$\det \begin{bmatrix} 1 & 0 & 3 \\ 1 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix} = 2$$

and so it has an inverse. The inverse turns out to equal

$$\begin{bmatrix} -\frac{1}{2} & \frac{3}{2} & 0 \\ \frac{2}{3} & -\frac{2}{9} & 1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}$$

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Some of the problems come from the end of Chapter 3 of Ken Kuttler's *A First Course in Linear Algebra*. (CC-BY)

Ken Kuttler, *A First Course in Linear Algebra*, Lyryx 2017, Open Edition, pp. 272–315.

18 Challenge Problems

Challenge Problems for Chapter 7

Problem 104 Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{bmatrix}$$

Does there exist a value of t for which this matrix fails to have an inverse? Explain.

Click the arrow to see answer. No. It has a nonzero determinant for all t .

Problem 105 If A , B , and C are each $n \times n$ matrices and ABC is invertible, show why each of A , B , and C are invertible.

Click the arrow to see answer.

This follows because $\det(ABC) = \det(A)\det(B)\det(C)$ and if this product is nonzero, then each determinant in the product is nonzero and so each of these matrices is invertible.

Problem 106 Suppose A , B are $n \times n$ matrices and that $AB = I$. Show that then $BA = I$.

Problem 107 Suppose A is an upper triangular matrix. Show that A^{-1} exists if and only if all elements of the main diagonal are non zero. Is it true that A^{-1} will also be upper triangular? Explain. Could the same be concluded for lower triangular matrices?

Problem 108 Let A , B , and C denote $n \times n$ matrices. Assume that $\det A = -1$, $\det B = 2$, and $\det C = 3$. Evaluate

1. $\det(A^3 BC^T B^{-1})$
2. $\det(B^2 C^{-1} AB^{-1} C^T)$
3. $\det(A^{-1} B^{-1} AB)$

Problem 109 If A and B are $n \times n$ matrices such that $AB = -BA$, and if n is odd, show that either A or B has no inverse.

Problem 110 Show that no 3×3 matrix A exists such that $A^2 + I = O$. Find a 2×2 matrix A with this property.

Problem 111 Show that $\det(A + B^T) = \det(A^T + B)$ for any $n \times n$ matrices A and B .

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W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2018, Open Edition, pp. 167.

19 Chapter 8 Problems

20 Solved Problems

Solved Problems for Chapter 8

Problem 112 If A is an invertible $n \times n$ matrix, compare the eigenvalues of A and A^{-1} . More generally, for m an arbitrary integer, compare the eigenvalues of A and A^m .

Click the arrow to see answer. $A^m X = \lambda^m X$ for any integer. In the case of -1 , $A^{-1} \lambda X = A A^{-1} X = X$ so $A^{-1} X = \lambda^{-1} X$. Thus the eigenvalues of A^{-1} are just λ^{-1} where λ is an eigenvalue of A .

Problem 113 If A is an $n \times n$ matrix and c is a nonzero constant, compare the eigenvalues of A and cA .

Click the arrow to see answer. Say $AX = \lambda X$. Then $cAX = c\lambda X$ and so the eigenvalues of cA are just $c\lambda$ where λ is an eigenvalue of A .

Problem 114 Let A, B be invertible $n \times n$ matrices which commute. That is, $AB = BA$. Suppose X is an eigenvector of B . Show that then AX must also be an eigenvector for B .

Click the arrow to see answer. $BAX = ABX = A\lambda X = \lambda AX$. Here it is assumed that $BX = \lambda X$.

Problem 115 Suppose A is an $n \times n$ matrix and it satisfies $A^m = A$ for some m a positive integer larger than 1. Show that if λ is an eigenvalue of A then $|\lambda|$ equals either 0 or 1.

Click the arrow to see answer. Let X be the eigenvector. Then $A^m X = \lambda^m X$, $A^m X = AX = \lambda X$ and so

$$\lambda^m = \lambda$$

Hence if $\lambda \neq 0$, then

$$\lambda^{m-1} = 1$$

and so $|\lambda| = 1$.

Problem 116 Show that if $AX = \lambda X$ and $AY = \lambda Y$, then whenever k, p are scalars,

$$A(kX + pY) = \lambda(kX + pY)$$

Does this imply that $kX + pY$ is an eigenvector? Explain.

Click the arrow to see answer. The formula follows from properties of matrix multiplications. However, this vector might not be an eigenvector because it might equal 0 and eigenvectors cannot equal 0.

Problem 117 Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 6 & 76 & 16 \\ -2 & -21 & -4 \\ 2 & 64 & 17 \end{bmatrix}$$

One eigenvalue is -2 .

Click the arrow to see answer. The characteristic polynomial of this matrix is $-\lambda^3 + 2\lambda^2 + 5\lambda - 6$. Knowing one of the eigenvalues gives us one factor, $(\lambda + 2)$. Use long division of polynomials to finish factoring $-\lambda^3 + 2\lambda^2 + 5\lambda - 6 = -(\lambda + 2)(\lambda - 1)(\lambda - 3)$. Now we have the following eigenvalues: $\lambda_1 = -2$, $\lambda_2 = 1$, and $\lambda_3 = 3$.

The corresponding eigenvectors are: $\mathbf{v}_1 = \begin{bmatrix} 7 \\ -2 \\ 6 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 8 \\ -2 \\ 7 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 4 \\ -1 \\ 4 \end{bmatrix}$.

Problem 118 Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 3 & 5 & 2 \\ -8 & -11 & -4 \\ 10 & 11 & 3 \end{bmatrix}$$

One eigenvalue is -3 .

Click the arrow to see answer. Characteristic polynomial: $-\lambda^3 - 5\lambda^2 - 7\lambda - 3 = -(\lambda + 3)(\lambda + 1)^2$.

$$\lambda_1 = -3, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

$$\lambda_2 = -1, \quad \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

Problem 119 Is it possible for a nonzero matrix to have only 0 as an eigenvalue?

Click the arrow to see answer. Yes. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ works.

Problem 120 Let T be the linear transformation which reflects vectors about the x axis. Find a matrix for T and then find its eigenvalues and eigenvectors.

Click the arrow to see answer.

The matrix of T is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

The eigenvalues and eigenvectors are:

$$\lambda_1 = -1, \quad \mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 1, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Problem 121 Let T be the linear transformation which reflects all vectors in $\mathbb{R}^3$ through the xy plane. Find a matrix for T and then obtain its eigenvalues and eigenvectors.

Click the arrow to see answer. The matrix of T is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ The eigenvalues are $\lambda_1 = -1, \lambda_2 = 1$. Bases for the corresponding eigenspaces are:

$$\mathcal{S}_{\lambda_1} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \mathcal{S}_{\lambda_2} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Problem 122 Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 5 & -18 & -32 \\ 0 & 5 & 4 \\ 2 & -5 & -11 \end{bmatrix}$$

One eigenvalue is 1. Diagonalize if possible.

Click the arrow to see answer. The eigenvalues are $-1, -1, 1$. The eigenvectors corresponding to the eigenvalues are:

$$\left\{ \begin{bmatrix} 10 \\ -2 \\ 3 \end{bmatrix} \right\} \leftrightarrow -1, \left\{ \begin{bmatrix} 7 \\ -2 \\ 2 \end{bmatrix} \right\} \leftrightarrow 1$$

Therefore this matrix is not diagonalizable.

Problem 123 Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} -13 & -28 & 28 \\ 4 & 9 & -8 \\ -4 & -8 & 9 \end{bmatrix}$$

One eigenvalue is 3. Diagonalize if possible.

Click the arrow to see answer. The eigenvectors and eigenvalues are:

$$\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\} \leftrightarrow 1, \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\} \leftrightarrow 1, \left\{ \begin{bmatrix} 7 \\ -2 \\ 2 \end{bmatrix} \right\} \leftrightarrow 3$$

The matrix P needed to diagonalize the above matrix is

$$\begin{bmatrix} 2 & -2 & 7 \\ 0 & 1 & -2 \\ 1 & 0 & 2 \end{bmatrix}$$

and the diagonal matrix D is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Problem 124 Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 89 & 38 & 268 \\ 14 & 2 & 40 \\ -30 & -12 & -90 \end{bmatrix}$$

One eigenvalue is -3 . Diagonalize if possible.

Click the arrow to see answer. The eigenvectors and eigenvalues are:

$$\left\{ \begin{bmatrix} -6 \\ -1 \\ -2 \end{bmatrix} \right\} \leftrightarrow 6, \left\{ \begin{bmatrix} -5 \\ -2 \\ 2 \end{bmatrix} \right\} \leftrightarrow -3, \left\{ \begin{bmatrix} -8 \\ -2 \\ 3 \end{bmatrix} \right\} \leftrightarrow -2$$

The matrix P needed to diagonalize the above matrix is

$$\begin{bmatrix} -6 & -5 & -8 \\ -1 & -2 & -2 \\ 2 & 2 & 3 \end{bmatrix}$$

and the diagonal matrix D is

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Problem 125 If $A \sim B$ and A has any of the following properties, show that B has the same property.

1. A is Idempotent, that is $A^2 = A$.
2. A is Nilpotent, that is $A^k = 0$ for some $k \geq 1$.

Click the arrow to see answer. If $B = P^{-1}AP$ and $A^k = 0$, then $B^k = (P^{-1}AP)^k = P^{-1}A^kP = P^{-1}0P = 0$.

3. A is Invertible.

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These problems come from Chapter 7 of Ken Kuttler's *A First Course in Linear Algebra*. (CC-BY)

Ken Kuttler, *A First Course in Linear Algebra*, Lyryx 2017, Open Edition, pp. 359–401.

21 Challenge Problems

Challenge Problems for Chapter 8

Problem 126 Suppose A is an $n \times n$ matrix consisting entirely of real entries but $a + ib$ is a complex eigenvalue having the eigenvector, $\mathbf{x} + i\mathbf{y}$. Here $\mathbf{x}$ and $\mathbf{y}$ are real vectors. Show that then $a - ib$ is also an eigenvalue with the eigenvector, $\mathbf{x} - i\mathbf{y}$.

Problem 127 If A is 2×2 and diagonalizable, show that $C(A) = \{X \mid XA = AX\}$ has dimension 2 or 4.

Problem 128 Let A be $n \times n$ with n distinct real eigenvalues. If $AC = CA$, show that C is diagonalizable.

Problem 129 Given a polynomial $p(z) = r_0 + r_1z + \cdots + r_nz^n$ and a square matrix A , the matrix $p(A) = r_0I + r_1A + \cdots + r_nA^n$ is called the evaluation of $p(z)$ at A . Let $B = P^{-1}AP$. Show that $p(B) = P^{-1}p(A)P$ for all polynomials $p(x)$.

Problem 130 If A is diagonalizable and $p(x)$ is a polynomial such that $p(\lambda) = 0$ for all eigenvalues λ of A , show that $p(A) = O$ (here, the final O is the zero matrix the same size as A).

Remark 1. The characteristic polynomial of A (see Definition ??) certainly satisfies the requirement that $p(\lambda) = 0$ for all eigenvalues λ of A . In solving this problem you have proved a special case of the Cayley-Hamilton theorem, see Theorem ??. In fact, if $p(\lambda)$ is the characteristic polynomial of A , then $p(A) = 0$ whether or not A is diagonalizable.

Problem 131 Let P be an invertible $n \times n$ matrix. If A is any $n \times n$ matrix, write $T_P(A) = P^{-1}AP$. Verify that:

1. $T_P(I) = I$
2. $T_P(AB) = T_P(A)T_P(B)$
3. $T_P(A + B) = T_P(A) + T_P(B)$
4. $T_P(rA) = rT_P(A)$
5. $T_P(A^k) = [T_P(A)]^k$ for $k \geq 1$
6. If A is invertible, $T_P(A^{-1}) = [T_P(A)]^{-1}$.
7. If Q is invertible, $T_Q[T_P(A)] = T_{PQ}(A)$.

Problem 132 Let $A = \begin{bmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{bmatrix}$ and $B = \begin{bmatrix} c & a & b \\ a & b & c \\ b & c & a \end{bmatrix}$.

1. Show that $x^3 - (a^2 + b^2 + c^2)x - 2abc$ has real roots by considering A .
2. Show that $a^2 + b^2 + c^2 \geq ab + ac + bc$ by considering B .

Problem 133 Assume the 2×2 matrix A is similar to an upper triangular matrix. If $\text{tr}A = 0 = \text{tr}A^2$, show that A^2 is equal to the zero matrix.

Problem 134 Show that A is similar to A^T for all 2×2 matrices A .

Problem 135 Suppose A is an $n \times n$ matrix and let $\mathbf{v}$ be an eigenvector such that $A\mathbf{v} = \lambda\mathbf{v}$. Also suppose the characteristic polynomial of A is

$$\det(zI - A) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0$$

Explain why

$$(A^n + a_{n-1}A^{n-1} + \cdots + a_1A + a_0I)\mathbf{v} = 0$$

Use this to prove that the Cayley-Hamilton theorem holds for any diagonalizable matrix A . (The Cayley-Hamilton theorem says that A satisfies its characteristic equation, i.e.,

$$A^n + a_{n-1}A^{n-1} + \cdots + a_1A + a_0I = 0$$

. (For a proof of the general case, see Theorem ??)

Problem 136 Suppose the characteristic polynomial of an $n \times n$ matrix A is $\lambda^n - 1$. Find A^{mn} where m is an integer.

Click the arrow to see answer. The eigenvalues are distinct because they are the n^{th} roots of 1. Hence if $\mathbf{x}$ is a given vector with

$$\mathbf{x} = \sum_{j=1}^n a_j \mathbf{v}_j$$

then

$$A^{nm}\mathbf{x} = A^{nm} \sum_{j=1}^n a_j \mathbf{v}_j = \sum_{j=1}^n a_j A^{nm} \mathbf{v}_j = \sum_{j=1}^n a_j \mathbf{v}_j = \mathbf{x}$$

so $A^{nm} = I$.

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22 Chapter 9 Problems

23 Solved Problems

Solved Problems for Chapter 9

Problem 137 Find an orthonormal basis for the span of the following set of vectors.

$$\begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ 1 \end{bmatrix}$$

Click the arrow to see answer. We will use Gram-Schmidt orthogonalization algorithm.

$$\mathbf{f}_1 = \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}$$

$$\mathbf{f}_2 = \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix} - \frac{25}{25} \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$$

$$\mathbf{f}_3 = \begin{bmatrix} 1 \\ 7 \\ 1 \end{bmatrix} - \frac{-25}{25} \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix} - \frac{25}{25} \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Normalizing each vector we get:

$$\begin{bmatrix} \frac{3}{5} \\ -\frac{4}{5} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{4}{5} \\ \frac{3}{5} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Problem 138 Using the Gram Schmidt process find an orthonormal basis for the following span:

$$\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Click the arrow to see the answer.

$$\begin{bmatrix} \frac{1}{6}\sqrt{6} \\ \frac{1}{3}\sqrt{6} \\ \frac{1}{6}\sqrt{6} \end{bmatrix}, \begin{bmatrix} \frac{3}{10}\sqrt{2} \\ -\frac{2}{5}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{bmatrix}, \begin{bmatrix} \frac{7}{15}\sqrt{3} \\ -\frac{1}{15}\sqrt{3} \\ -\frac{1}{3}\sqrt{3} \end{bmatrix}$$

Problem 139 Write $\mathbf{x}$ as the sum of a vector in W and a vector in $W^\perp$ if

$$\mathbf{x} = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}, \quad W = \text{span} \left(\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right)$$

Click the arrow to see the answer. Observe that the two vectors that span W are orthogonal. We will refer to these vectors as $\mathbf{f}_1$ and $\mathbf{f}_2$.

$$\mathbf{w} = \text{proj}_W(\mathbf{x}) = \text{proj}_{\mathbf{f}_1}(\mathbf{x}) + \text{proj}_{\mathbf{f}_2}(\mathbf{x}) = \frac{12}{14} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + \frac{11}{3} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{21} \begin{bmatrix} -59 \\ 41 \\ 131 \end{bmatrix}$$

$$\mathbf{w}^\perp = \mathbf{x} - \mathbf{w} = \frac{1}{21} \begin{bmatrix} 80 \\ 64 \\ 16 \end{bmatrix}$$

$$\mathbf{x} = \mathbf{w} + \mathbf{w}^\perp$$

Problem 140 Write $\mathbf{x}$ as the sum of a vector in W and a vector in $W^\perp$ if

$$\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 9 \end{bmatrix}, \quad W = \text{span} \left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right)$$

Click the arrow to see the answer. Observe that the three vectors that span W form an orthogonal set. We will refer to these vectors as $\mathbf{f}_1$, $\mathbf{f}_2$ and $\mathbf{f}_3$.

$$\mathbf{w} = \text{proj}_W(\mathbf{x}) = \text{proj}_{\mathbf{f}_1}(\mathbf{x}) + \text{proj}_{\mathbf{f}_2}(\mathbf{x}) + \text{proj}_{\mathbf{f}_3}(\mathbf{x}) =$$

$$= \frac{17}{3} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \frac{5}{3} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} + \frac{8}{6} \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 9 \\ 5 \\ 16 \\ 26 \end{bmatrix}$$

$$\mathbf{w}^\perp = \mathbf{x} - \mathbf{w} = \frac{1}{3} \begin{bmatrix} 0 \\ -2 \\ -1 \\ 1 \end{bmatrix}$$

$$\mathbf{x} = \mathbf{w} + \mathbf{w}^\perp$$

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Ken Kuttler, *A First Course in Linear Algebra*, Lyryx 2017, Open Edition, pp. 433–438.

24 Challenge Problems

Challenge Problems for Chapter 9

Problem 141 If U is a subspace of $\mathbb{R}^n$, show that $(U^\perp)^\perp = U$.

Problem 142 If W is a subspace of $\mathbb{R}^n$, show how to find an $n \times n$ matrix A such that $W = \{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\}$.

Problem 143 If U and W are subspaces, define $U + W$ to be the set of all possible sums of elements of U with elements of W .

1. Is $U + W$ a subspace? Show that $(U + W)^\perp = U^\perp \cap W^\perp$.
2. Illustrate this result with a diagram in $\mathbb{R}^3$ where U and W are two distinct lines through the origin. What does $U + W$ look like? What do $U^\perp$ and $W^\perp$ look like?

Problem 144 A square matrix E satisfying $E^2 = E = E^T$ is called a projection matrix. If E is a projection matrix, show that $I - E$ is also a projection matrix.

Problem 145 Let E be an $n \times n$ matrix, and let $W = \{\mathbf{x}E \mid \mathbf{x} \text{ in } \mathbb{R}^n\}$. Show that the following are equivalent.

1. $E^2 = E = E^T$ (E is a projection matrix).
2. $(\mathbf{x} - \mathbf{x}E) \cdot (\mathbf{y}E) = 0$ for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$.
3. $\text{proj}_W(\mathbf{x}) = \mathbf{x}E$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

Problem 146 If $EF = 0 = FE$ and E and F are projection matrices, show that $E + F$ is also a projection matrix.

Problem 147 If A is $m \times n$ and AA^T is invertible, show that $E = A^T(AA^T)^{-1}A$ is a projection matrix.

Click the arrow to see answer.

$$E^T = A^T[(AA^T)^{-1}]^T(A^T)^T = A^T[(AA^T)^T]^{-1}A = A^T[AA^T]^{-1}A = E$$

Problem 148 Consider $A = \begin{bmatrix} 0 & a & 0 \\ a & 0 & c \\ 0 & c & 0 \end{bmatrix}$ where one of $a, c \neq 0$. Show that the characteristic polynomial (see

Definition ??) is given by $c_A(z) = z(z - k)(z + k)$, where $k = \sqrt{a^2 + c^2}$, and find an orthogonal matrix Q such that $Q^{-1}AQ$ is diagonal.

Click the arrow to see the answer. $Q = \frac{1}{\sqrt{2}k} \begin{bmatrix} c\sqrt{2} & a & a \\ 0 & k & -k \\ -a\sqrt{2} & c & c \end{bmatrix}$

Problem 149 Given $A = \begin{bmatrix} b & a \\ a & b \end{bmatrix}$, show that the characteristic polynomial (see Definition ??) is given by $c_A(z) = (z - a - b)(z + a - b)$, and find an orthogonal matrix Q such that $Q^{-1}AQ$ is diagonal.

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W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2018, Open Edition, p. 422–423

25 Chapter 10 Problems

26 Solved Problems

Solved Problems for Chapter 10

Problem 150 Let M, N be subspaces of a vector space V and consider $M + N$ defined as the set of all $m + n$ where $m \in M$ and $n \in N$. Show that $M + N$ is a subspace of V .

Click the arrow to see answer. We need to show that $M + N$ is closed under addition and scalar multiplication. (See Theorem ??.) Suppose $\mathbf{x}$ and $\mathbf{y}$ are in $M + N$. Then $\mathbf{x} = m_1 + n_1$ and $\mathbf{y} = m_2 + n_2$, where m_i is in M and n_i is in N . To see that $M + N$ is closed under scalar multiplication, consider $a\mathbf{x} = a(m_1 + n_1) = am_1 + an_1$. Since am_1 is in M , and an_1 is in N , $a\mathbf{x}$ is in $M + N$. Next, we show that $M + N$ is closed under vector addition. $\mathbf{x} + \mathbf{y} = (m_1 + m_2) + (n_1 + n_2)$. Since $(m_1 + m_2)$ is an element of M , and $(n_1 + n_2)$ is an element of N we have closure under addition.

Problem 151 Let M, N be subspaces of a vector space V . Then $M \cap N$ consists of all vectors which are in both M and N . Show that $M \cap N$ is a subspace of V .

Click the arrow to see answer. If a vector is in both subspaces M and N , then its scalar multiple must also be in both. Same is true about the sum of two elements of $M \cap N$.

Problem 152 Let M, N be subspaces of a vector space $\mathbb{R}^2$. Then $N \cup M$ consists of all vectors which are in either M or N . Show that $N \cup M$ is not necessarily a subspace of $\mathbb{R}^2$ by giving an example where $N \cup M$ fails to be a subspace.

Click the arrow to see answer. As our example, let M be the first quadrant, and let N be the third quadrant.

Problem 153 Define $T : \mathbb{P}^2 \rightarrow \mathbb{P}^3$ by $T(p(x)) = xp(x)$.

1. Find $T(-x^2 + 2x - 4)$.
2. Is T a linear transformation? If so, prove it. If not, give a counterexample.

Click the arrow to see answer.

1. $T(-x^2 + 2x - 4) = -x^3 + 2x^2 - 4x$
2. $T(ap(x) + bq(x)) = x(ap(x) + bq(x)) = axp(x) + bxq(x) = aT(p(x)) + bT(p(x))$. T is a linear transformation.

Problem 154 Let $p(x) = 4x^2 - x$. Is $p(x)$ in $\text{span}(x^2 + x, x^2 - 1, -x + 2)$? Consider the question in two different ways, then compare your work for each approach.

1. Do this directly using the definition of span (Definition ??).

2. Do this using isomorphisms.

Click the arrow to see answer.

1. Apply the definition of span directly. We are looking for coefficients a , b and c such that

$$ax^2 + ax + bx^2 - b - cx + 2c = 4x^2 - x$$

Collecting like terms on the left and setting their coefficients equal to their counterparts on the right gives us the following system of equations.

$$a + b = 4$$

$$a - c = -1$$

$$-b + 2c = 0$$

This gives rise to the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 1 & 0 & -1 & -1 \\ 0 & -1 & 2 & 0 \end{array} \right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & 10/3 \\ 0 & 0 & 1 & 5/3 \end{array} \right]$$

This shows that our system has a unique solution and gives us the specific coefficients to express $p(x)$ as a linear combination of the vectors in the given set.

2. We can also look at this problem in light of isomorphisms. Let's start by mapping

$$x^2 + x \mapsto \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad x^2 - 1 \mapsto \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad -x + 2 \mapsto \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

(Why is this an isomorphism?)

Is the vector $\begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$ in the span of the three vectors above? Set up an augmented matrix to answer this question. Compare this matrix to the matrix in part (a).

Problem 155 Let $p(x) = -x^2 + x + 2$. Is $p(x)$ in $\text{span}(x^2 + x + 1, 2x^2 + x)$?

Click the arrow to see answer. $p(x)$ is not in the span of the given vectors. The system

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & 2 & -1 \end{array} \right]$$

is infeasible.

Problem 156 Consider the vector space of polynomials of degree at most 2, $\mathbb{P}_2$. Determine whether the following is a basis for $\mathbb{P}_2$.

$$\{x^2 + x + 1, 2x^2 + 2x + 1, x + 1\}$$

Click the arrow to see answer. There is an isomorphism from $\mathbb{R}^3$ to $\mathbb{P}_2$, defined as follows:

$$T(\mathbf{e}_1) = 1, \quad T(\mathbf{e}_2) = x, \quad T(\mathbf{e}_3) = x^2$$

Thus,

$$T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = x^2 + x + 1, \quad T\left(\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}\right) = 2x^2 + 2x + 1, \quad T\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) = 1 + x$$

It follows that if

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

is a basis for $\mathbb{R}^3$, then the polynomials will be a basis for $\mathbb{P}_2$ because they will be independent. Recall that an isomorphism takes a linearly independent set to a linearly independent set. Also, since T is an isomorphism, it preserves all linear relations.

Problem 157 Find a basis in $\mathbb{P}_2$ for the subspace

$$\text{span}(1 + x + x^2, 1 + 2x, 1 + 5x - 3x^2)$$

If the above three vectors do not yield a basis, exhibit one of them as a linear combination of the others.

Click the arrow to see answer. This is the situation in which you have a spanning set and you want to cut it down to form a linearly independent set which is also a spanning set. Use the same isomorphism as above. Since T is an isomorphism, it preserves all linear relations so if such can be found in $\mathbb{R}^3$, the same linear relations will be present in $\mathbb{P}_2$.

Problem 158 Define $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ as follows.

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x}$$

Show that T is one to one.

Click the arrow to see answer.

We need to show that $T(\mathbf{v}) = T(\mathbf{w})$ implies that $\mathbf{v} = \mathbf{w}$. Suppose

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

Performing matrix-vector multiplication on both sides shows that $v_1 = w_1$, and $v_2 = w_2$. Therefore, $\mathbf{v} = \mathbf{w}$.

Problem 159 Define $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ as follows.

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x}$$

Is T onto?

Click the arrow to see answer.

T is not onto. One way to show this is by finding a counter-example. We need a vector in $\mathbb{R}^3$ that is not an image of any element of $\mathbb{R}^2$ under T . Try $\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$.

Problem 160 Find the coordinates of $\mathbf{v}$ with respect to the ordered basis $\mathcal{B}$ of $\mathbb{M}_{2,2}$.

$$\mathbf{v} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$$

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Click the arrow to see answer.

We need to find coefficients a , b , c , and d such that

$$a \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + d \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$$

This leads to the following system of equations:

$$a + b + d = 1$$

$$a = 2$$

$$b + c = -1$$

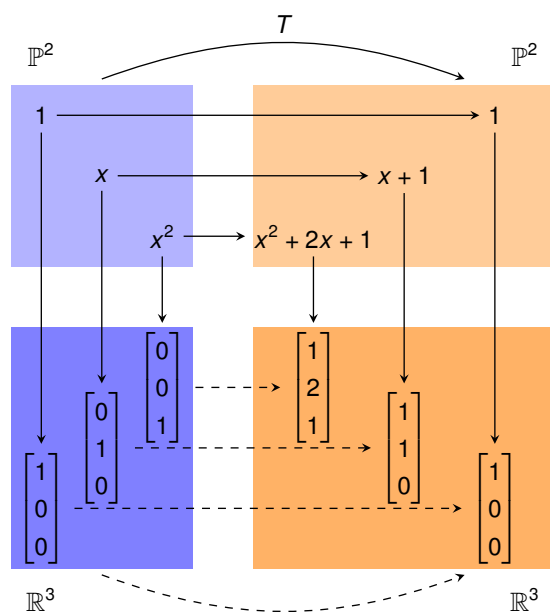
$$c + d = 0$$

We get $a = 2$, $b = -1$, $c = 0$, $d = 0$. Therefore $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}$.

Problem 161 Define $T : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ by $T(p(x)) = p(x+1)$. (Verify that T is a linear transformation.) Find the matrix of T if the basis for the domain and the codomain is $\{1, x, x^2\}$.

Click the arrow to see answer.

We start with a diagram:



Based on where the standard unit vectors of $\mathbb{R}^2$ map to, we have the following matrix.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Bibliography

Some of the problems come from the end of Chapter 9 of Ken Kuttler's *A First Course in Linear Algebra*. (CC-BY)

Ken Kuttler, *A First Course in Linear Algebra*, Lyryx 2017, Open Edition, pp. 469–535.

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W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2018, Open Edition, pp. 501.

27 Challenge Problems

Challenge Problems for Chapter 10

Problem 162 Given linear transformations

$$\begin{array}{ccccc} \mathbb{R}^2 & \xrightarrow{T_1} & \mathbb{R}^2 & \xrightarrow{T_2} & \mathbb{R}^2 \\ & \searrow & & \nearrow & \\ & & T & & \end{array}$$

1. If T_1 and T_2 are both one-to-one, show that T is one-to-one.
2. If T_1 and T_2 are both onto, show that T is onto.

Problem 163 Let $T : V \rightarrow W$ be a linear transformation.

1. If T is one-to-one and $TR_1 = TR_2$ for linear transformations $R_1, R_2 : U \rightarrow V$, show that $R_1 = R_2$.
2. If T is onto and $S_1T = S_2T$ for linear transformations $S_1, S_2 : W \rightarrow U$, show that $S_1 = S_2$.

Problem 164 Consider functions defined on $\{1, 2, \dots, n\}$ having values in $\mathbb{R}$. Explain how, if V is the set of all such functions, V can be considered as $\mathbb{R}^n$.

Problem 165 Let $T : V \rightarrow U$ and $S : U \rightarrow W$ be linear transformations.

1. If ST is one-to-one, show that T is one-to-one and that $\dim V \leq \dim U$.
2. If ST is onto, show that S is onto and that $\dim W \leq \dim U$.

Problem 166 Let $\mathbb{D}_n$ denote the space of all functions $f : \{1, 2, \dots, n\} \rightarrow \mathbb{R}^n$ (see Problem ??). If $T : \mathbb{D}_n \rightarrow \mathbb{R}^n$ is defined by

$$T(f) = (f(1), f(2), \dots, f(n))$$

show that T is an isomorphism.

Problem 167 Let $\mathbb{R}^{\mathbb{N}}$ denote the set of all real valued sequences. For $\mathbf{a} \equiv \{a_n\}_{n=1}^{\infty}$, $\mathbf{b} \equiv \{b_n\}_{n=1}^{\infty}$ two of these, define their sum to be given by

$$\mathbf{a} + \mathbf{b} = \{a_n + b_n\}_{n=1}^{\infty}$$

and define scalar multiplication by

$$c\mathbf{a} = \{ca_n\}_{n=1}^{\infty} \text{ where } \mathbf{a} = \{a_n\}_{n=1}^{\infty}$$

Is this a vector space?

Problem 168 Let $\mathbb{C}^2$ be the set of ordered pairs of complex numbers. Define addition and scalar multiplication in the usual way.

$$(z, w) + (\hat{z}, \hat{w}) = (z + \hat{z}, w + \hat{w}), \quad u(z, w) \equiv (uz, uw)$$

Here the scalars are from $\mathbb{C}$. Show this is a vector space.

Bibliography

Several problems come from Chapter 9 of Ken Kuttler's *A First Course in Linear Algebra*. (CC-BY)

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