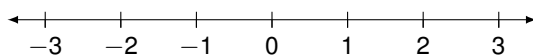

Week 1

June 5, 2026

Contents

1 A Brief Introduction to $\mathbb{R}^n$

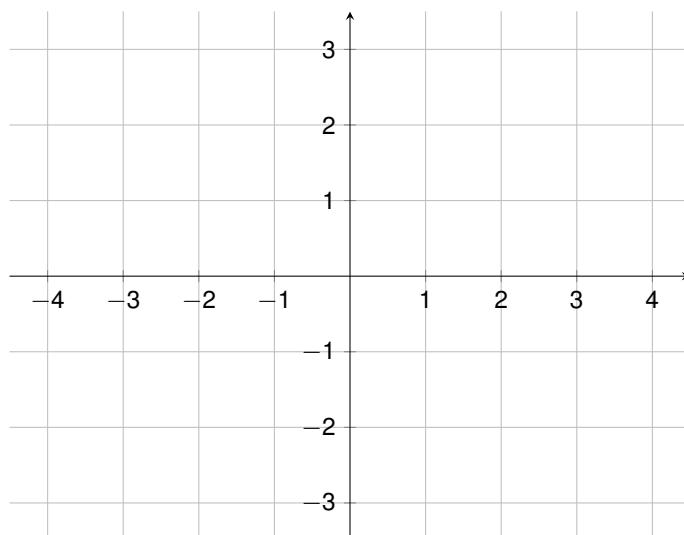
The set of all real numbers is denoted by $\mathbb{R}$. It is convenient to associate real numbers with points on a line, called the *real number line*.



The set of all ordered pairs (x, y) , where x and y are real numbers, is called $\mathbb{R}^2$. Using set notation we write:

$$\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$$

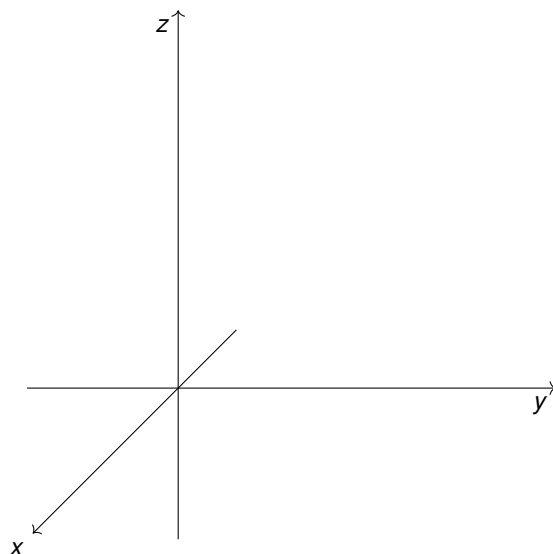
Geometrically speaking, $\mathbb{R}^2$ can be associated with a coordinate plane in which we refer to each point by its x and y coordinates.



The set of all ordered triples (x, y, z) , where x , y and z are real numbers, is called $\mathbb{R}^3$.

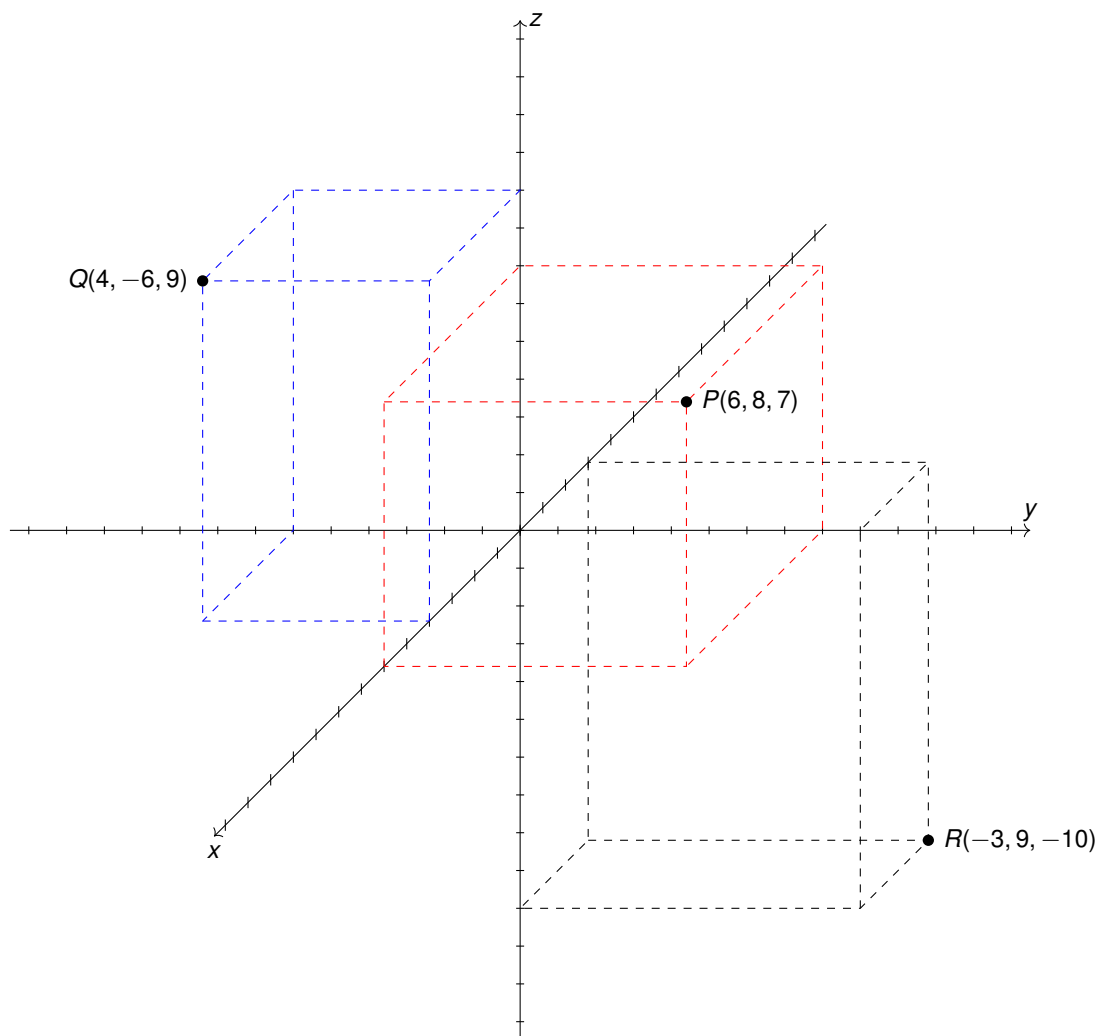
$$\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$$

Geometrically, an ordered triple of $\mathbb{R}^3$ is associated with a point of a three-dimensional space whose position is given by x , y and z coordinates.

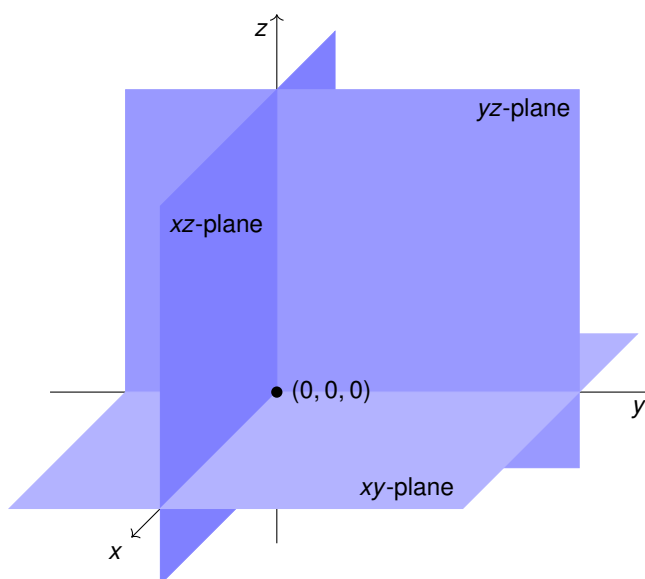


Example 1. The following points are shown plotted in $\mathbb{R}^3$.

1. $P(6, 8, 7)$
2. $Q(4, -6, 9)$
3. $R(-3, 9, -10)$



Each pair of axes in $\mathbb{R}^3$ determines a plane. The resulting three planes are called *coordinate planes*. Each coordinate plane is named after the axes that determine it. Thus, we have the xy -plane, xz -plane, and yz -plane. Coordinate planes intersect at the point $(0, 0, 0)$, called the *origin*, and subdivide $\mathbb{R}^3$ into eight regions, called *octants*.



The set of all ordered n -tuples $(x_1, x_2, \dots, x_n)$, where x_i is a real number for $1 \leq i \leq n$, is called $\mathbb{R}^n$.

$$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) \mid x_i \in \mathbb{R} \text{ for } 1 \leq i \leq n\}$$

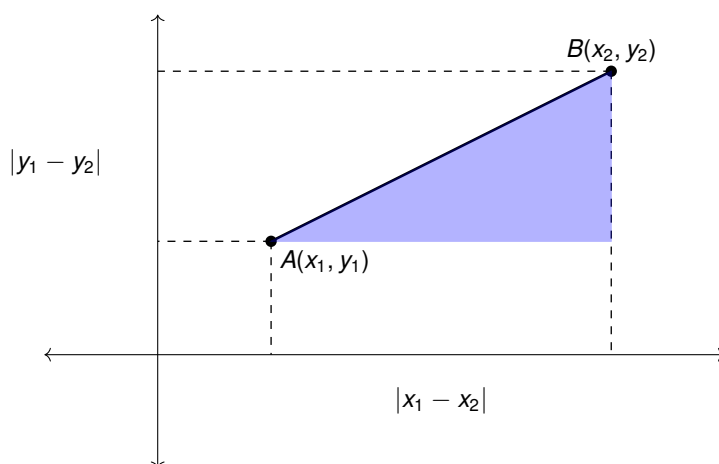
The point $(0, 0, \dots, 0)$ in $\mathbb{R}^n$ is called the *origin*.

$\mathbb{R}^n$ cannot be visualized for $n > 3$, but many familiar ideas, such as the distance formula, can be generalized to $\mathbb{R}^n$.

Distance in $\mathbb{R}^n$

In this section we will establish a formula for the distance between two points in $\mathbb{R}^n$. We begin by observing that the distance between two numbers (points) x_1 and x_2 on the number line is given by $|x_1 - x_2|$. (Why do we use the absolute value brackets?).

We can use the Pythagorean Theorem to establish the distance formula for points of $\mathbb{R}^2$.



Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be points in $\mathbb{R}^2$. By the Pythagorean Theorem we have

$$AB^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

$$AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Why were we able to drop the absolute value brackets?

The distance formula for points in $\mathbb{R}^3$ can also be derived using the Pythagorean Theorem. Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be points of $\mathbb{R}^3$. Use the following GeoGebra interactive to walk through the steps of the derivation of the distance formula. RIGHT-CLICK and DRAG to rotate the image.

Access GeoGebra interactives through the online version of this text at

<https://ximera.osu.edu/linearalgebradv3/LinearAlgebraInteractiveIntro>.

Let points $A'(x_1, y_1, 0)$ and $B'(x_2, y_2, 0)$ be projections of A and B onto the xy -plane. By the distance formula in $\mathbb{R}^2$, the distance between A' and B' is

$$A'B' = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Let $C = (x_2, y_2, z_1)$. Observe that $\triangle ABC$ is a right triangle with $AC = A'B'$, and $BC = |z_1 - z_2|$. By the Pythagorean theorem we have

$$AB^2 = AC^2 + BC^2 = \left(\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \right)^2 + (z_1 - z_2)^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$$

This gives us the following formula.

Formula 1. Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be points in $\mathbb{R}^3$. The distance between A and B is given by

$$AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

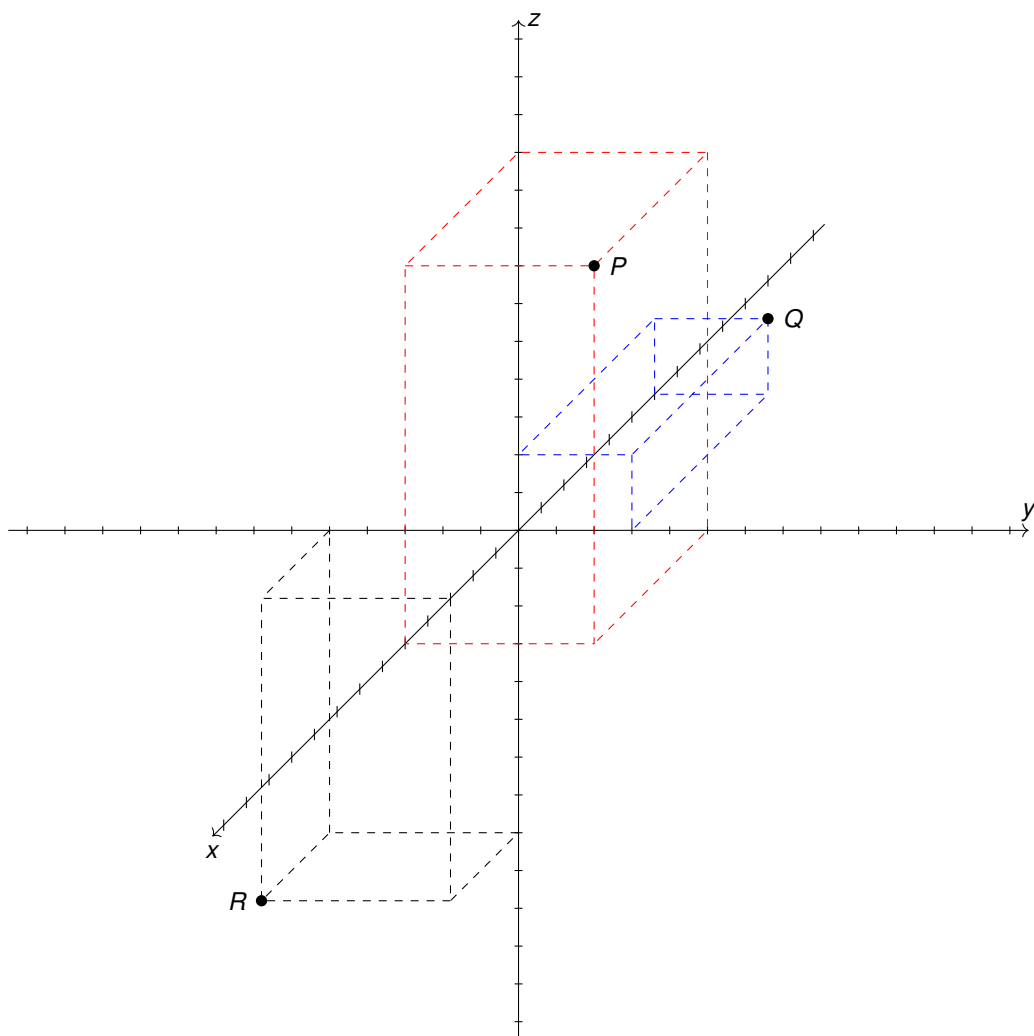
Observe the similarity of pattern in the distance formulas for $\mathbb{R}^1$, $\mathbb{R}^2$ and $\mathbb{R}^3$. We will take advantage of this pattern to define the distance between two points of $\mathbb{R}^n$.

Formula 2. Let $A(a_1, a_2, \dots, a_n)$ and $B(b_1, b_2, \dots, b_n)$ be points in $\mathbb{R}^n$. The distance between A and B is given by

$$AB = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + \dots + (a_n - b_n)^2}$$

Practice Problems

Problem 1 Find the coordinates of each point.



Answer:

$P(\dots, \dots, \dots)$

$Q(\dots, \dots, \dots)$

$R(\dots, \dots, \dots)$

Problem 2 Find the coordinates of each point. RIGHT-CLICK and DRAG to rotate the image.

Access GeoGebra interactives through the online version of this text at

<https://ximera.osu.edu/linearalgebradv3/LinearAlgebraInteractiveIntro>.

Answer:

$A(\dots, \dots, \dots)$

$$B(\dots, \dots, \dots)$$

$$C(\dots, \dots, \dots)$$

$$D(\dots, \dots, \dots)$$

Problem 3 Find the distance between $A(-2, -1, 4)$ and $B(1, -5, -8)$.

$$AB = \dots$$

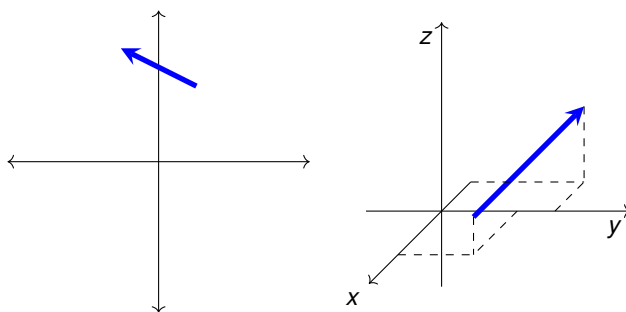
2 Introduction to Vectors

What is a Vector?

A *scalar* is a quantity that has size, often called magnitude, but no direction. For example, temperature, mass and speed are scalars. In this course, scalars will typically be real numbers, but we will also see complex numbers on a few occasions.

A *vector* has magnitude and direction. For example, velocity is a vector because it tells us how fast the object is traveling and also the direction of travel.

If an object is traveling along a number line, the direction of travel is given by the sign of its velocity (positive or negative), while the speed is given by the absolute value of the velocity. If the object is traveling in a plane or in space, direction of travel can be described by an arrow, while the speed can be represented by the length of the arrow. Graphically speaking, vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ look like this:



A vector can be denoted by a lower-case letter with an arrow over the top (like this: $\vec{u}$), or a bold lower-case letter (like this: $\mathbf{u}$).

The magnitude, or length, of a vector is denoted by double absolute value brackets. For example, the magnitude of $\mathbf{u}$, is denoted by $\|\mathbf{u}\|$. A vector of zero length and no direction is called the *zero vector*. We denote the zero vector by $\vec{0}$ or $\mathbf{0}$. Going forward, we will use the terms magnitude of a vector and length of a vector interchangeably.

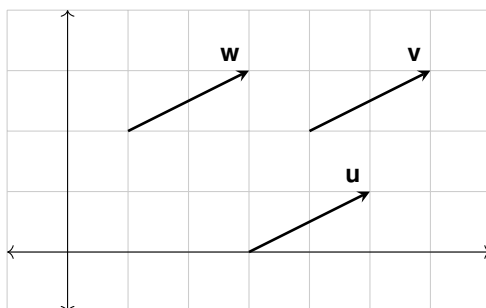
Sometimes it is convenient to refer to a vector by naming the endpoints of the arrow. In the figure below, point A is the *tail*, and point B is the *head* of the vector.



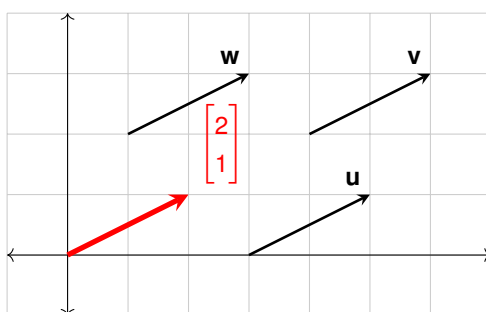
We refer to this vector as $\vec{AB}$.

Vectors in Standard Position

Vectors that point in the same direction and have the same length are said to be *equivalent*. For example, vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in the figure below are equivalent. We write $\mathbf{u} = \mathbf{v} = \mathbf{w}$.



For the purpose of developing standard, convenient notation, we observe that every vector is equivalent to some vector whose tail is at the origin. Vectors with tails at the origin are said to be in *standard position*. We will refer to each vector in standard position by the coordinates of its head. For example, a vector in standard position whose head is located at the point (2, 1) will be referred to as $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.



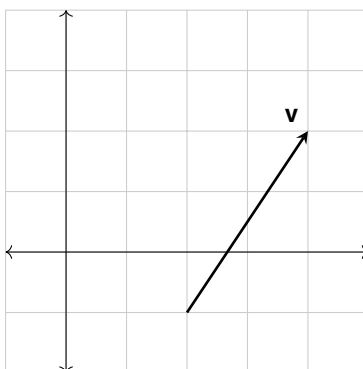
Vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in the figure are equivalent to vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. We write $\mathbf{u} = \mathbf{v} = \mathbf{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Number 2 is called the *first component* of the vector (or the *x*-component) while number 1 is the *second component* (or the *y*-component). The form $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is called the *component form*.

Vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an example of a *column vector*. Occasionally, we will find that representing this vector as a *row vector* $[2, 1]$ is more convenient.

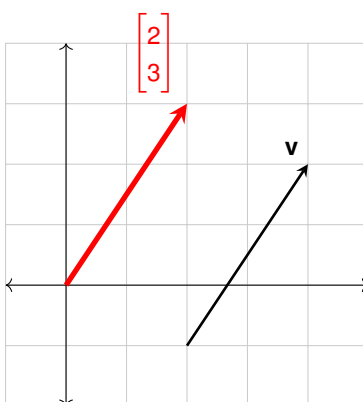
Column (or row) representation of vectors in component form allows us to go beyond the physical and geometric definition, and think of vectors more abstractly as arrays of numbers.

Our next goal is to find a process for writing any vector in the coordinate plane in component form.

Example 2. Consider vector $\mathbf{v}$ shown below. We will express $\mathbf{v}$ in component form.



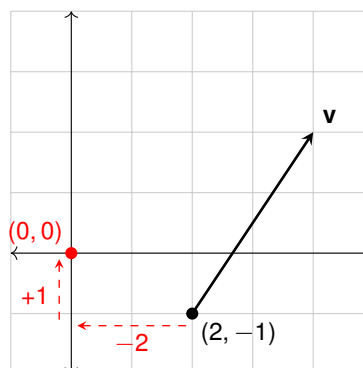
Explanation. Note that the vector has a “run” of 2 and a “rise” of 3. If we construct a vector with tail at the origin, a “run” of 2 and a “rise” of 3, we will have a vector in standard position equivalent to vector $\mathbf{v}$.



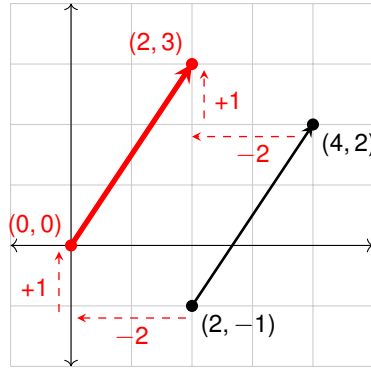
The component form for the vector we constructed is $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$. This gives us $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

The approach we used in Example ?? is applicable to specific vectors that can easily be visualized. What we need now is an algebraic approach that can be generalized to higher dimensions and more abstract situations.

Let’s return to vector $\mathbf{v}$ of Example ??. Suppose we were to slide vector $\mathbf{v}$ into standard position. Consider what would happen to the tail of $\mathbf{v}$ as we do so.



What happens to the tail of the vector has to happen to the head

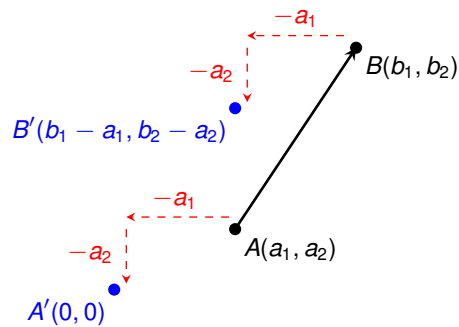


We subtracted 2 from the x -coordinate and added 1 to the y -coordinate of the tail. To find the new location of the head we subtract 2 from the x -coordinate of the head, and add 1 to the y -coordinate of the head. This gives us $(4 - 2, 2 + 1)$. So, the new location of the head is $(2, 3)$, and $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

If you look back at what we did you will find that the components of $\mathbf{v}$ were computed by subtracting the coordinates of the tail from the coordinates of the head

$$\mathbf{v} = \begin{bmatrix} 4 - 2 \\ 2 - (-1) \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

The following diagram summarizes and generalizes our findings.



Let $\overrightarrow{AB}$ be a vector in $\mathbb{R}^2$, with tail at point $A(a_1, a_2)$ and head at point $B(b_1, b_2)$. As we slide $\overrightarrow{AB}$ into standard position by moving point A to the origin, point B travels along with point A by undergoing the same horizontal and vertical shifts. We now have an equivalent vector $\overrightarrow{A'B'}$ in standard position. The diagram suggests the following formula.

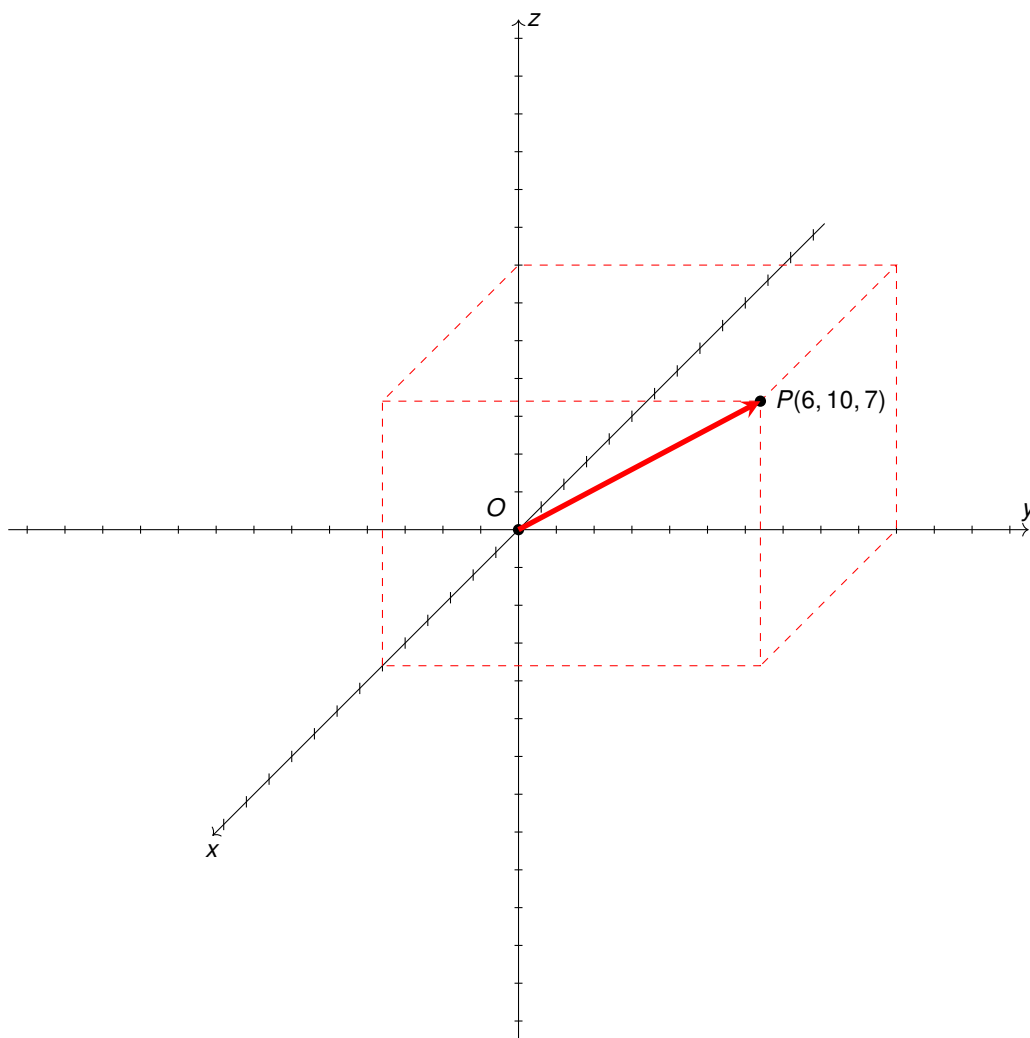
Formula 3 (“Head - Tail” Formula in $\mathbb{R}^2$). Suppose a vector’s tail is at point $A(a_1, a_2)$ and the vector’s head is at $B(b_1, b_2)$, then

$$\overrightarrow{AB} = \begin{bmatrix} b_1 - a_1 \\ b_2 - a_2 \end{bmatrix} \quad (1)$$

Vectors in $\mathbb{R}^3$

Definitions of standard position and component form for vectors in $\mathbb{R}^3$ are analogous to their counterparts for vectors in $\mathbb{R}^2$. For example, vector $\vec{OP}$ in the figure below, is in standard position and can be written in component form as

$$\vec{OP} = \begin{bmatrix} 6 \\ 10 \\ 7 \end{bmatrix}.$$



If a vector is not in standard position but the location of its head and tail are known, a three-dimensional version of the “Head - Tail” formula can be used to express the vector in component form.

Formula 4 (“Head - Tail” Formula in $\mathbb{R}^3$). Suppose a vector’s tail is at point $A(a_1, a_2, a_3)$ and the vector’s head is at

$B(b_1, b_2, b_3)$, then

$$\overrightarrow{AB} = \begin{bmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{bmatrix} \quad (2)$$

Vectors in $\mathbb{R}^n$

We cannot see $\mathbb{R}^n$ for $n > 3$, but we can conceptualize it by generalizing what we know about $\mathbb{R}^2$ and $\mathbb{R}^3$. A vector $\mathbf{v}$

in standard position whose head is located at $(v_1, v_2, \dots, v_n)$ can be written in component form as $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$.

Recall that we defined the *zero vector* as a vector that has length 0 and no direction. In component form, the zero vector is a vector all of whose components are 0.

We conclude this section by stating the generalized “Head - Tail” formula.

Formula 5 (“Head - Tail” Formula in $\mathbb{R}^n$). Suppose a vector’s tail is at point $A(a_1, a_2, \dots, a_n)$ and the vector’s head is at $B(b_1, b_2, \dots, b_n)$, then

$$\overrightarrow{AB} = \begin{bmatrix} b_1 - a_1 \\ b_2 - a_2 \\ \vdots \\ b_n - a_n \end{bmatrix} \quad (3)$$

Practice Problems

Problem 4 Express each vector in component form.

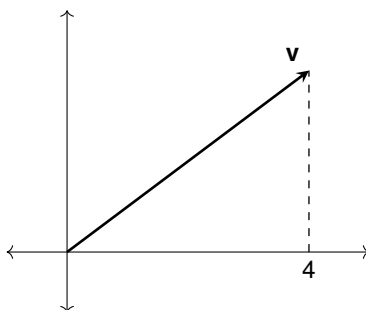
1. Vector $\mathbf{v}$ has a tail at $(-2, 4)$ and a head at $(4, 5)$.

$$\mathbf{v} = \begin{bmatrix} \dots \dots \\ \dots \dots \end{bmatrix}$$

2. Vector $\mathbf{p}$ has a tail at $(2, 4, 5)$ and a head at $(-2, 4, 8)$.

$$\mathbf{p} = \begin{bmatrix} \dots \dots \\ \dots \dots \\ \dots \dots \end{bmatrix}$$

Problem 5 Express vector $\mathbf{v}$ shown in the figure in component form if the magnitude (length) of $\mathbf{v}$ is 5.



Answer:

$$\mathbf{v} = \begin{bmatrix} \dots \dots \\ \dots \dots \end{bmatrix}$$

Problem 6 Let $A(-2, 3)$, $B(3, 1)$ and $C(-1, -1)$. Find point D such that $\overrightarrow{AB} = \overrightarrow{CD}$.

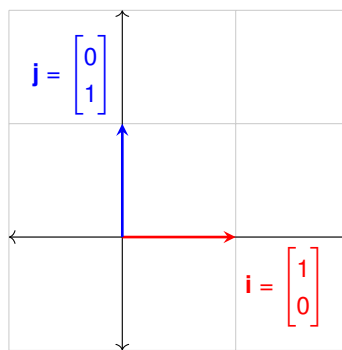
Answer:

$$D(\dots, \dots)$$

3 Standard Unit Vectors in $\mathbb{R}^n$

Standard Unit Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$

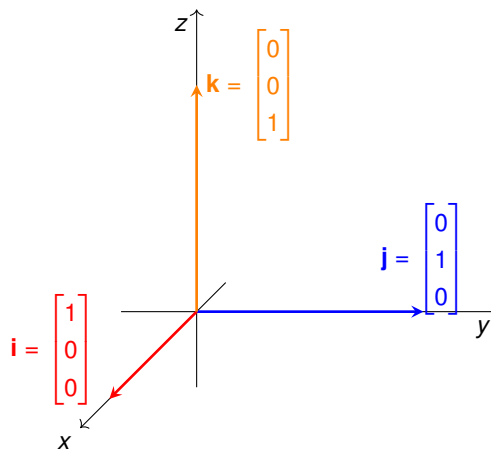
A *unit vector* is a vector of length 1. A unit vector in the positive direction of a coordinate axis is called a *standard unit vector*. There are two standard unit vectors in $\mathbb{R}^2$. The vector $\mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is parallel the x -axis, and the vector $\mathbf{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is parallel the y -axis.



Vector names $\mathbf{i}$ and $\mathbf{j}$ are reserved for standard unit vectors in the direction of x and y axes, respectively. We chose to express $\mathbf{i}$ and $\mathbf{j}$ as column vectors, instead of row vectors, because the context in which we will encounter them in the future will require them to be column vectors. You may see them presented as row vectors in a different course.

There are three standard unit vectors in $\mathbb{R}^3$:

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



A Vector as a Linear Combination of Standard Unit Vectors

Every vector in $\mathbb{R}^2$ and $\mathbb{R}^3$ can be written as a sum of scalar multiples of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. For example, if $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix}$, then

$$\mathbf{v} = \begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (-2) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 7 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 3\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}$$

The expression $3\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}$ is called a *linear combination* of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

Standard Unit Vectors in $\mathbb{R}^n$

When working with vectors in $\mathbb{R}^n$, we often use a different notation to denote the standard unit vectors.

Definition 1. Let $\mathbf{e}_i$ denote a vector that has 1 as the i^{th} component and zeros elsewhere. In other words,

$$\mathbf{e}_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

where 1 is in the i^{th} position. We say that $\mathbf{e}_i$ is a *standard unit vector* of $\mathbb{R}^n$.

Practice Problems

Problem 7 Express the given vector as a linear combination of appropriate standard unit vectors.

$$\mathbf{u} = \begin{bmatrix} 0 \\ 4 \\ -3 \end{bmatrix}$$

Answer:

$$\mathbf{u} = \dots \mathbf{i} + \dots \mathbf{j} + \dots \mathbf{k}$$

Problem 8 Express the given vector as a linear combination of appropriate standard unit vectors.

$$\mathbf{w} = \begin{bmatrix} 5 \\ -3 \\ 1 \\ 7 \end{bmatrix}$$

Answer:

$$\mathbf{w} = \dots \mathbf{e}_1 + \dots \mathbf{e}_2 + \dots \mathbf{e}_3 + \dots \mathbf{e}_4$$

Problem 9 Express the given vector in component form.

$$\mathbf{u} = \mathbf{i} + 3\mathbf{j}$$

is a vector in $\mathbb{R}^2$.

Answer:

$$\mathbf{u} = \begin{bmatrix} \dots \\ \dots \end{bmatrix}$$

Problem 10 Express the given vector in component form.

$$\mathbf{v} = -\mathbf{j} + 5\mathbf{k}$$

is a vector in $\mathbb{R}^3$.

Answer:

$$\mathbf{v} = \begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix}$$

Problem 11 Express the given vector in component form.

$$\mathbf{w} = \mathbf{e}_1 - 2\mathbf{e}_3 + 4\mathbf{e}_4$$

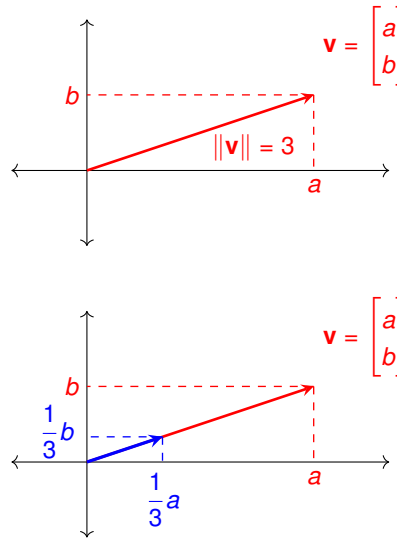
is a vector in $\mathbb{R}^4$.

Answer:

$$\mathbf{w} = \begin{bmatrix} \dots \\ \dots \\ \dots \\ \dots \end{bmatrix}$$

4 Unit Vector in the Direction of a Given Vector

Recall that a *unit* vector is a vector of length 1. Given a non-zero vector $\mathbf{v}$, we can find a unit vector in the same direction by multiplying $\mathbf{v}$ by an appropriate scalar. For example, if $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\|\mathbf{v}\| = 3$, then a unit vector $\mathbf{u}$ in the same direction is given by $\mathbf{u} = \begin{bmatrix} a/3 \\ b/3 \end{bmatrix} = \begin{bmatrix} a/\|\mathbf{v}\| \\ b/\|\mathbf{v}\| \end{bmatrix}$.



In general, dividing a non-zero vector by its own magnitude produces a unit vector in the same direction. We summarize this observation in a theorem.

Theorem 1. Let $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ be a non-zero vector in $\mathbb{R}^n$. Vector $\mathbf{u}$ given by

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|} \mathbf{v} = \begin{bmatrix} v_1 / \|\mathbf{v}\| \\ v_2 / \|\mathbf{v}\| \\ \vdots \\ v_n / \|\mathbf{v}\| \end{bmatrix}$$

is a unit vector in the direction of $\mathbf{v}$.

Proof Because $\mathbf{u}$ is a positive scalar multiple of $\mathbf{v}$, $\mathbf{u}$ points in the direction of $\mathbf{v}$. We now show that $\|\mathbf{u}\| = 1$.

$$\begin{aligned}\|\mathbf{u}\| &= \sqrt{\left(\frac{v_1}{\|\mathbf{v}\|}\right)^2 + \left(\frac{v_2}{\|\mathbf{v}\|}\right)^2 + \dots + \left(\frac{v_n}{\|\mathbf{v}\|}\right)^2} \\ &= \frac{1}{\|\mathbf{v}\|} \sqrt{v_1^2 + v_2^2 + \dots + v_n^2} \\ &= \frac{\|\mathbf{v}\|}{\|\mathbf{v}\|} = 1\end{aligned}$$

■

Example 3. Find a unit vector in the direction of $\mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

Explanation. We first compute $\|\mathbf{v}\|$.

$$\|\mathbf{v}\| = \sqrt{4 + 9 + 1 + 1} = \sqrt{15}$$

By Theorem ??,

$$\mathbf{u} = \begin{bmatrix} 2/\sqrt{15} \\ -3/\sqrt{15} \\ 1/\sqrt{15} \\ 0 \\ 1/\sqrt{15} \end{bmatrix} = \frac{1}{\sqrt{15}} \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Practice Problems

Problem 12 Find a unit vector in the direction of the given vector $\mathbf{v}$.

$$\mathbf{v} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

Answer: $\mathbf{u} = \begin{bmatrix} \dots \\ \dots \end{bmatrix}$

Problem 13 Find a unit vector in the direction of the given vector $\mathbf{v}$.

$$\mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix}$$

Enter exact answers. No decimal approximations.

Answer: $\mathbf{u} = \begin{bmatrix} \dots \dots \\ \dots \dots \\ \dots \dots \end{bmatrix}$

Problem 14 Let $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ \sqrt{7} \end{bmatrix}$. Apply the concepts from this section to find a vector $\mathbf{w}$ that points in the same direction as $\mathbf{v}$ and whose length is 5.

Enter exact answers. No decimal approximations.

Answer: $\mathbf{w} = \begin{bmatrix} \dots \dots \\ \dots \dots \\ \dots \dots \end{bmatrix}$

5 Dot Product and its Properties

Definition 2. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. The *dot product* of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} \cdot \mathbf{v}$, is given by

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n.$$

Example 4. Find $\mathbf{u} \cdot \mathbf{v}$ if $\mathbf{u} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$.

Explanation.

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} = (-2)(3) + (0)(2) + (1)(-4) = -6 - 4 = -10$$

Note that the dot product of two vectors is a scalar. For this reason, the dot product is sometimes called a *scalar product*.

Properties of the Dot Product

A quick examination of Example ?? will convince you that the dot product is *commutative*. In other words, $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$. This and other properties of the dot product are stated below.

Theorem 2. The following properties hold for vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$ and scalar k in $\mathbb{R}$.

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
4. $(k\mathbf{u}) \cdot \mathbf{v} = k(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (k\mathbf{v})$
5. $\mathbf{u} \cdot \mathbf{u} \geq 0$, and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.
6. $\|\mathbf{u}\|^2 = \mathbf{u} \cdot \mathbf{u}$

We will prove Property ?. The remaining properties are left as exercises.

Proof of Property ??:

$$\begin{aligned}
 (\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} &= \left(\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \right) \cdot \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \\
 &= (u_1 + v_1)w_1 + (u_2 + v_2)w_2 + \dots + (u_n + v_n)w_n \\
 &= u_1 w_1 + v_1 w_1 + u_2 w_2 + v_2 w_2 + \dots + u_n w_n + v_n w_n \\
 &= (u_1 w_1 + u_2 w_2 + \dots + u_n w_n) + (v_1 w_1 + v_2 w_2 + \dots + v_n w_n) \\
 &= \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}
 \end{aligned}$$

■

We will illustrate Property ?? with an example.

Example 5. Let $\mathbf{u} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. Use $\mathbf{u}$ to illustrate Property ?? of Theorem ??.

Explanation.

$$\|\mathbf{u}\|^2 = (-2)^2 + 3^2 = (-2)(-2) + (3)(3) = \begin{bmatrix} -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \mathbf{u} \cdot \mathbf{u}$$

If we take the square root of both sides of the equation in Property ??, we get an alternative way to think of the **length** of a vector.

Corollary 1 (Length of a Vector). Let $\mathbf{v}$ be a vector in $\mathbb{R}^n$, then the *length*, or the *magnitude*, of $\mathbf{v}$ is given by

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$$

Practice Problems

Problem 15 Find the dot product of $\mathbf{u}$ and $\mathbf{v}$ if

$$\mathbf{u} = \begin{bmatrix} -1 \\ -2 \\ 5 \\ 4 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ -2 \\ -3 \\ 1 \end{bmatrix}$$

Answer:

$$\mathbf{u} \cdot \mathbf{v} = \dots$$

Problem 16 Find the dot product of $\mathbf{u}$ and $\mathbf{v}$ if

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

Answer:

$$\mathbf{u} \cdot \mathbf{v} = \dots \dots$$

Problem 17 Use vector $\mathbf{u} = \begin{bmatrix} 2 \\ 5 \\ -7 \end{bmatrix}$ to illustrate Property ?? of Theorem ??.

Problem 18 Prove Properties ??, ??, ??, ?? and ?? of Theorem ??.

Problem 19 From the given list of vector pairs, identify ALL pairs of vectors that lie on perpendicular lines.

Select All Correct Answers:

1. $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$

2. $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$

3. $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

4. $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$

Compute $\mathbf{u} \cdot \mathbf{v}$ for each pair. What do you observe?

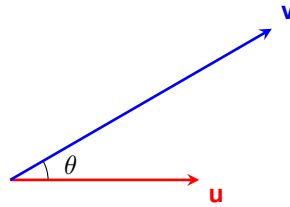
Problem 20 1. Vector $\mathbf{u}$ parallel to the line $y = mx$ has the form $\mathbf{u} = k \begin{bmatrix} 1 \\ m \end{bmatrix}$. Assuming that $m \neq 0$, find the general form for a vector $\mathbf{v}$ perpendicular to $y = mx$.

2. Find $\mathbf{u} \cdot \mathbf{v}$.

3. What is true about the dot product of perpendicular vectors?

6 Dot Product and Angle

Given two vectors $\mathbf{u}$ and $\mathbf{v}$, let θ be the angle between them such that $0 \leq \theta \leq \pi$. We will refer to θ as the *included angle*.

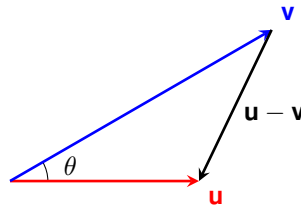


The following theorem establishes a relationship between the dot product and the included angle.

Theorem 3. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$, and let θ be the included angle. Then

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$$

Proof Consider the triangle formed by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{u} - \mathbf{v}$.



By the Law of Cosines we have:

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2 \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$$

By Theorem ???? of Dot Product and its Properties

$$(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} - 2 \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$$

By Theorem ???? of Dot Product and its Properties

$$(\mathbf{u} - \mathbf{v}) \cdot \mathbf{u} - (\mathbf{u} - \mathbf{v}) \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} - 2 \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$$

By Theorem ???? of Dot Product and its Properties

$$\mathbf{u} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} - 2 \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$$

By Theorem ???? of Dot Product and its Properties

$$\begin{aligned} -2(\mathbf{u} \cdot \mathbf{v}) &= -2 \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta \\ \mathbf{u} \cdot \mathbf{v} &= \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta \end{aligned}$$

■

Example 6. Find the included angle between vectors $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$.

Explanation. By Theorem ??, $\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}$.

$$\mathbf{u} \cdot \mathbf{v} = 2 - 2 - 12 = -12$$

$$\|\mathbf{u}\| = \sqrt{4 + 1 + 16} = \sqrt{21}$$

$$\|\mathbf{v}\| = \sqrt{1 + 4 + 9} = \sqrt{14}$$

$$\cos \theta = \frac{-12}{\sqrt{21}\sqrt{14}}$$

$$\theta \approx 134.4^\circ$$

Orthogonal Vectors

Definition 3. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. We say $\mathbf{u}$ and $\mathbf{v}$ are *orthogonal* if $\mathbf{u} \cdot \mathbf{v} = 0$.

We can use Theorem ?? to show that two non-zero *orthogonal* vectors of $\mathbb{R}^n$ are simply *perpendicular* vectors (the included angle is 90°). To see this, suppose that $\mathbf{u} \cdot \mathbf{v} = 0$ for nonzero vectors $\mathbf{u}, \mathbf{v}$. Then from Theorem ?? we have

$$0 = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta.$$

Since $\mathbf{u}, \mathbf{v}$ are nonzero vectors, we have $0 = \cos \theta$, which implies $\theta = 90^\circ$. The converse also holds. If $\theta = 90^\circ$, then the dot product is clearly 0.

The reason we prefer the term “orthogonal” to “perpendicular” in this course is because $\mathbb{R}^n$ is only one example of a *vector space*, and the dot product is only one example of a more general product, called an *inner product*. For vectors in $\mathbb{R}^n$ a zero dot product happens to coincide with the geometric idea of perpendicularity, but there are many vector spaces that do not possess the visual geometry of $\mathbb{R}^n$. (Later in the text, you will encounter vector spaces whose vectors are polynomial functions!) In these more abstract settings, a zero inner product still signals a special relationship between vectors. The term *orthogonal* captures this relationship.

Remark 1. By our definition, the zero vector is orthogonal to any vector. However, we will not use the word *perpendicular* when the zero vector is involved, as it is not possible to talk about an “included angle”.

Practice Problems

Problem 21 Find the degree measure of the included angle, θ for each pair of vectors. Round your answers to the nearest tenth.

1. $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ -1 \end{bmatrix}$.

Answer: $\theta = \dots^\circ$

2. $\begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$

Answer: $\theta = \dots\dots^\circ$

3. $\begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -5 \\ -2 \\ 4 \end{bmatrix}$

Answer: $\theta = \dots\dots^\circ$

Problem 22 What does the sign of the dot product tell us about the included angle?

Problem 23 Find all values of a so that $\begin{bmatrix} a^2 \\ 2a \\ 1 \end{bmatrix}$ is orthogonal to $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. List your answers in increasing order.

Answer: $\dots\dots, \dots\dots$

Problem 24 Find the value of x for which the vector $\begin{bmatrix} x \\ -4 \end{bmatrix}$ is parallel to the vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$. What is the measure of the included angle, θ ? Find the measure of the included angle using Theorem ???. Do the two results agree?

Answer:

$$x = \dots\dots$$

$$\theta = \dots\dots^\circ$$

Problem 25 Prove that if $\mathbf{u}$ is a unit vector, then $\mathbf{u} \cdot \mathbf{u} = 1$.

Problem 26 Prove that if $\mathbf{u}_1$ and $\mathbf{u}_2$ are unit vectors, then $-1 \leq \mathbf{u}_1 \cdot \mathbf{u}_2 \leq 1$. In what cases are the extreme values of 1 and -1 attained?

Problem 27 Imagine a clock with hands represented by vectors $\mathbf{m}$ and $\mathbf{h}$, as shown below. At what whole hour will $\mathbf{m} \cdot \mathbf{h}$ attain its maximum value? At what whole hour will $\mathbf{m} \cdot \mathbf{h}$ be as small as possible?



Answer:

$\mathbf{m} \cdot \mathbf{h}$ is greatest at : 00 o'clock

$\mathbf{m} \cdot \mathbf{h}$ is smallest at : 00 o'clock

Photo Credits

The following images are courtesy of [Wikimedia Commons](#)

Hannes Grobe, *Wall clock manufactured by Telefonbau & Normalzeit*. CC-BY 3.0

7 Parametric Equations of Lines

At this point you should be very familiar with graphing linear equations of the form $y = mx + b$, where m is the slope of the line and b is the y -intercept. Unfortunately, there is no equivalent way of representing lines in $\mathbb{R}^3$, or better yet, $\mathbb{R}^n$. In this section we will develop an alternative way of representing lines.

Parametric Lines in $\mathbb{R}^2$

Exploration 28 *Imagine a ladybug crawling around a plane. At every instant in time, the ladybug's position in the plane can be described by an ordered pair (x, y) . Coordinates x and y are functions of time. Suppose the position of the ladybug at time t is given by:*

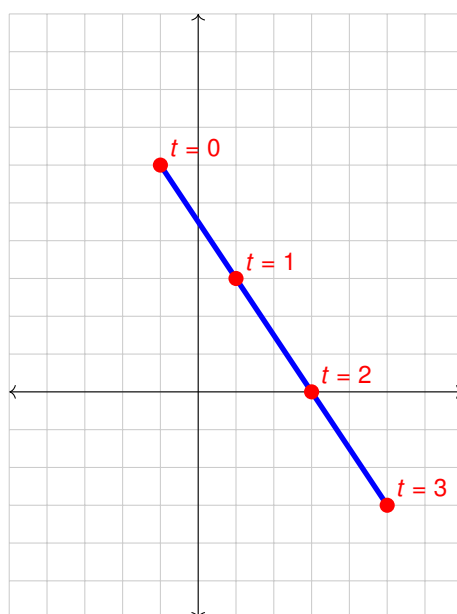
$$x = 2t - 1$$

$$y = -3t + 6$$

Our goal is to sketch the path of the ladybug during the time interval $0 \leq t \leq 3$. To do this we will make a table of values, just like you did when you first started sketching graphs of functions.

t	x	y
0	-1	6
1	1	3
2	3	0
3	5	-3

The point (x, y) , corresponding to each value of t is the location of the ladybug in the coordinate plane at time t .



It appears that the points corresponding to $t = 0, 1, 2, 3$ lie on a line with slope $-\frac{3}{2}$. In the next problem we will find the equation of the line that contains the path of the ladybug.

Exploration 29 In Exploration ?? we considered equations

$$x = 2t - 1$$

$$y = -3t + 6$$

that described the path of a ladybug in the plane. We will now consider these equations in a broader context as simply describing a curve in the plane. We conjectured that the curve described by these equations is a line with slope $-\frac{3}{2}$. We will now find the equation of this line.

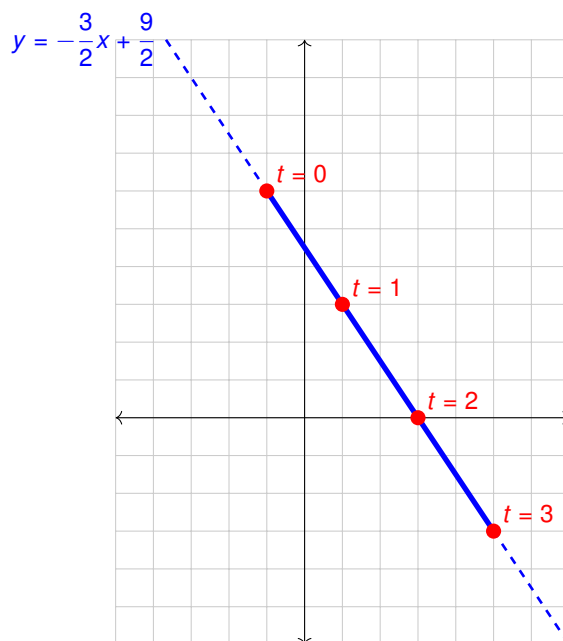
One approach to finding the equation is to solve one of the given equations for t , then substitute into the other equation. Solving $x = 2t - 1$ for t gives us

$$t = \frac{1}{2}x + \frac{1}{2}$$

Substituting this expression into $y = -3t + 6$ we get

$$y = -3\left(\frac{1}{2}x + \frac{1}{2}\right) + 6 = -\frac{3}{2}x + \frac{9}{2}$$

Plotting the line $y = -\frac{3}{2}x + \frac{9}{2}$ together with the path we plotted earlier, we see that our original path lies on the line.



If we do not restrict t to values between 0 and 3, we can get every point on the line $y = -\frac{3}{2}x + \frac{9}{2}$.

Equations such as

$$\begin{aligned}x &= 2t - 1 \\y &= -3t + 6\end{aligned}$$

are called *parametric equations*, and t is called a *parameter*.

When given an equation of the form $y = mx + b$, we recognize it as an equation whose graph is a line and we don't need to make a table of values to sketch the graph of the equation. We should be able to do the same for parametric equations of lines. In the next Exploration we will examine our equations carefully to see if we can discern any patterns that would help us plot the line without making a table of values.

Exploration 30 *Let's return to parametric equations from the previous problem.*

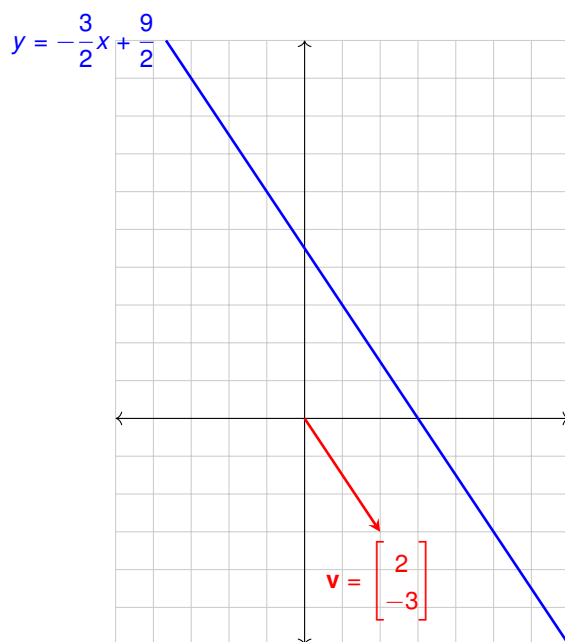
$$\begin{aligned}x &= 2t - 1 \\y &= -3t + 6\end{aligned}$$

Consider the table of values we constructed in Exploration ??.

t	x	y
0	-1	6
1	1	3
2	3	0
3	5	-3

Observe that every time t increases by 1, x increases by 2 and y decreases by 3. An increase by 2 in the x -coordinate combined with a decrease of 3 in the y -coordinate corresponds to the coefficients 2 and -3 of t in the parametric equations. Our findings are in agreement with the fact that the slope of the line is $-\frac{2}{3}$.

We will capture the “rise” and “run” aspect of the line by using vector $\mathbf{v} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$. Sketching $\mathbf{v}$ together with the line, we find that the vector is parallel to the line.



Vector $\mathbf{v}$ is called a direction vector. Observe that the components of $\mathbf{v}$ are the coefficients of t in the parametric equations.

$$x = 2t - 1$$

$$y = -3t + 6$$

Next, let's turn our attention to constants -1 and 6 . They are the values of x and y when $t = 0$, and correspond to the point $(-1, 6)$.

Recall that in Exploration ?? we said that the given parametric equations describe the position of a ladybug crawling in the coordinate plane. If we are concerned with the position of the ladybug at time t , the point $(-1, 6)$ is very important - this is where the bug is located when $t = 0$. But, if we are using parametric equations simply to describe the line $y = -\frac{3}{2}x + \frac{9}{2}$, without regard for when the bug is located at each point, then the point $(-1, 6)$ is not any more special than any other point on the line. In fact, we can use any other point on this line to find another set of parametric equations that describe the same line. In Practice Problem ?? you will be asked to show that equations

$$x = 2t + 1$$

$$y = -3t + 3$$

describe the same line. Thus, parametric representations are not unique.

We can generalize our observations in Exploration ?? as follows.

Formula 6. Let $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ be a direction vector for line l , and let (x_0, y_0) be an arbitrary point on l . Then the following parametric equations describe l :

$$x = at + x_0$$

$$y = bt + y_0$$

Example 7. Find parametric equations for line l if l contains the point $(2, -5)$ and is parallel to line k with parametric equations

$$x = 3t + 4$$

$$y = -8t + 1$$

Explanation. To find a set of parametric equations for l , we need a point on l and a direction vector. Point $(2, -5)$ is on l . Line l is parallel to k , so we can use $\begin{bmatrix} 3 \\ -8 \end{bmatrix}$, a direction vector for k , as a direction vector for l . This gives us the following parametric equations for l :

$$x = 3t + 2$$

$$y = -8t - 5$$

Parametric Lines in $\mathbb{R}^3$

Exploration 31 Consider the following set of parametric equations

$$x = 4t + 1$$

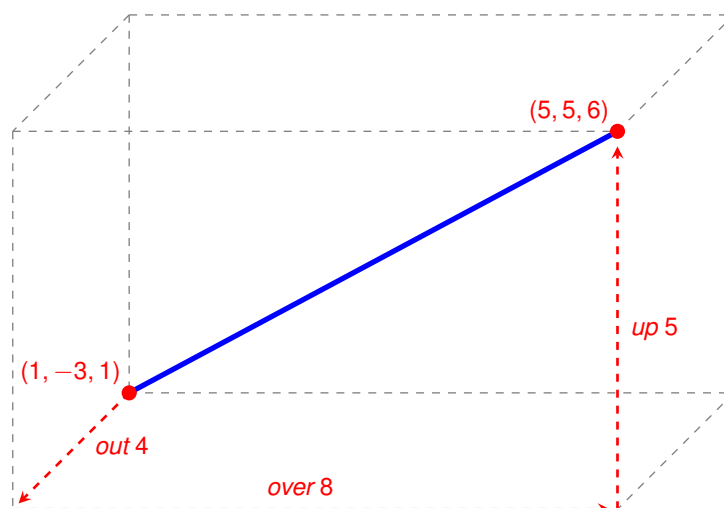
$$y = 8t - 3$$

$$z = 5t + 1$$

We will begin by making a table of values.

t	x	y	z
-1	-3	-11	-4
0	1	-3	1
1	5	5	6
2	9	13	11

Observe that every time t increases by 1, x increases by 4, y increases by 8, and z increases by 5. The pattern “out 4, over 8, up 5” is illustrated in the diagram.



We will use a direction vector $\mathbf{v} = \begin{bmatrix} 4 \\ 8 \\ 5 \end{bmatrix}$ to represent the direction of the line. Note that the components of $\mathbf{v}$ are the same as the coefficients of t in the parametric equations.

$$\begin{aligned} x &= 4t + 1 \\ y &= 8t - 3 \\ z &= 5t + 1 \end{aligned}$$

Also, observe that the constants 1, -3 , 1 correspond to the coordinates of the point on the line for which $t = 0$.

$$\begin{aligned} x &= 4t + 1 \\ y &= 8t + (-3) \\ z &= 5t + 1 \end{aligned}$$

As before, the point $(1, -3, 1)$ may be replaced with any other point on the line to produce a different set of parametric equations describing the same line.

We can generalize our observations in Exploration ?? as follows.

Formula 7. Let $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be a direction vector for line l , and let (x_0, y_0, z_0) be an arbitrary point on l . Then the following parametric equations describe l :

$$\begin{aligned} x &= at + x_0 \\ y &= bt + y_0 \\ z &= ct + z_0 \end{aligned}$$

Example 8. Find a set of parametric equations for a line in $\mathbb{R}^3$ that passes through $(1, -2, 4)$ and $(-3, 1, 1)$.

Explanation. We need a point and a direction vector. To find the direction vector we will use the “head-tail” formula (Formula ??) to find the vector whose tail is at $(1, -2, 4)$ and whose head is at $(-3, 1, 1)$. The direction vector is

$$\begin{bmatrix} -3 - 1 \\ 1 - (-2) \\ 1 - 4 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ -3 \end{bmatrix}$$

For our point, we can pick either of the two given points. The equations will differ depending on the point we pick, but they will describe the same line. Remember, parametric representations are not unique! If we choose $(1, -2, 4)$ for our point we get the following set of parametric equations:

$$\begin{aligned} x &= -4t + 1 \\ y &= 3t - 2 \\ z &= -3t + 4 \end{aligned}$$

Parametric Lines in $\mathbb{R}^n$

After working with parametric equations of lines in $\mathbb{R}^2$ and $\mathbb{R}^3$, it should be easy to surmise what parametric equations of lines look like for lines in $\mathbb{R}^n$.

Formula 8. Let $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ be a direction vector for line l in $\mathbb{R}^n$, and let $(a_1, a_2, \dots, a_n)$ be an arbitrary point on l . Then

the following parametric equations describe l :

$$x_1 = v_1 t + a_1$$

$$x_2 = v_2 t + a_2$$

$$\vdots$$

$$x_n = v_n t + a_n$$

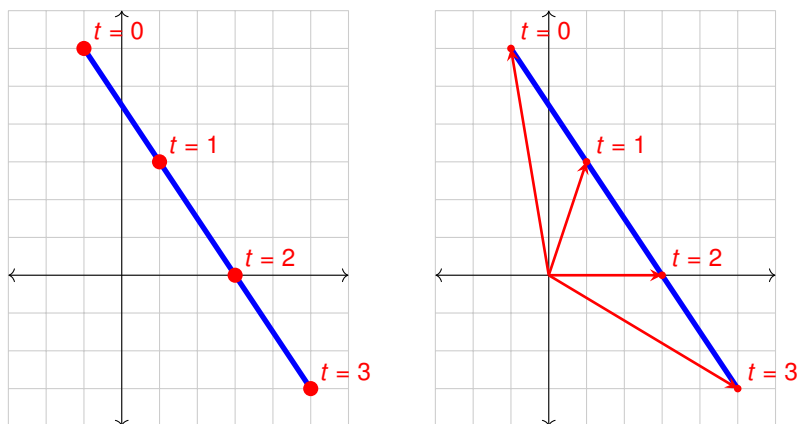
From Parametric Equations to Vector Equations

Consider again the parametric equations from Exploration ??.

$$x = 2t - 1$$

$$y = -3t + 6$$

These equations describe the line by providing coordinates of individual points on the line, as shown in the left figure below.



If we consider the points on the line to be heads of vectors, then we can think of the line as being “traced” by a vector as t changes (See above, right). We can capture this algebraically by using a *vector equation* for the line:

$$\mathbf{x}(t) = \begin{bmatrix} 2t - 1 \\ -3t + 6 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} t + \begin{bmatrix} -1 \\ 6 \end{bmatrix}$$

Note that the direction vector $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and the point $(-1, 6)$, now vector $\begin{bmatrix} -1 \\ 6 \end{bmatrix}$, are easy to identify in the equation.

We can use vector equations to describe lines in $\mathbb{R}^n$ using the following formula.

Formula 9. Let $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ be a direction vector for line l in $\mathbb{R}^n$, and let $(a_1, a_2, \dots, a_n)$ be an arbitrary point on l . Then the following vector equation describes l :

$$\mathbf{x}(t) = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} t + \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

Practice Problems

Problem 32 In Explorations ??, ?? and ?? you encountered a set of parametric equations

$$\begin{aligned} x &= 2t - 1 \\ y &= -3t + 6 \end{aligned}$$

Show that the following parametric equations describe the same line.

$$\begin{aligned} x &= 2t + 1 \\ y &= -3t + 3 \end{aligned}$$

Problem 33 Find two different parametric representations for a line with slope 3 and y -intercept $(0, -4)$.

Problem 34 Suppose line l has a parametric representation:

$$\begin{aligned} x_1 &= t + 3 \\ x_2 &= t - 4 \\ x_3 &= 3t \\ x_4 &= 7t + 1 \end{aligned}$$

Find the missing components for the two vectors below if we know that both vectors can be considered direction vectors for line l .

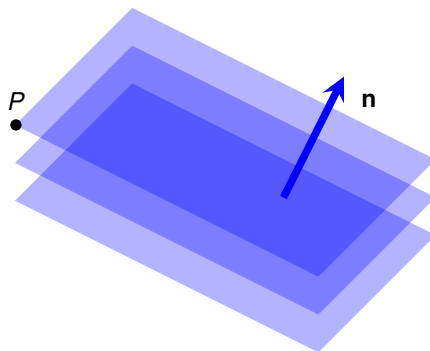
$$\begin{bmatrix} \dots \\ \dots \\ \dots \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ \dots \\ \dots \\ \dots \end{bmatrix}$$

8 Planes in $\mathbb{R}^3$

You are probably familiar with the expression “two points determine a line.” This means that given two distinct points, there is exactly one line that passes through both of them. There are other ways to describe a line. For example, as we saw in [Parametric Equations of Lines](#), a single point together with a direction vector also determine a line.

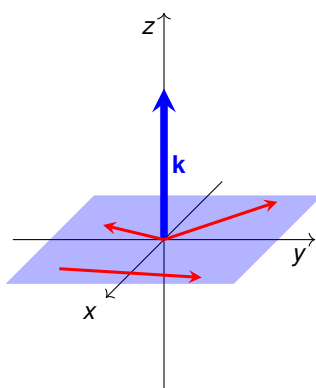
You are probably also familiar with the fact that three non-collinear points determine a plane. (This is why photographers use tripods for stability, while four-legged chairs often wobble!) Is there another way to determine a plane?

The diagram below shows several planes perpendicular to vector $\mathbf{n}$. In fact, there are infinitely many such planes. Vector $\mathbf{n}$ does not determine a plane, but if we know of a point P contained in the plane, then together, P and $\mathbf{n}$ describe a unique plane.

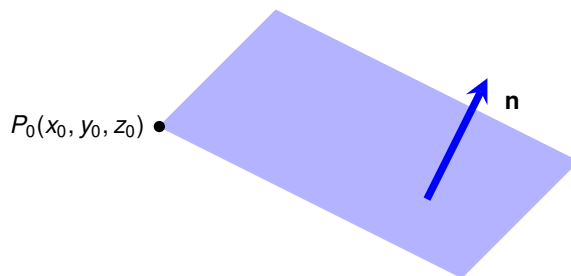


Definition 4 (Normal Vector). A nonzero vector $\mathbf{n}$ is called a *normal* for a plane if it is orthogonal to every vector in the plane.

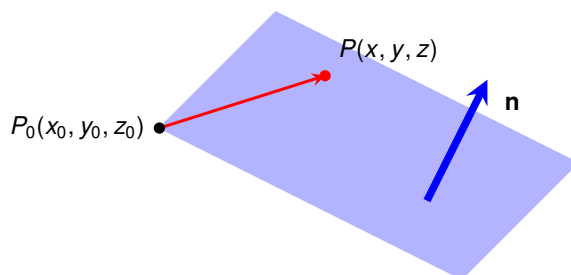
Example 9. The standard unit vector $\mathbf{k}$ is orthogonal to every vector in the xy -plane, therefore it is a normal for the xy -plane.



Given a point $P_0 = P_0(x_0, y_0, z_0)$ and a nonzero vector $\mathbf{n}$, there is a unique plane through P_0 with normal $\mathbf{n}$.



This fact can be used to give a very simple description of a plane. Observe that a point $P = P(x, y, z)$ lies on this plane if and only if the vector $\overrightarrow{P_0P}$ is orthogonal to $\mathbf{n}$ (i.e. P lies in the plane if and only if $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$).



Let $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. By “head-tail” formula (Formula ??), we have: $\overrightarrow{P_0P} = \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix}$. So, $P(x, y, z)$ lies in the plane if and only if

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

We summarize this result as a theorem.

Theorem 4. The plane through $P_0(x_0, y_0, z_0)$ with a normal vector

$$\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \mathbf{0}$$

is given by

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad (4)$$

Example 10. Find an equation of the plane through $P_0(1, -1, 3)$ with $\mathbf{n} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ as a normal vector.

Explanation. Here the equation becomes

$$3(x - 1) - (y + 1) + 2(z - 3) = 0$$

This simplifies to $3x - y + 2z = 10$.

GeoGebra interactive below shows the plane, together with point P_0 and the normal vector. RIGHT-CLICK and DRAG to rotate the image for a better view.

Access GeoGebra interactives through the online version of this text at

<https://ximera.osu.edu/linearalgebradv3/LinearAlgebraInteractiveIntro>.

As demonstrated in Example ??, we can distribute coefficients a , b and c of equation (??) as follows:

$$ax - ax_0 + by - by_0 + cz - cz_0 = 0$$

Setting $d = ax_0 + by_0 + cz_0$, shows that every plane with a normal vector $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ has a linear equation of the form

$$ax + by + cz = d \quad (5)$$

for some constant d . Conversely, the graph of this equation is a plane with $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ as a normal vector (assuming that a , b , and c are not all zero).

Example 11. Find an equation of the plane through $P_0(3, -1, 2)$ that is parallel to the plane with equation $2x - 3y = 6$.

Explanation. The plane with equation $2x - 3y = 6$ has a normal vector $\mathbf{n} = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$. Because the two planes are parallel, $\mathbf{n}$ serves as a normal for the plane we seek, so the equation is $2x - 3y = d$ for some d by (??). Insisting that $P_0(3, -1, 2)$ lies on the plane determines d ; that is, $d = 2 \cdot 3 - 3(-1) = 9$. Hence, the equation is $2x - 3y = 9$.

GeoGebra interactive below shows the two planes, together with point P_0 and the normal vector. RIGHT-CLICK and DRAG to rotate the image for a better view.

Access GeoGebra interactives through the online version of this text at

<https://ximera.osu.edu/linearalgebradv3/LinearAlgebraInteractiveIntro>.

Linear Equations and their Graphs: From Lines to Hyperplanes

An equation of the form $ax + by = d$ is a linear equation whose graph is a line in $\mathbb{R}^2$. If we solve for y in terms of x , we obtain a more familiar form of this equation, $y = -\frac{a}{b}x + \frac{d}{b}$. The slope of the corresponding line is $m = -\frac{a}{b}$. Observe that a line perpendicular to this line has the slope $\frac{b}{a}$. If we interpret a as a horizontal “run” and b as a vertical “rise”,

we see that the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ is perpendicular to the line $ax + by = d$. You can use the following GeoGebra interactive to solidify your understanding of this.

Access GeoGebra interactives through the online version of this text at

<https://ximera.osu.edu/linearalgebradv3/LinearAlgebraInteractiveIntro>.

The idea of coefficients in front of variables forming components of a normal vector should be very familiar to you.

Recall that the graph of $ax + by + cz = d$ is a plane in $\mathbb{R}^3$ with a normal vector $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

Lines and planes may seem very different, but they are all graphs of linear equations, just in different dimensions. In general, an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = d$$

is called a *linear equation* in n variables. The graph of such an equation, for $n > 3$, is called a *hyperplane*. The vector in $\mathbb{R}^n$ whose components are the coefficients $a_1, a_2, \dots, a_n$ is orthogonal to every vector in the hyperplane. Unfortunately, hyperplanes are impossible to see, but we can often use insights we gain from working with lines and planes and generalize them to the invisible world of higher dimensions.

Practice Problems

Problem 35 Find an equation of the form $ax + by + cz = d$ for the plane described below.

The plane has a normal vector $\mathbf{n} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ and the plane passes through the point $(4, 4, -3)$.

Answer:

$$\dots\dots x + \dots\dots y + \dots\dots z = \dots\dots$$

Problem 36 Find an equation of the form $ax + by + cz = d$ for the plane described below.

The plane contains the point $(-1, 3, 0)$ and is parallel to the plane described by $2x - 5y + 4z = 7$.

Answer:

$$\dots\dots x + \dots\dots y + \dots\dots z = \dots\dots$$

Problem 37 Find an equation of the form $ax + by + cz = d$ for the plane described below.

The plane contains the point $(2, 0, 5)$ and is parallel to the xz -plane.

Answer:

$$\dots\dots x + \dots\dots y + \dots\dots z = \dots\dots$$

Problem 38 How many planes satisfying the given conditions are there?

Containing points $(1, 2, 3)$ and $(4, 5, 6)$, and having a normal vector $\mathbf{n} = \begin{bmatrix} -9 \\ -9 \\ -9 \end{bmatrix}$.

Multiple Choice:

1. 0

2. 1

3. Infinitely Many

Problem 39 How many planes satisfying the given conditions are there?

Containing $(0, 2, 4)$, $(-1, 5, 3)$, $(4, 9, 1)$.

Multiple Choice:

1. 0
2. 1
3. Infinitely Many

Problem 40 How many planes satisfying the given conditions are there?

Planes containing $(-1, 3, 5)$, $(1, 3, 6)$, $(3, 3, 7)$.

Multiple Choice:

1. 0
2. 1
3. Infinitely Many

Text Source

The text in this section is an adaptation of Section 4.2 of Keith Nicholson's *Linear Algebra with Applications*. (CC-BY-NC-SA)

W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2018, Open Edition, p 233-234.

9 Introduction to Systems of Linear Equations

You are probably familiar with the concept of a *system of linear equations* and with some methods for solving such systems. In this section, we will look at the algebra and geometry of finding and interpreting solutions of systems of linear equations. We will start with two-variable and three-variable systems, then move on to systems involving more variables.

Algebra of Linear Systems

When you were first introduced to systems of equations, you learned to solve for one variable in terms of the other(s), then substitute. Here, we will introduce another method. This alternative method involves adding multiples of one equation to another equation in order to eliminate one of the variables. This method will form the foundation for an algorithm we will develop for solving linear systems and performing other computations related to systems. Exploration ?? illustrates how the second method works.

Exploration 41 *The purpose of this problem is to formalize what you may already know (perhaps under a different name) about elementary row operations as means of solving systems of linear equations. Consider the system*

$$\begin{array}{rcl} 2x & - & y = -4 \\ 3x & + & 2y = 1 \end{array}$$

We will begin by adding twice the first row to the second row, and replacing the second row with the sum.

$$R_2 + 2R_1 \rightarrow R_2$$

$$\begin{array}{rcl} 2x & - & y = -4 \\ 7x & + & 0y = -7 \end{array}$$

Note that this step eliminates y from the second equation.

Next we divide both sides of the second equation by 7.

$$\frac{1}{7}R_2 \rightarrow R_2$$

$$\begin{array}{rcl} 2x & - & y = -4 \\ x & & = -1 \end{array}$$

We now know what x is. Our next goal is to eliminate x from the first equation. To this end, we subtract twice the second row from the first row and replace the first row with the difference.

$$R_1 - 2R_2 \rightarrow R_1$$

$$\begin{array}{rcl} 0x & - & y = -2 \\ x & & = -1 \end{array}$$

Next we multiply both sides of the first equation by -1 .

$$\begin{array}{rcl} -R_1 & \rightarrow & R_1 \\ y & = & 2 \\ x & = & -1 \end{array}$$

Finally, we can switch the order of equations in order to display x in the top row.

$$\begin{array}{rcl} R_1 & \leftrightarrow & R_2 \\ x & = & -1 \\ y & = & 2 \end{array}$$

This solution can be written as an ordered pair $(-1, 2)$.

In Exploration ?? we introduced elementary row operations and the notation associated with them. We now make these definitions formal.

Definition 5 (Elementary Row Operations). The following three operations performed on a linear system are called *elementary row operations*.

1. Switching the order of equations (rows) i and j :

$$R_i \leftrightarrow R_j$$

2. Multiplying both sides of equation (row) i by the same non-zero constant, k , and replacing equation i with the result:

$$kR_i \rightarrow R_i$$

3. Adding k times equation (row) i to equation (row) j , and replacing equation j with the result:

$$R_j + kR_i \rightarrow R_j$$

As we applied elementary row operations to the system in Exploration ??, the system changed, but a quick check will convince you that all six systems have the same solution: $(-1, 2)$. The six systems are said to be *equivalent*.

It turns out that if a system of equations is transformed into another system through a sequence of elementary row operations, the new system will be equivalent to the original system, in other words, both systems will have the same solution(s). We will formalize this statement as Theorem ?? at the end of this section.

Example 12. Solve the system of equations using elementary row operations.

$$\begin{array}{rclclcl} 3x & - & y & + & z & = & 0 \\ 2x & + & y & + & 2z & = & 2 \\ x & + & 4y & - & 2z & = & 11 \end{array}$$

Explanation. It may be daunting to think about how to begin. But keep in mind the desired end-result. What we want is to use elementary row operations to transform the given system into something like this

$$\begin{array}{rcl} x & & = a \\ & y & = b \\ & & z = c \end{array}$$

We will accomplish this by using a convenient variable in one row to “wipe out” this variable from the other two rows. For example, we can use x in the third equation to wipe out $3x$ in the first equation and $2x$ in the second equation. To do this, multiply the third row by -3 and add it to the top row, then multiply the third row by -2 and add it to the second row. We now have:

$$\begin{array}{rcl} \xrightarrow{R_1 - 3R_3} & 0x & - 13y + 7z = -33 \\ \xrightarrow{R_2 - 2R_3} & 0x & - 7y + 6z = -20 \\ & x & + 4y - 2z = 11 \end{array}$$

In the previous step x was a convenient variable to use because the coefficient in front of x was 1. We no longer have a variable with coefficient 1. We could create a coefficient of 1 using division, but that would lead to fractions, making computations cumbersome. Instead, we will subtract twice the second row from the first row. This gives us:

$$\begin{array}{rcl} \xrightarrow{R_1 - 2R_2} & 0x & + y - 5z = 7 \\ & 0x & - 7y + 6z = -20 \\ & x & + 4y - 2z = 11 \end{array}$$

Next we add seven times the first row to the second row, and subtract four times the first row from the third row.

$$\begin{array}{rcl} & 0x & + y - 5z = 7 \\ \xrightarrow{R_2 + 7R_1} & 0x & + 0y - 29z = 29 \\ \xrightarrow{R_3 - 4R_1} & x & + 0y + 18z = -17 \end{array}$$

Now we divide both sides of the second row by -29 .

$$\begin{array}{rcl} & 0x & + y - 5z = 7 \\ \xrightarrow{-\frac{1}{29}R_2} & 0x & + 0y + z = -1 \\ & x & + 0y + 18z = -17 \end{array}$$

Adding 5 times the second row to the first row and subtracting 18 times the second row from the third row gives us

$$\begin{array}{rcl} \xrightarrow{R_1 + 5R_2} & 0x & + y + 0z = 2 \\ & 0x & + 0y + z = -1 \\ \xrightarrow{R_3 - 18R_2} & x & + 0y + 0z = 1 \end{array}$$

Finally, rearranging the rows gives us

$$\begin{array}{rcl} x & + & 0y + 0z = 1 \\ 0x & + & y + 0z = 2 \\ 0x & + & 0y + z = -1 \end{array}$$

$$\begin{array}{rcl} x & & = 1 \\ & y & = 2 \\ & & z = -1 \end{array}$$

Thus the system has a unique solution $(1, 2, -1)$.

At this point you may be wondering whether it will always be possible to take a system of three equations and three unknowns and use elementary row operations to transform it to a system of the form

$$\begin{array}{rcl} x & & = a \\ & y & = b \\ & & z = c \end{array}$$

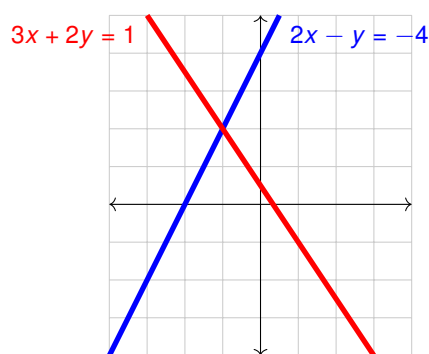
The short answer to this question is NO. The existence of an equivalent system of this form implies that the original system has a unique solution (a, b, c) . However, it is possible for a system to have no solutions or to have infinitely many solutions. We will study these different possibilities from an algebraic perspective in subsequent sections. For now, we will attempt to gain insight into existence and uniqueness of solutions through geometry.

Geometry of Linear Systems in Two Variables

Exploration Problem ?? offers an example of a linear system of two equations and two unknowns (variables) with a unique solution.

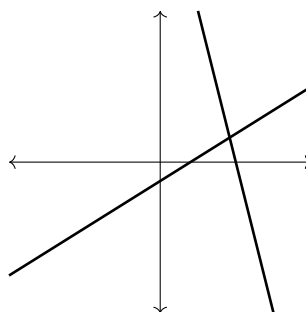
$$\begin{array}{rcl} 2x & - & y = -4 \\ 3x & + & 2y = 1 \end{array}$$

Geometrically, the graph of each equation is a line in $\mathbb{R}^2$. The point $(-1, 2)$ is a solution to both equations, so it must lie on both lines. The graph below shows the two lines intersecting at $(-1, 2)$.

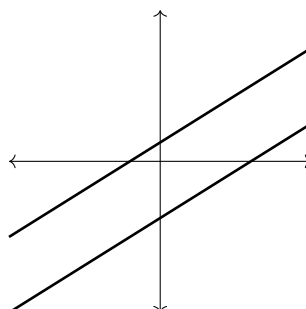


Given a system of two equations with two unknowns, there are three possible geometric outcomes.

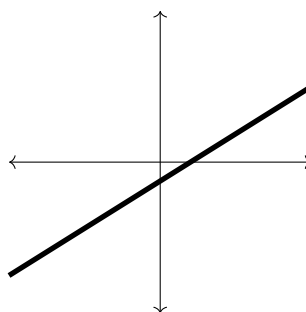
- First, the graphs of the two equations intersect at a point. If this is the case, the system has exactly one solution. We say that the system is *consistent* and has a *unique solution*.



- Second, the two lines may have no points in common. If this is the case, the system has no solutions. We say that the system is *inconsistent*.



- Finally, the two lines may coincide. In this case, there are infinitely many points that satisfy both equations simultaneously. We say that the system is consistent and has infinitely many solutions.



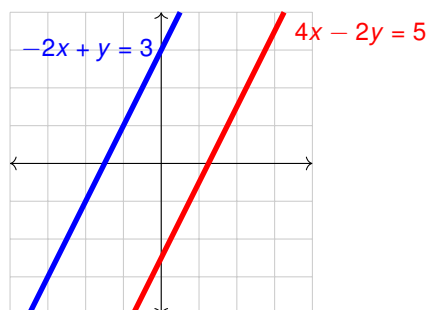
Example 13. Solve the system of equations and interpret your results geometrically.

$$\begin{aligned} -2x + y &= 3 \\ 4x - 2y &= 5 \end{aligned}$$

Explanation. We will use elementary row operations. Adding twice the first equation to the second equation gives us

$$\begin{aligned} -2x + y &= 3 \\ 0x - 0y &= 11 \quad \leftarrow \text{Never true} \end{aligned}$$

This is where we run into a problem: there are no values of x and y that satisfy the second equation. We conclude that the system is inconsistent. Plotting the two lines in the same coordinate plane shows that the two lines are parallel.



Example 14. Solve the system of equations and interpret your results geometrically.

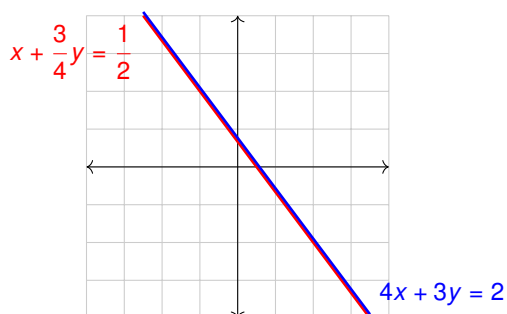
$$\begin{array}{rcl} 4x & + & 3y = 2 \\ x & + & \frac{3}{4}y = \frac{1}{2} \end{array}$$

Explanation. To eliminate x from the second equation, we subtract one quarter of the first equation from the second. This gives us

$$\begin{array}{rcl} 4x & + & 3y = 2 \\ 0x & + & 0y = 0 \quad \leftarrow \text{Always true} \end{array}$$

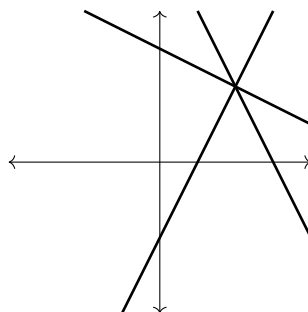
Unlike the situation in Example ??, any combination of x and y satisfies the second equation. So, any ordered pair (x, y) that satisfies the first equation will satisfy the second equation. Thus, the solution set for this system is the same as the set of all solutions of $4x + 3y = 2$.

When we plot the two equations of the original system, we find that the two lines coincide.

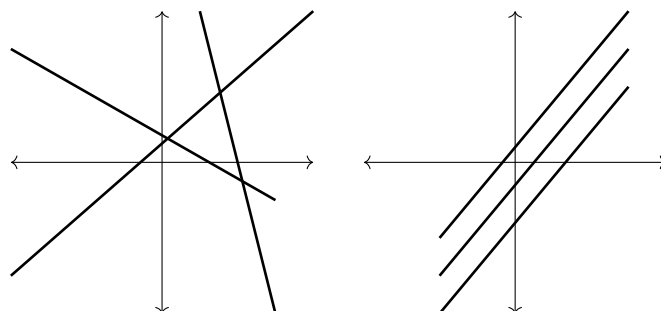


Given a linear system in two variables and more than two equations, we have a variety of geometric possibilities. In terms of the number of solutions, there are three possibilities.

- First, it is possible for the graphs of all equations in the system to intersect at a single point, giving us a unique solution.



- Second, it is possible for the graphs to have no points common to all of them. If this is the case, the system is inconsistent.



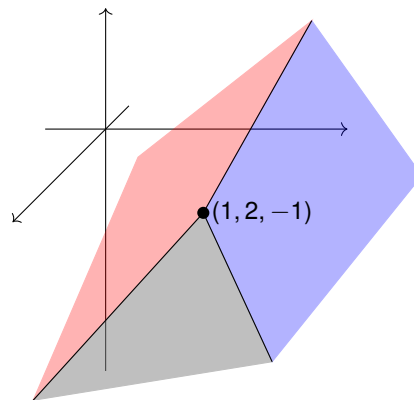
- Finally, it is possible for all of the lines to coincide, giving us infinitely many solutions.

Geometry of Linear Systems in Three Variables

In Example ?? we solved the following linear system of three equations and three unknowns

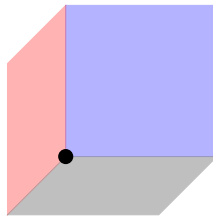
$$\begin{array}{rcrcrcrcrcl} 3x & - & y & + & z & = & 0 \\ 2x & + & y & + & 2z & = & 2 \\ x & + & 4y & - & 2z & = & 11 \end{array}$$

We found that the system has a unique solution $(1, 2, -1)$. The graph of each equation is a plane. The three planes intersect at a single point, as shown in the figure.

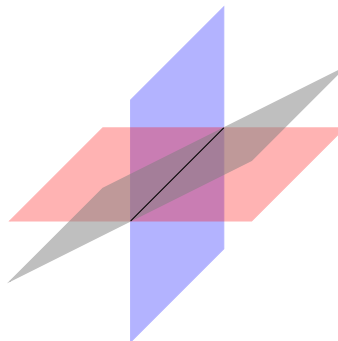


Given a linear system of three equations in three variables, there are three ways in which the system can be consistent.

- First, the three planes could intersect at a single point, giving us a unique solution.



- Second, the three planes can intersect in a line, forming a paddle-wheel shape. In this case, every point along the line of intersection is a solution to the system, giving us infinitely many solutions.



- Finally, the three planes can coincide. If this is the case, there are infinitely many solutions.



There are four ways for a system to be inconsistent. They are depicted below.



General Systems of Linear Equations

Definition 6. A *linear equation* in variables $x_1, \dots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$$

where $a_1, \dots, a_n$ and b are constants.

An n -tuple $(x_1, x_2, \dots, x_n)$ is a *solution* to the equation $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$ provided that it turns the equation into a true statement. The set of all n -tuples that are solutions to a given equation is called the *graph* of the equation. The graph of a linear equation in two variables is a line in $\mathbb{R}^2$. The graph of a linear equation in three variables is a plane in $\mathbb{R}^3$. In $\mathbb{R}^n$, for $n > 3$, we say that the graph of a linear equation is a *hyperplane*. A hyperplane cannot be visualized, but we can still talk about intersections of hyperplanes and their other attributes in algebraic terms.

A linear system of m equations and n unknowns is typically written as follows

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ & & & & \vdots & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \end{array}$$

A *solution to a system* of linear equations in n variables is an n -tuple that satisfies every equation in the system. All solutions to a system of equations, taken together, form a *solution set*.

Definition 7. Two systems of linear equations are said to be equivalent if they have the same solution set.

Recall that to solve systems of equations in this section, we utilized three *elementary row operations*. These operations are:

1. Switching the order of two equations
2. Multiplying both sides of an equation by the same non-zero constant
3. Adding a multiple of one equation to another

Theorem 5. Given a system of linear equations, any of the three elementary row operations performed on the system produces an equivalent system.

Proof Clearly, the order of equations does not affect the solution set, so ?? produces an equivalent system. Next, you learned years ago that multiplying both sides of an equation by a non-zero constant does not change its solution set, which establishes that ?? produces an equivalent system. To see that ?? produces an equivalent system, note that if we add a multiple of an equation to another equation in the system, we are adding the same thing to both sides, which does not change the solution set of that equation, nor of the system. ■

Practice Problems

Problem 42 Give a graphical illustration (draw a picture) for each of the following scenarios for a system of THREE equations and TWO unknowns:

1. The system of three equations is inconsistent, but a combination of any two of the three equations forms a consistent system.
2. The system is consistent and has a unique solution.
3. The system is consistent and has infinitely many solutions.
4. The system is inconsistent and no two equations form a consistent system.

Problem 43 Solve the given system of linear equations algebraically or algebraically demonstrate that a solution does not exist. Interpret your results geometrically.

$$\begin{aligned}x + 3y &= 4 \\x - 2y &= -6\end{aligned}$$

Solution:

$$(\dots, \dots)$$

Problem 44 Solve the given system of linear equations algebraically or algebraically demonstrate that a solution does not exist. Interpret your results geometrically.

$$\begin{aligned}-3x + 2y &= 7 \\6x - 4y &= 5\end{aligned}$$

Problem 45 Solve the given system of linear equations algebraically or algebraically demonstrate that a solution does not exist. Interpret your results geometrically.

$$\begin{aligned}x - 2y + z &= 0 \\3x - 2y + 4z &= 2 \\2x - y + 2z &= 3\end{aligned}$$

Solution:

$$(\dots, \dots, \dots)$$

Problem 46 Consider the following system of equations.

$$\begin{aligned}kx + 8y &= 4 \\2x + ky &= -2\end{aligned}$$

1. Find all possible values of k such that this system has no solutions.

Answer:

$$k = \dots$$

2. Find all possible values of k such that this system has infinitely many solutions.

Answer:

$$k = \dots\dots$$

Problem 47 Why is there a non-zero provision in Part ?? of Definition ?? ? Why is there not a non-zero provision in Part ?? ?

Problem 48 Suppose the following system was obtained from system (A) by adding twice the second row of (A) to the first row.

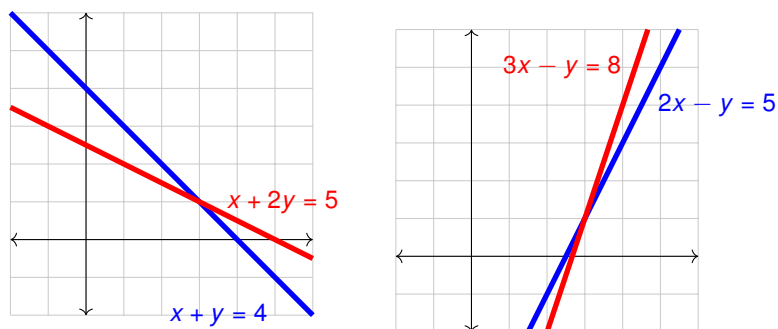
$$\begin{array}{rcrcrcrcl} 8x & + & 3y & = & 11 \\ 3x & + & 2y & = & 5 \end{array}$$

Find system (A).

Answer:

$$\begin{array}{rcrcrcrcl} \dots\dots x & + & \dots\dots y & = & \dots\dots \\ 3x & + & 2y & = & 5 \end{array}$$

Problem 49 The following figures show a geometric depiction of two equivalent systems. (The systems are equivalent because they have the same solution set.) Can the first system be transformed into the second system by elementary row operations? If so, how?



Problem 50 Consider the system of equations

$$\begin{array}{rcrcrcrcl} ax & + & by & = & e \\ cx & + & dy & = & f \end{array}$$

Show that if (x_0, y_0) is a solution to this system, and if we apply elementary row operation ?? to the system, then (x_0, y_0) will be a solution to the new system of equations.

Problem 51 Demonstrate that elementary row operations are reversible by answering the following questions. Be specific about the elementary row operation that you would use.

1. Suppose we obtained system (B) from system (A) by swapping two equations. How would we obtain system (A) from system (B)?

2. Suppose we obtained system (B) from system (A) by multiplying one of the equations of (A) by a non-zero constant k . How would we obtain system (A) from system (B)?
3. Suppose we obtained system (B) from system (A) by adding a multiple of one of the equations of (A) to another. How would we obtain system (A) from system (B)?