
Sandbox

June 5, 2026

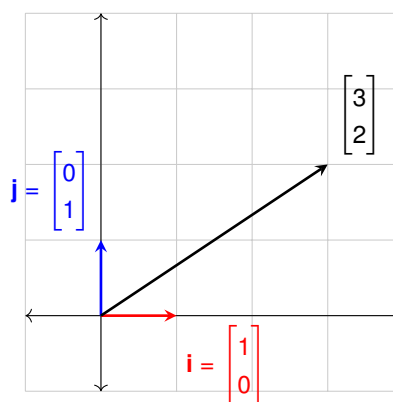
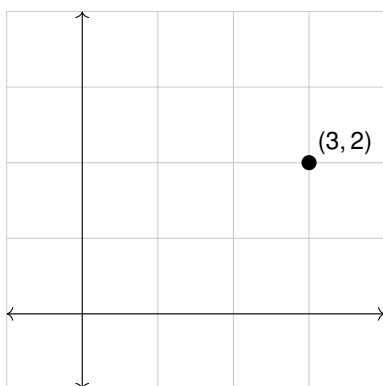
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1 Linear Combinations and Alternative Coordinate Systems

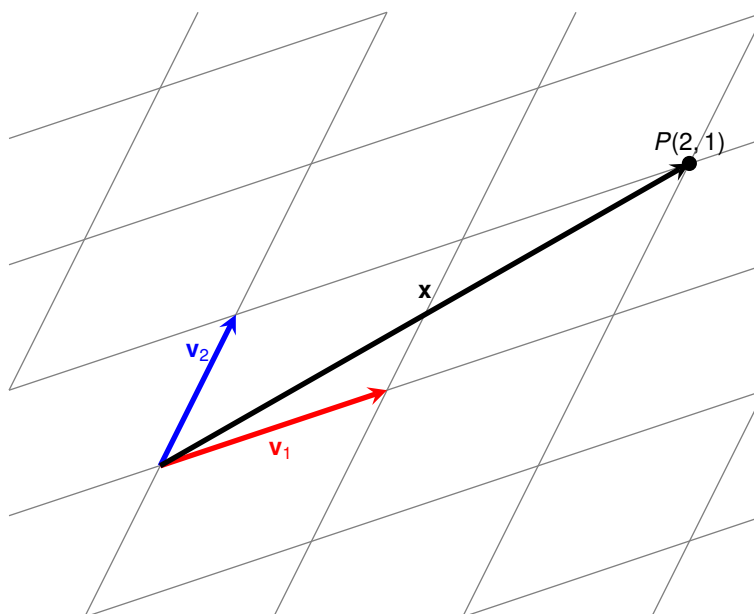
How to Read a Coordinate System

Let's take a fresh look at the familiar rectangular coordinate system. In the diagram on the left, the point $(3, 2)$ can be located by moving your finger 3 units along the x -axis, then moving your finger 2 units parallel to the y -axis. We can think of the x and y axes as being determined by the standard unit vectors $\mathbf{i}$ and $\mathbf{j}$. The vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ can be written as a linear combination of $\mathbf{i}$ and $\mathbf{j}$ as

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = 3\mathbf{i} + 2\mathbf{j}$$



Orthogonal unit vectors are often a convenient choice for establishing a coordinate system, but occasionally, using other vectors is preferable. An example of an alternative coordinate grid determined by vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ is shown below.



Fortunately, finding points and constructing vectors in such coordinate systems relies on the same principles as we use in the familiar rectangular coordinate system. To reach point P , we first travel 2 units along the line containing $\mathbf{v}_1$, then one unit along the line parallel to $\mathbf{v}_2$. We can say that in this coordinate system, P has coordinates $(2, 1)$. Also, observe that vector $\mathbf{x}$ can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ as follows

$$\mathbf{x} = 2\mathbf{v}_1 + \mathbf{v}_2$$

Idea 1. We take it for granted that the x -coordinate (the coefficient in front of $\mathbf{i}$) is the first component in the ordered pair of coordinates, and the y -coordinate (the coefficient in front of $\mathbf{j}$) is the second component.

In the above setup, we chose to list the coefficient in front of $\mathbf{v}_1$ as the first coordinate of P and the coefficient in front of $\mathbf{v}_2$ as the second coordinate.

How would the coordinates of P be impacted if we had chosen a different order? Think about why establishing (or knowing) the order in which $\mathbf{v}_1$ and $\mathbf{v}_2$ are used is important.

The following interactive demonstrates how to find the coordinates for a point using a coordinate grid defined using vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ (in that order). To use the interactive, you can either (1) move the sliders to change the coordinates of P within this coordinate system, or (2) move the tips of vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ to change the coordinate system. For (2), explain why the coordinates of P do not change while P moves around the plane.

GeoGebra link: <https://www.geogebra.org/m/qxjtdgkc>

Explorations

Exploration 1 Use the interactive below to answer the questions

GeoGebra link: <https://www.geogebra.org/m/qw5dpmqq>

Question 1.1 List the coordinates for each P_i with respect to the given coordinate system. Your coordinates should be of the form (A-coordinate, B-coordinate).

$$P_1 = (\dots, \dots)$$

$$P_2 = (\dots, \dots)$$

$$P_3 = (\dots, \dots)$$

$$P_4 = (\dots, \dots)$$

$$P_5 = (\dots, \dots)$$

Question 1.2 Move point B to coincide with P_3 . List the coordinates for each P_i with respect to the new coordinate system. Your coordinates should be of the form (A-coordinate, B-coordinate).

$$P_1 = (\dots, \dots)$$

$$P_2 = (\dots, \dots)$$

$$P_3 = (\dots, \dots)$$

$$P_4 = (\dots, \dots)$$

$$P_5 = (\dots, \dots)$$

How do these coordinates compare to the coordinates in the previous question? Explain why this is happening. (Hint: you can use the RESET button to return to the original coordinate system to compare.)

Question 1.3 Press RESET to return to the original coordinate system. Move point A to coincide with P_1 . List the coordinates for each P_i with respect to the new coordinate system. Your coordinates should be of the form (A-coordinate, B-coordinate).

$$P_1 = (\dots, \dots)$$

$$P_2 = (\dots, \dots)$$

$$P_3 = (\dots, \dots)$$

$$P_4 = (\dots, \dots)$$

$$P_5 = (\dots, \dots)$$

How do these coordinates compare to the coordinates in the original coordinate system? Explain why this is happening. (Hint: you can use the reset button to return to the original coordinate system to compare.)

Question 1.4 Press RESET to return to the original coordinate system. Express each $\vec{OP}_i$ as a linear combination of $\vec{OA}$ and $\vec{OB}$.

$$\vec{OP}_1 = \dots \vec{OA} + \dots \vec{OB}$$

$$\vec{OP}_2 = \dots \vec{OA} + \dots \vec{OB}$$

$$\vec{OP}_3 = \dots \vec{OA} + \dots \vec{OB}$$

$$\vec{OP}_4 = \dots \vec{OA} + \dots \vec{OB}$$

$$\vec{OP}_5 = \dots \vec{OA} + \dots \vec{OB}$$

Discuss the relationship between your answers to the first question and your answers here.

Question 1.5 Move point B to coincide with P_2 . What do you observe? Do vectors $\vec{OA}$ and $\vec{OB}$ determine a good coordinate system for the plane? Can we express every point in the plane as a linear combination of $\vec{OA}$ and $\vec{OB}$? Can we express some points in the plane as linear combinations of $\vec{OA}$ and $\vec{OB}$?

Exploration 2 We will use the same set-up as in the previous exploration but introduce an additional vector $\vec{OC}$.

GeoGebra link: <https://www.geogebra.org/m/b3k96x2w>

Question 2.1 Suppose we want to express P_1 using a coordinate system determined by three vectors $\vec{OA}$, $\vec{OB}$, and $\vec{OC}$. If the coordinates are to be of the form (A-coordinate, B-coordinate, C-coordinate), how many ways do you think there would be to express P_1 ? Fill in the missing coordinates for P_1 below.

$$P_1 = (\dots, \dots, 0)$$

$$P_1 = (0, \dots, \dots)$$

$$P_1 = (\dots, 0, \dots)$$

$$P_1 = (\dots, -1, -1)$$

Question 2.2 Based on your work above, express $\vec{OP_1}$ as a linear combination of $\vec{OA}$, $\vec{OB}$, and $\vec{OC}$. How many ways do you think there are to do this?

Question 2.3 Compare and contrast the coordinate systems in this exploration and Exploration ??.

Existence and Uniqueness of Representation

Idea 2. What makes a good coordinate system?

- Every point has coordinates associated with it;
- Coordinates associated with each point are unique.

2 Coordinate Systems

We will start our Linear Algebra journey with a very important point.

• P = very important point

How would you describe the location of P ? Whether you reference the edges of the page or introduce your own set of axes, you will find that (1) a reference system is required to describe the location, and (2) the resulting description may differ from someone else's, since different choices of coordinate system lead to different representations.

Idea 3. A mathematical object is distinct from its representation. A single object may have a multitude of representations dictated by convention and convenience.

Creating and Using a Coordinate System

When you first encountered vectors in your earlier courses it was probably assumed that these vectors exist in some established rectangular coordinate system. Such a coordinate system implicitly postulates the following structure:

- Location of the origin
- Unit of length
- Orthogonality of axes

The representation of points and vectors is driven by the established coordinate system.

What if we started from scratch? Take two vectors, choose the location of the origin, then use the two vectors to create a coordinate grid. The interactive below allows you to move points A and B to create two vectors. Move the slider to see how different points in the plane can be represented using this coordinate grid, and how the representation of a fixed point changes depending on what grid is chosen.

GeoGebra link: <https://www.geogebra.org/m/edcyushm>

When using an alternative coordinate system, it is customary to identify the vectors that determine it. For now, we will say that the coordinates of point P are stated with respect to $\{\vec{OA}, \vec{OB}\}$. We will refine this statement later in the text.

Exploration 3 Use the interactive below to answer questions about the coordinates of points with respect to various coordinate systems.

GeoGebra link: <https://www.geogebra.org/m/qw5dpmqq>

Question 3.1 List the coordinates for each P_i with respect to the given coordinate system. Your coordinates should be of the form $(\vec{OA}\text{-coordinate}, \vec{OB}\text{-coordinate})$.

$$P_1 = (\dots, \dots)$$

$$P_2 = (\dots, \dots)$$

$$P_3 = (\dots, \dots)$$

$$P_4 = (\dots, \dots)$$

$$P_5 = (\dots, \dots)$$

Question 3.2 REFRESH your browser to return to the original coordinate system. Move point A to coincide with P_1 . List the coordinates for each P_i with respect to the new coordinate system.

$$P_1 = (\dots, \dots)$$

$$P_2 = (\dots, \dots)$$

$$P_3 = (\dots, \dots)$$

$$P_4 = (\dots, \dots)$$

$$P_5 = (\dots, \dots)$$

How do these coordinates compare to the coordinates in the original coordinate system? Explain why this is happening. (Hint: REFRESH your browser to return to the original coordinate system to compare.)

Question 3.3 REFRESH your browser to return to the original coordinate system. Move point B to coincide with P_3 . List the coordinates for each P_i with respect to the new coordinate system.

$$P_1 = (\dots, \dots)$$

$$P_2 = (\dots, \dots)$$

$$P_3 = (\dots, \dots)$$

$$P_4 = (\dots, \dots)$$

$$P_5 = (\dots, \dots)$$

How do these coordinates compare to the coordinates in the previous question? Explain why this is happening. (Hint: REFRESH your browser to return to the original coordinate system to compare.)

Question 3.4 Move point B to coincide with P_2 . What do you observe? Do vectors $\vec{OA}$ and $\vec{OB}$ determine a good coordinate system for the plane? Can we express every point in the plane with respect to the coordinate system determined by $\{\vec{OA}, \vec{OB}\}$? Can we express some points in the plane with respect to the coordinate system determined by $\{\vec{OA}, \vec{OB}\}$?

Exploration 4 We will use the same set-up as in the previous exploration but introduce an additional vector $\vec{OC}$.

GeoGebra link: <https://www.geogebra.org/m/b3k96x2w>

Question 4.1 Suppose we want to express P_1 using a “coordinate system” determined by three vectors $\vec{OA}$, $\vec{OB}$, and $\vec{OC}$. Fill in the missing coordinates for P_1 below, if the coordinates are to be of the form ($\vec{OA}$ -coordinate, $\vec{OB}$ -coordinate, $\vec{OC}$ -coordinate).

$$P_1 = (\dots, \dots, 0)$$

$$P_1 = (0, \dots, \dots)$$

$$P_1 = (\dots, 0, \dots)$$

$$P_1 = (\dots, -1, -1)$$

How many ways are there to express P_1 using this “coordinate system”?

Multiple Choice:

1. 1
2. 4
3. Infinitely many

Question 4.2 Compare and contrast the coordinate systems in this exploration and Exploration ??.

What we discovered is that not all collections of vectors are adequate for making a useful coordinate system. In Exploration ?? we learned that if two vectors are collinear, then not all points in the plane can be represented with respect to those two vectors. In Exploration ?? we found that having too many vectors results in a point having infinitely many representations, which would be computationally confusing.

Idea 4. What makes a good coordinate system?

- Every point can be represented using the coordinate system;
- Each point has a unique representation.