
Week 10

June 5, 2026

Contents

1 Finding the Determinant

In this section we will define a function that assigns to each square matrix A a scalar output called the *determinant* of A . We will denote the determinant of A by $\det A$. For a matrix with real number entries, the output of the determinant function will always be a real number.

One important property of the determinant is its connection to matrix inverses. We will find that a matrix A is singular if and only if $\det A = 0$.

Geometrically speaking, the determinant of a matrix of a linear transformation is the factor by which the area (or volume or hypervolume) is scaled by the transformation.

Cofactor Expansion Along the Top Row

To start from the beginning, let us define the determinant of a 1×1 matrix.

Definition 1. Let $A = \begin{bmatrix} a \end{bmatrix}$. Define the *determinant* of A by $\det A = a$.

It is important to note that this definition is consistent with our goal of making a connection between determinants and invertibility. Observe that $A^{-1} = \begin{bmatrix} a^{-1} \end{bmatrix}$ exists if and only if $a \neq 0$.

Now we proceed to 2×2 matrices. According to Formula ??, the inverse of a nonsingular matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Observe that A^{-1} exists if and only if $ad - bc \neq 0$. We will call the number $ad - bc$ the determinant of A .

Definition 2. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The determinant of A is defined by

$$\det A = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \quad (1)$$

Note the distinction between the square bracket notation associated with the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and the vertical bar notation $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ used to denote the determinant in expression (??).

Example 1.

$$\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1)(4) - (2)(3) = -2$$

The easiest way to understand the definition of the determinant for a 3×3 matrix is to start with an example.

Example 2. Find $\det A$ if

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 5 & -1 & 2 \\ 1 & 4 & 1 \end{bmatrix}$$

Explanation. We will use three color-coded copies of the matrix A

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 5 & -1 & 2 \\ 1 & 4 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 3 & -2 & 1 \\ 5 & -1 & 2 \\ 1 & 4 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 3 & -2 & 1 \\ 5 & -1 & 2 \\ 1 & 4 & 1 \end{bmatrix}$$

$$\begin{aligned} \det A &= (3) \begin{vmatrix} -1 & 2 \\ 4 & 1 \end{vmatrix} - (-2) \begin{vmatrix} 5 & 2 \\ 1 & 1 \end{vmatrix} + (1) \begin{vmatrix} 5 & -1 \\ 1 & 4 \end{vmatrix} \\ &= (3)(-1 - 8) - (-2)(5 - 2) + (1)(20 + 1) \\ &= -27 + 6 + 21 \\ &= 0 \end{aligned}$$

We now formalize what we learned in Example ??.

Definition 3. Let

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

The *determinant* of A is given by

$$\det A = |A| = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix} \quad (2)$$

We will now reiterate several important features of this definition and introduce some vocabulary:

- The coefficients a , b and c are the entries of the first row of matrix A . Coefficients in the formula follow an alternating sign pattern: $+a$, $-b$, $+c$. This pattern will persist in the determinant formulas for determinants of larger matrices.
- When using equation (??), we compute determinants of three matrices:

$$\begin{bmatrix} e & f \\ h & i \end{bmatrix}, \quad \begin{bmatrix} d & f \\ g & i \end{bmatrix}, \quad \begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

These matrices are called *minor* matrices. To form each minor matrix, cross out the row and column that the corresponding coefficient is in. For example, the minor matrix corresponding to coefficient b is found by crossing out the row and column that b is in.

$$\begin{bmatrix} \cancel{a} & \cancel{b} & \cancel{c} \\ d & e & f \\ g & h & i \end{bmatrix}$$

- The process of finding the determinant described in Definition ?? is referred to as a *cofactor expansion along the top row*.

Example 3. Find $\det A$ if

$$A = \begin{bmatrix} 4 & 3 & -2 \\ 1 & -5 & 3 \\ -4 & 1 & 1 \end{bmatrix}$$

Explanation.

$$\det A = (4) \begin{vmatrix} \cdots & \cdots \\ \cdots & \cdots \end{vmatrix} - (3) \begin{vmatrix} \cdots & \cdots \\ \cdots & \cdots \end{vmatrix} + (-2) \begin{vmatrix} \cdots & \cdots \\ \cdots & \cdots \end{vmatrix} = \dots$$

We are starting to observe a certain pattern in the process of computing the determinant. This pattern will persist for larger matrices. Let's take a look at a 4×4 matrix.

Example 4. Find $\det A$ if

$$A = \begin{bmatrix} 2 & 3 & -2 & -5 \\ 0 & 1 & -2 & 0 \\ 1 & 3 & 0 & -1 \\ 2 & 0 & 1 & 1 \end{bmatrix}$$

Explanation. As you watch the video below, pay particular attention to the same patterns as you saw in the case of 3×3 matrices: the alternating sign pattern and the process of forming minor matrices.

$$\begin{aligned} \det A &= 2 \begin{vmatrix} 1 & -2 & 0 \\ 3 & 0 & -1 \\ 0 & 1 & 1 \end{vmatrix} - 3 \begin{vmatrix} 0 & -2 & 0 \\ 1 & 0 & -1 \\ 2 & 1 & 1 \end{vmatrix} - 2 \begin{vmatrix} 0 & 1 & 0 \\ 1 & 3 & -1 \\ 2 & 0 & 1 \end{vmatrix} - (-5) \begin{vmatrix} 0 & 1 & -2 \\ 1 & 3 & 0 \\ 2 & 0 & 1 \end{vmatrix} \\ &= 2(7) - 3(6) - 2(-3) + 5(11) \\ &= 57 \end{aligned}$$

Example 5. Find $\det A$ if

$$A = \begin{bmatrix} 4 & -1 & 2 & 1 \\ 3 & 0 & 1 & -2 \\ 2 & 1 & 5 & 1 \\ -2 & 1 & 3 & -1 \end{bmatrix}$$

Explanation. We will use the entries in the top row as coefficients in front of 3×3 determinants. As before, we will use the alternating sign pattern for the coefficients:

$$+(4), -(-1), +(2), -(1)$$

Just like in the case of a 3×3 matrix in Example ??, each of the smaller determinants is obtained by crossing out the row and the column where the coefficient is located.

$$\begin{aligned} \det A &= 4 \begin{vmatrix} 0 & 1 & -2 \\ 1 & 5 & 1 \\ 1 & 3 & -1 \end{vmatrix} + \begin{vmatrix} 3 & 1 & -2 \\ 2 & 5 & 1 \\ -2 & 3 & -1 \end{vmatrix} + 2 \begin{vmatrix} 3 & 0 & -2 \\ 2 & 1 & 1 \\ -2 & 1 & -1 \end{vmatrix} - \begin{vmatrix} 3 & 0 & 1 \\ 2 & 1 & 5 \\ -2 & 1 & 3 \end{vmatrix} \\ &= 4(\dots) + (\dots) + 2(\dots) - (\dots) \\ &= \dots \end{aligned}$$

Cofactor Expansion Along the First Column

We defined the determinant of a matrix in terms of cofactor expansion along the top row. We will now see what happens when we expand along the first *column* instead. We will refer to this process as *cofactor expansion along the first column*. Surprisingly, both expansions yield the same result. To illustrate this, let's revisit Examples ?? and ??.

Example 6. Let

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 5 & -1 & 2 \\ 1 & 4 & 1 \end{bmatrix}$$

In Example ?? we found that $\det A = 0$. Let's try to mimic what we did earlier, but instead of expanding along the first row, we will expand along the first column.

$$\begin{aligned} & (3) \begin{vmatrix} -1 & 2 \\ 4 & 1 \end{vmatrix} - (5) \begin{vmatrix} -2 & 1 \\ 4 & 1 \end{vmatrix} + (1) \begin{vmatrix} -2 & 1 \\ -1 & 2 \end{vmatrix} \\ &= (3)(-1 - 8) - (5)(-2 - 4) + (1)(-4 + 1) \\ &= -27 + 30 - 3 \\ &= 0 \\ &= \det A \end{aligned}$$

Let's go through this process again for a larger matrix.

Exploration 1 Let

$$A = \begin{bmatrix} 4 & -1 & 2 & 1 \\ 3 & 0 & 1 & -2 \\ 2 & 1 & 5 & 1 \\ -2 & 1 & 3 & -1 \end{bmatrix}$$

In Example ?? we found that $\det A = -58$. We will now try to expand along the first column.

When computing determinants of the four 3×3 matrices below, try different approaches. You might want to expand along the first row for some of them, and along the first column for others. Looking for where zeros are located will help you decide what to try.

$$\begin{aligned} & 4 \begin{vmatrix} 0 & 1 & -2 \\ 1 & 5 & 1 \\ 1 & 3 & -1 \end{vmatrix} - 3 \begin{vmatrix} -1 & 2 & 1 \\ 1 & 5 & 1 \\ 1 & 3 & -1 \end{vmatrix} + 2 \begin{vmatrix} -1 & 2 & 1 \\ 0 & 1 & -2 \\ 1 & 3 & -1 \end{vmatrix} - (-2) \begin{vmatrix} -1 & 2 & 1 \\ 0 & 1 & -2 \\ 1 & 5 & 1 \end{vmatrix} \\ &= 4(\dots) - 3(\dots) + 2(\dots) + 2(\dots) \\ &= \dots \\ &= \det A \end{aligned}$$

In Example ?? and Exploration ?? we were careful not to claim at the outset that we were finding the determinant of the matrix by cofactor expansion along the first column; we merely observed that the resulting value was equal to the determinant. It is possible to prove that both expansions produce the same result (Theorem ??). Therefore the determinant of a matrix can be defined in terms of cofactor expansion along the first row or column.

Cofactor Expansion Along Any Row or Column

We originally defined the determinant of a matrix via expansion along the top row of the matrix. We later observed that expansion along the first column produces the same result. It turns out that the value of the determinant can be computed by expanding along any row or column. This result is known as the *Laplace Expansion Theorem* (??).

When expanding along an arbitrary row or column, we will continue to follow the two patterns we observed earlier.

- The alternating sign pattern for coefficients will follow the checkerboard pattern below.

$$\begin{bmatrix} + & - & + & - & + & \dots \\ - & + & - & + & - & \dots \\ + & - & + & - & + & \dots \\ - & + & - & + & - & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

- Minor matrices will be formed by eliminating the row and column that the corresponding coefficient is in.

To illustrate this, let's take another look at matrix A from Example ??.

Example 7. Let

$$A = \begin{bmatrix} 4 & -1 & 2 & 1 \\ 3 & 0 & 1 & -2 \\ 2 & 1 & 5 & 1 \\ -2 & 1 & 3 & -1 \end{bmatrix}$$

Follow the rules described above to expand along the second row. Compare your result with the determinant you found in Example ??.

Explanation. The second row has the advantage over other rows in that it contains a zero. This will simplify our calculations. Following the checkerboard sign pattern along the second row we get

$$\begin{aligned} \det A &= -3 \begin{vmatrix} -1 & 2 & 1 \\ 1 & 5 & 1 \\ 1 & 3 & -1 \end{vmatrix} - \begin{vmatrix} 4 & -1 & 1 \\ 2 & 1 & 1 \\ -2 & 1 & -1 \end{vmatrix} + (-2) \begin{vmatrix} 4 & -1 & 2 \\ 2 & 1 & 5 \\ -2 & 1 & 3 \end{vmatrix} \\ &= -3(\dots\dots) - (\dots\dots) - 2(\dots\dots) \\ &= \dots\dots \end{aligned}$$

This answer is the same as the answer we got using expansion along the first row in Example ??.

It is clear that having zeros as entries in the matrix reduces the number of computations necessary to find the determinant. The following example demonstrates how to use zeros to our advantage.

Example 8. Find $\det A$ if

$$A = \begin{bmatrix} 4 & 0 & 0 & 0 & 2 \\ 0 & -1 & 1 & 0 & 0 \\ 2 & 0 & 0 & -5 & 3 \\ 0 & 1 & 4 & 0 & -1 \\ 1 & 1 & 5 & 0 & 0 \end{bmatrix}$$

Explanation. The fourth column contains the most zeros, so we will expand along that column. The (3, 4)-entry is the only non-zero entry in the fourth column. Following the checkerboard pattern, we see that the sign in front of -5 is a minus.

$$\det A = -(-5) \begin{vmatrix} 4 & 0 & 0 & 2 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 4 & -1 \\ 1 & 1 & 5 & 0 \end{vmatrix}$$

Next we will expand the minor matrix along the top row.

$$\det A = 5 \left(4 \begin{vmatrix} -1 & 1 & 0 \\ 1 & 4 & -1 \\ 1 & 5 & 0 \end{vmatrix} - 2 \begin{vmatrix} 0 & -1 & 1 \\ 0 & 1 & 4 \\ 1 & 1 & 5 \end{vmatrix} \right)$$

Try the next step on your own. We suggest that you expand the first matrix along the last column and expand the second matrix along the first column.

$$\det(A) = 5(4(\dots) - 2(\dots)) = \dots$$

A Note on Equivalency

We initially introduced the determinant of a matrix via cofactor expansion along the top row. We later observed that cofactor expansion along any row or column produces the same result. We have to be careful, however, not to use a few examples as “proof” that all cofactor expansions are equivalent. Such claims need to be carefully supported with general proofs. Unfortunately, in this case, the proofs are tedious and conceptually unenlightening. An interested reader can find them in [Tedious Proofs Concerning Determinants](#).

Determinants of Some Special Matrices

We know that we can find the determinant of a matrix by cofactor expansion along the top row or the first column. (See Theorem ?? of [Tedious Proofs Concerning Determinants](#) for proof.) This property gives rise to a useful result.

Theorem 1. Let A be a square matrix, then

$$\det A^T = \det A$$

Proof See Practice Problem ??.

■

As we observed earlier, having zeros in a matrix makes it easier for us to compute its determinant. Recall that a square matrix is *upper-triangular* if all of the entries below the main diagonal are zero. Similarly, a square matrix is called *lower-triangular* if all of the entries above the main diagonal are zero. Together, upper and lower triangular matrices are categorized as *triangular* matrices.

Theorem 2. If A is a triangular matrix, then $\det A$ is equal to the product of its diagonal entries.

Proof We proceed by induction on n , where A is an $n \times n$ matrix. It is easy to see that this result holds for $n = 1, 2$. Suppose that the result holds for $(n - 1) \times (n - 1)$ triangular matrices. We need to show that it holds for $n \times n$ triangular matrices.

Suppose $A = [a_{ij}]$ is a triangular matrix. Then, with the exception of a_{11} , the entries in the first row (or column) of A are guaranteed to be zeros. We will take advantage of these zeros and expand along the first row (or column) of A . As we do so, we obtain a single product of a_{11} and the determinant of a minor matrix obtained by crossing out the first row and column of A . But this minor $(n-1) \times (n-1)$ matrix is also a triangular matrix with diagonal entries $a_{22}, a_{33}, \dots, a_{nn}$. By induction hypothesis, its determinant is equal to the product of its diagonal entries, $a_{22} \cdot a_{33} \cdot \dots \cdot a_{nn}$. Therefore

$$\det A = a_{11}(a_{22} \cdot a_{33} \cdot \dots \cdot a_{nn})$$

This completes the proof. ■

As an immediate consequence of this theorem, we have the following result.

Corollary 1. Let I be the identity matrix, then

$$\det I = 1$$

We first introduced block matrices in [Block Matrix Multiplication](#). Matrices of the form $\begin{bmatrix} A & C \\ O & B \end{bmatrix}$ and $\begin{bmatrix} A & O \\ D & B \end{bmatrix}$, where A, B are square matrices and O is the zero matrix, are said to be *block triangular*. The following theorem makes it easy to compute determinants of such matrices.

Theorem 3. Consider block triangular matrices $\begin{bmatrix} A & C \\ O & B \end{bmatrix}$ and $\begin{bmatrix} A & O \\ D & B \end{bmatrix}$, where A and B are square matrices. Then

$$\det \begin{bmatrix} A & C \\ O & B \end{bmatrix} = \det A \det B \quad \text{and} \quad \det \begin{bmatrix} A & O \\ D & B \end{bmatrix} = \det A \det B$$

Proof Write $T = \begin{bmatrix} A & C \\ O & B \end{bmatrix}$ and proceed by induction on k , where A is $k \times k$. If $k = 1$, then the result follows from cofactor expansion along the first column. In general, let $S_i(T)$ denote the matrix obtained from T by deleting row i and column 1. Then the cofactor expansion along the first column is

$$\det T = a_{11} \det S_1(T) - a_{21} \det S_2(T) + \dots + (-1)^{k+1} a_{k1} \det S_k(T)$$

where $a_{11}, a_{21}, \dots, a_{k1}$ are the entries in the first column of A . Observe that

$$S_i(T) = \begin{bmatrix} S_i(A) & C_i \\ O & B \end{bmatrix}$$

where $i = 1, 2, \dots, k$, $S_i(A)$ denotes matrix A with column 1 and row i deleted, and C_i denotes matrix C with row i deleted.

Since $S_i(A)$ is a $(k-1) \times (k-1)$ matrix, by the induction hypothesis,

$$\det S_i(T) = \det S_i(A) \cdot \det B$$

This gives us

$$\begin{aligned} \det T &= a_{11} \det S_1(T) - a_{21} \det S_2(T) + \dots + (-1)^{k+1} a_{k1} \det S_k(T) \\ &= a_{11} \det S_1(A) \cdot \det B - a_{21} \det S_2(A) \cdot \det B + \dots \\ &\quad \dots + (-1)^{k+1} a_{k1} \det S_k(A) \cdot \det B \\ &= (a_{11} \det S_1(A) - a_{21} \det S_2(A) + \dots + (-1)^{k+1} a_{k1} \det S_k(A)) \det B \\ &= \det A \det B \end{aligned}$$

The lower triangular case is similar. ■

Example 9. Find $\det A$ if

$$A = \begin{bmatrix} 2 & 1 & 3 & 3 \\ 1 & -1 & -2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 4 & 1 \end{bmatrix}$$

Explanation.

$$\begin{vmatrix} 2 & 1 & 3 & 3 \\ 1 & -1 & -2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 4 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} \begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix} = \dots\dots$$

Practice Problems

Problem 2 Find the determinant.

$$A = \begin{bmatrix} 4 & -2 \\ 3 & 7 \end{bmatrix}$$

Answer:

$$\det A = \dots\dots$$

Problem 3 Find the determinant.

$$B = \begin{bmatrix} 5 & -1 & 0 \\ 0 & 3 & -2 \\ 1 & -1 & 2 \end{bmatrix}$$

Answer:

$$\det(B) = \dots\dots$$

Problem 4 Find the determinant.

$$C = \begin{bmatrix} 1 & -2 & 0 & 0 & 0 \\ 0 & -4 & 1 & 1 & 0 \\ 3 & 0 & -1 & 0 & 1 \\ 0 & 0 & 4 & 1 & 0 \\ -1 & -2 & 0 & 0 & 0 \end{bmatrix}$$

Answer:

$$\det(C) = \dots\dots$$

Problem 5 Prove that for a square matrix A , $\det A^T = \det A$. (Theorem ??)

Problem 6 Let

$$A = \begin{bmatrix} -2 & 1 & 3 \\ 0 & 4 & 0 \\ -1 & 1 & 2 \end{bmatrix}$$

1.

$$\det A = \dots$$

2. Let A_1 be a matrix obtained from A by switching the first and the second row of A .

$$\det A_1 = \dots$$

What do you observe?

3. Let A_2 be a matrix obtained from A by multiplying the bottom row by -3 .

$$\det A_2 = \dots$$

What do you observe?

4. Let A_3 be a matrix obtained from A by multiplying A by 2.

$$\det A_3 = \dots$$

What do you observe? Relate your observations to what you found in the previous part.

5. Let A_4 be a matrix obtained from A by adding twice the third row to the first.

$$\det A_4 = \dots$$

What do you observe?

Problem 7 Is it true that $\det(A + B) = \det A + \det B$?

Text Source

The text in this section partially comes from Section 3.1 of Keith Nicholson's *Linear Algebra with Applications* (CC-BY-NC-SA).

2 Cross Product and its Properties

In [Dot Product and its Properties](#) we introduced the *dot product*, one of two important products for vectors. We will now introduce the second type of product, called the *cross product*. There are several important distinctions to keep in mind. First, the dot product is defined for two vectors of $\mathbb{R}^n$, for any natural number n ; the cross-product will only be defined for vectors of $\mathbb{R}^3$. Second, the dot product is a scalar; the cross product of two vectors will be a vector. Finally, we will find that unlike the dot product, the cross product is not commutative.

The cross product has many applications in physics and engineering. It also has important geometric properties which will be addressed in this section and in [Determinants as Areas and Volumes](#).

Preliminaries

In order to define the cross product in a convenient way we need to define 2×2 and 3×3 determinants. If you know how to find such determinants you may skip this section and proceed directly to the definition.

Definition 4 (2×2 Determinant). A 2×2 *determinant* is a number associated with a 2×2 matrix

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example 10.

$$\begin{vmatrix} 2 & 4 \\ -5 & 3 \end{vmatrix} = (2)(3) - (4)(-5) = 6 + 20 = 26$$

Definition 5 (3×3 Determinant). A 3×3 *determinant* is a number associated with a 3×3 matrix

$$\det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Example 11.

$$\begin{aligned} \begin{vmatrix} 4 & 2 & 1 \\ 5 & -3 & 0 \\ -2 & 3 & 2 \end{vmatrix} &= 4 \begin{vmatrix} -3 & 0 \\ 3 & 2 \end{vmatrix} - 2 \begin{vmatrix} 5 & 0 \\ -2 & 2 \end{vmatrix} + 1 \begin{vmatrix} 5 & -3 \\ -2 & 3 \end{vmatrix} \\ &= 4((-3)(2) - (0)(3)) - 2((5)(2) - (0)(-2)) + ((5)(3) - (-3)(-2)) \\ &= (4)(-6) - (2)(10) + (15 - 6) \\ &= -35 \end{aligned}$$

For more help on 3×3 determinants, see Example ?? and the video located there.

Definition of the Cross Product

Definition 6. Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ be vectors in $\mathbb{R}^3$. The *cross product* of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} \times \mathbf{v}$, is given by

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= (u_2 v_3 - u_3 v_2)\mathbf{i} - (u_1 v_3 - u_3 v_1)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k} \\ &= (u_2 v_3 - u_3 v_2) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - (u_1 v_3 - u_3 v_1) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (u_1 v_2 - u_2 v_1) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ -u_1 v_3 + u_3 v_1 \\ u_1 v_2 - u_2 v_1 \end{bmatrix} \end{aligned}$$

This formula is much easier to remember when stated symbolically in terms of determinants.

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

Example 12. Find the cross product of $\mathbf{u} = \begin{bmatrix} 3 \\ -10 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ 4 \\ 7 \end{bmatrix}$.

Explanation.

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -10 & 2 \\ -2 & 4 & 7 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -10 & 2 \\ 4 & 7 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 3 & 2 \\ -2 & 7 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 3 & -10 \\ -2 & 4 \end{vmatrix} \\ &= \mathbf{i}((-10)(7) - (2)(4)) - \mathbf{j}((3)(7) - (2)(-2)) + \mathbf{k}((3)(4) - (-10)(-2)) \\ &= -78\mathbf{i} - 25\mathbf{j} - 8\mathbf{k} = \begin{bmatrix} -78 \\ -25 \\ -8 \end{bmatrix} \end{aligned}$$

Properties of the Cross Product

Exploration 8 What would happen if we took the cross product of the vectors in Example ?? but reversed the order?

Let $\mathbf{v} = \begin{bmatrix} -2 \\ 4 \\ 7 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 3 \\ -10 \\ 2 \end{bmatrix}$. Recall that $\mathbf{u} \times \mathbf{v} = \begin{bmatrix} -78 \\ -25 \\ -8 \end{bmatrix}$. We need to compute $\mathbf{v} \times \mathbf{u}$. If you have already studied the effect that switching two rows of a matrix has on its determinant, you should be able to guess the outcome of the

upcoming computation.

$$\begin{aligned}
 \mathbf{v} \times \mathbf{u} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 4 & 7 \\ 3 & -10 & 2 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 4 & 7 \\ -10 & 2 \end{vmatrix} - \mathbf{j} \begin{vmatrix} -2 & 7 \\ 3 & 2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -2 & 4 \\ 3 & -10 \end{vmatrix} \\
 &= \mathbf{i}((4)(2) - (7)(-10)) - \mathbf{j}((-2)(2) - (7)(3)) + \mathbf{k}((-2)(-10) - (4)(3)) \\
 &= 78\mathbf{i} + 25\mathbf{j} + 8\mathbf{k} = \begin{bmatrix} 78 \\ 25 \\ 8 \end{bmatrix} = -(\mathbf{u} \times \mathbf{v})
 \end{aligned}$$

This computation shows that the cross product is an operation that is not commutative. It also suggests that switching the order of the vectors changes the sign of the result.

Warning 1. The Cross Product is not a commutative operation.

Theorem 4. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^3$, then

$$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$$

Proof The proof is left to the reader. (See Practice Problem ??) ■

The next theorem lists two additional properties of the cross product. Proofs of these properties are routine and are left to the reader.

Theorem 5. Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^3$, and k be a scalar, then

1. Scalar Associativity

$$(k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v}) = k(\mathbf{u} \times \mathbf{v})$$

2. Distributivity

$$(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$$

$$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$$

The cross product has several important geometric properties. The following problems give us a glimpse of these properties.

Exploration 9 Compute the following products:

$$\mathbf{i} \times \mathbf{j} = \begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix}, \quad \mathbf{j} \times \mathbf{k} = \begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix}$$

For the two vectors in each product, sketch the vectors together with the product vector. What do you observe about the relationship between the cross product and the plane determined by the two vectors in the product?

Orthogonality Property

Exploration 10 In this problem we will return to vectors of $\mathbf{v} = \begin{bmatrix} -2 \\ 4 \\ 7 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 3 \\ -10 \\ 2 \end{bmatrix}$ of Example ?? and Exploration Problem ??. We know that

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} -78 \\ -25 \\ -8 \end{bmatrix} \quad \text{and} \quad \mathbf{v} \times \mathbf{u} = \begin{bmatrix} 78 \\ 25 \\ 8 \end{bmatrix}$$

We will now compute the dot product of $\mathbf{v} \times \mathbf{u}$ with each of the original vectors $\mathbf{u}$ and $\mathbf{v}$.

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{u}) = \begin{bmatrix} 3 \\ -10 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 78 \\ 25 \\ 8 \end{bmatrix} = 0$$

$$\mathbf{v} \cdot (\mathbf{v} \times \mathbf{u}) = \begin{bmatrix} -2 \\ 4 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} 78 \\ 25 \\ 8 \end{bmatrix} = 0$$

It is also easy to verify that $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ and $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$. Recall that the dot product of two vectors is 0 if and only if the two vectors are orthogonal. (Definition ??) We conclude that, at least in this case, the cross product of two vectors is orthogonal to each of the vectors.

It turns out that the orthogonality property illustrated by Exploration Problems ?? and ?? holds in general. We state it as a theorem.

Theorem 6. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors of $\mathbb{R}^3$, then $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

Proof This proof can be done by direct computation and is left to the reader. (See Practice Problem ??) ■

Cross Product and the Angle between Vectors

Recall that the dot product of $\mathbf{u}$ and $\mathbf{v}$ is related to the angle θ between $\mathbf{u}$ and $\mathbf{v}$ by the following formula

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta \quad (3)$$

We will derive an analogous result for the cross product. To do so, we will need the following Lemma.

Lemma 1. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^3$. Then

$$\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$$

Proof The proof is left to the reader. ■

The following theorem establishes a relationship between the magnitude of the cross product, the magnitudes of the two vectors involved in the cross product and the angle between the two vectors. It is important to note that the identity in this theorem involves the magnitude of the cross product, not the cross product itself.

Theorem 7. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^3$. Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$ such that $0 \leq \theta \leq \pi$. Then

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta$$

Proof By Lemma ?? and (??) we have

$$\begin{aligned} \|\mathbf{u} \times \mathbf{v}\|^2 &= \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2 \\ &= \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\|\mathbf{u}\| \|\mathbf{v}\| \cos \theta)^2 \\ &= \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \cos^2 \theta \\ &= \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 (1 - \cos^2 \theta) \\ &= \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \sin^2 \theta \end{aligned}$$

Observe that all magnitudes are non-negative. Also, $\sin \theta \geq 0$ because $0 \leq \theta \leq \pi$. Taking the square root of both sides give us

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta$$

■

Practice Problems

Problem 11 Find the cross product $\mathbf{u} \times \mathbf{v}$, and verify that $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$.

Answer:

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix}$$

Problem 12 Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^3$, prove that

$$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$$

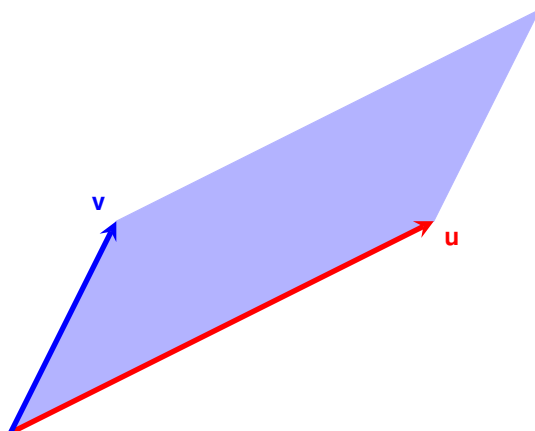
Problem 13 Let $\mathbf{u}$ and $\mathbf{v}$ be vectors of $\mathbb{R}^3$. Prove that $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. (Theorem ??)

Problem 14 Prove that the cross product of any vector with itself is the zero vector.

3 Determinants, Areas, and Volumes

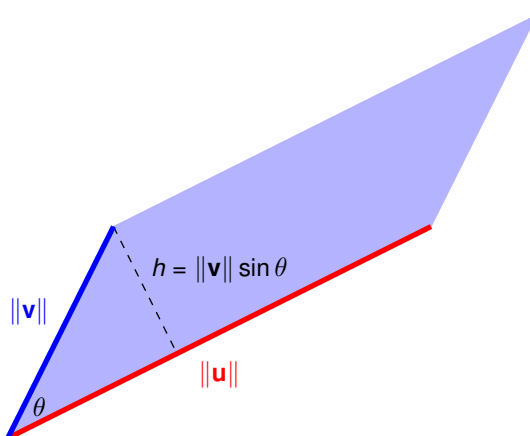
2×2 Determinant and the Area of a Parallelogram

Consider the parallelogram determined by vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$.



Recall that the area of a parallelogram is given by the product of the length of the base and the height. As shown in the diagram below, the length of the base is the magnitude of $\mathbf{u}$. The height, h , can be found using trigonometry

$$h = \|\mathbf{v}\| \sin \theta$$



Using the area of a parallelogram formula together with Theorem ?? we get

$$\text{Area} = (\text{base})(\text{height}) = \|\mathbf{u}\| h = \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta = \|\mathbf{u} \times \mathbf{v}\|$$

We have established the following formula.

Formula 1. The area of a parallelogram determined by vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$ is given by

$$\text{Area} = \|\mathbf{u} \times \mathbf{v}\|$$

Example 13. Use Formula ?? to find the area of a parallelogram determined by vectors

$$\mathbf{u} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix}$$

Explanation. We can start by visualizing the parallelogram in GeoGebra. RIGHT-CLICK and DRAG to rotate the image below. The area of the parallelogram, rounded to two decimal places, is displayed inside the parallelogram.

Access GeoGebra interactives through the online version of this text at

<https://ximera.osu.edu/linearalgebradv3>.

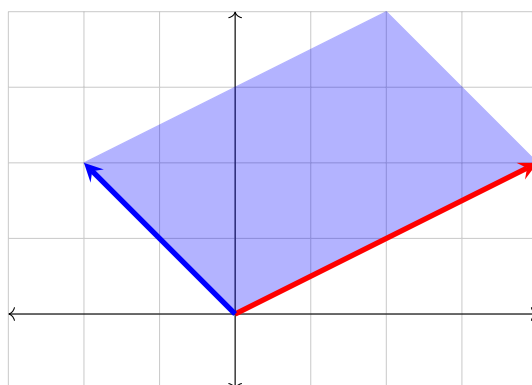
To find the exact area we compute

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \times \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 3 \\ 1 & 6 & 2 \end{vmatrix} = \begin{bmatrix} \dots\dots \\ \dots\dots \\ \dots\dots \end{bmatrix}$$

$$\text{Area} = \|\mathbf{u} \times \mathbf{v}\| = \sqrt{\dots\dots}$$

Formula ?? can be easily adapted to parallelograms determined by vectors in $\mathbb{R}^2$, as illustrated by the following example.

Example 14. Find the area of the parallelogram in the diagram.



Explanation. The vectors that determine the parallelogram are

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

The problem we run into is that these vectors are in $\mathbb{R}^2$, whereas the cross product is defined only for vectors in $\mathbb{R}^3$. We will get around this difficulty by “padding” our vectors with zeros on the bottom. In other words, we will consider

them as vectors sitting in the xy -coordinate plane in $\mathbb{R}^3$. This allows us to compute the cross product

$$\begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} \times \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 2 & 0 \\ -2 & 2 & 0 \end{vmatrix} = \mathbf{k}((4)(2) - (2)(-2)) = 12\mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}$$

The area of the parallelogram is then given by

$$\text{Area} = \left\| \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix} \right\| = 12$$

Example ?? illustrates an important phenomenon. Observe that the zeros in the last column of the determinant ensure that the $\mathbf{i}$ and $\mathbf{j}$ components of the cross product are zero, while the last component is the determinant of the 2×2 matrix whose rows (or columns) are the two vectors that determine the parallelogram in $\mathbb{R}^2$. In general, if the parallelogram is determined by vectors

$$\begin{bmatrix} a \\ b \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c \\ d \end{bmatrix}$$

then the area of the parallelogram can be computed as follows:

$$\begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \times \begin{bmatrix} c \\ d \\ 0 \end{bmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & 0 \\ c & d & 0 \end{vmatrix} = \mathbf{k} \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \mathbf{k}((a)(d) - (b)(c)) = \begin{bmatrix} 0 \\ 0 \\ ad - bc \end{bmatrix}$$

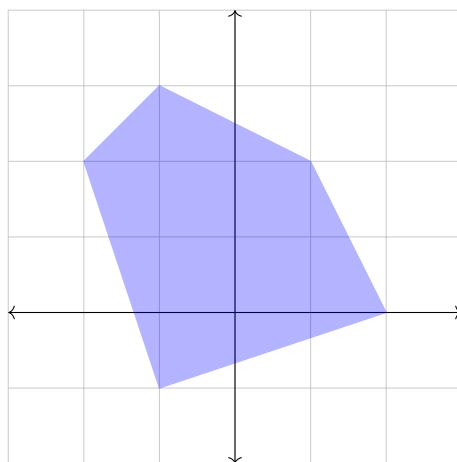
$$\text{Area} = \left\| \begin{bmatrix} 0 \\ 0 \\ ad - bc \end{bmatrix} \right\| = |ad - bc| = \left| \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right| = \left| \det \begin{bmatrix} a & c \\ b & d \end{bmatrix} \right|$$

So the area of the parallelogram turns out to be the absolute value of the determinant of the matrix whose rows (or columns) are the two vectors that determine the parallelogram. The following formula summarizes our discussion.

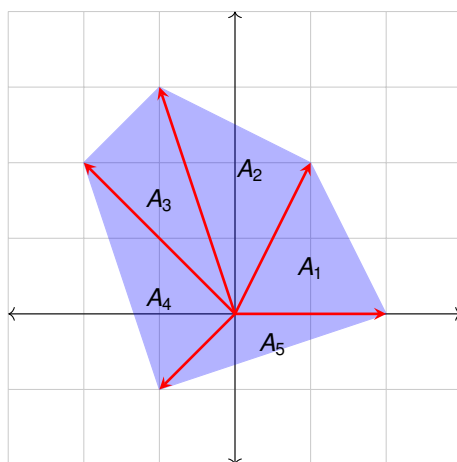
Formula 2. Let $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} c \\ d \end{bmatrix}$ be vectors of $\mathbb{R}^2$. The area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$ is given by

$$\text{Area} = \left| \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right| = \left| \det \begin{bmatrix} a & c \\ b & d \end{bmatrix} \right|$$

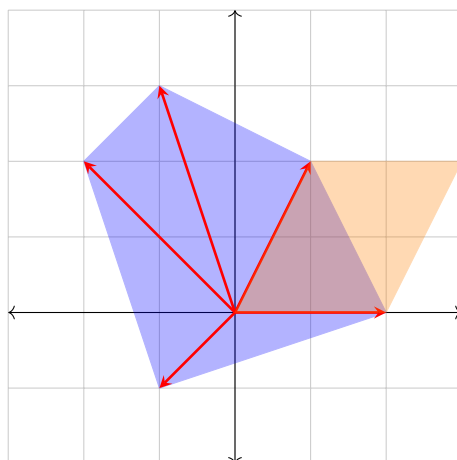
Example 15. Use Formula ?? to find the area of the polygon shown below.



Explanation. We will start by splitting this region into triangles.



We can find the total area of the polygon by finding the area of each triangle. The area of each triangle is half of the area of the corresponding parallelogram. For instance, A_1 is half of the area of the parallelogram depicted below.



We compute

$$A_1 = \frac{1}{2} \left| \det \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \right| = \dots$$

$$A_2 = \frac{1}{2} \left| \det \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \right| = \dots$$

$$A_3 = \frac{1}{2} \left| \det \begin{bmatrix} -1 & -2 \\ 3 & 2 \end{bmatrix} \right| = \dots$$

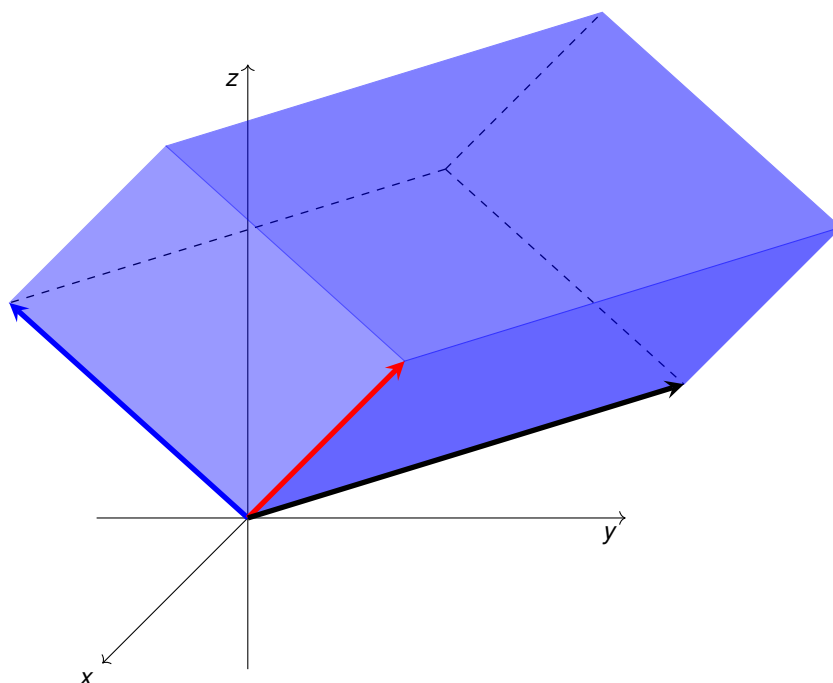
$$A_4 = \frac{1}{2} \left| \det \begin{bmatrix} -2 & -1 \\ 2 & -1 \end{bmatrix} \right| = \dots$$

$$A_5 = \frac{1}{2} \left| \det \begin{bmatrix} -1 & 2 \\ -1 & 0 \end{bmatrix} \right| = \dots$$

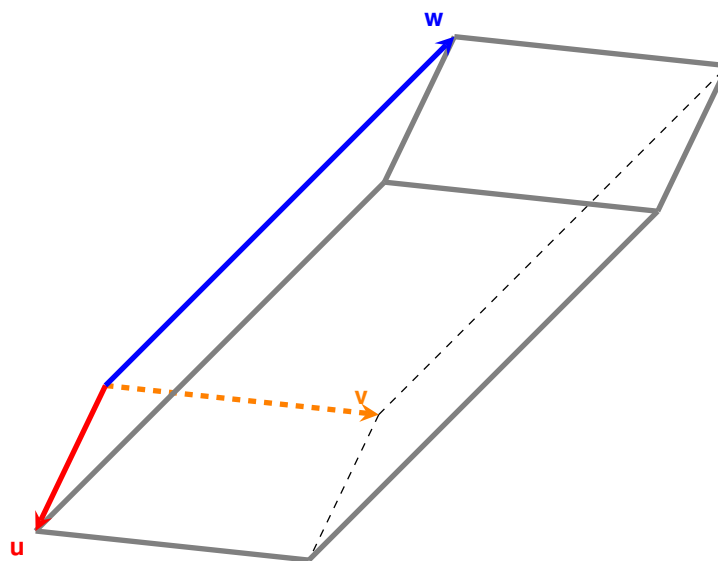
The total area of the polygon is ...

3×3 Determinant and the Volume of a Parallelepiped

Our next goal is to find the volume of a three-dimensional figure called a *parallelepiped*. A parallelepiped is a six-faced figure whose opposite faces are congruent parallelograms located in parallel planes. A parallelepiped is a three-dimensional counterpart of a parallelogram, and is determined by three non-coplanar vectors in $\mathbb{R}^3$. The figure below shows a parallelepiped determined by three vectors.



Consider a parallelepiped determined by vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, as shown below.



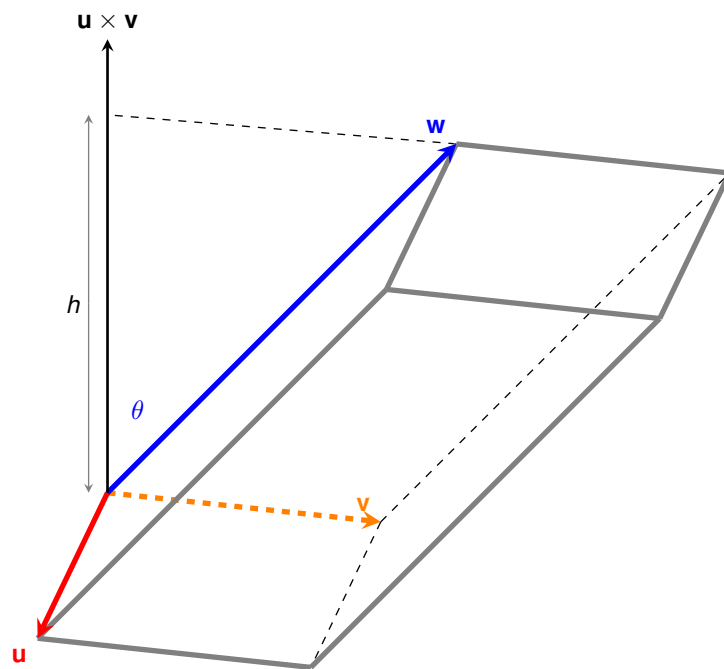
The volume of a parallelepiped is given by

$$\text{Volume} = (\text{area of base})(\text{height})$$

We will consider the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$ to be the base of the parallelepiped. Thus, the area of the

base is given by

$$\text{Area of Base} = \|\mathbf{u} \times \mathbf{v}\|$$



The height of the parallelepiped is measured along a line perpendicular to the base. By Theorem ??, $\mathbf{u} \times \mathbf{v}$ lies on such a line. Let θ be the angle between $\mathbf{w}$ and $\mathbf{u} \times \mathbf{v}$, $0 \leq \theta \leq \pi$. Then the height, h , of the parallelepiped is given by

$$h = \|\mathbf{w}\| |\cos \theta|$$

It may be difficult to visualize this in two dimensions. Below is a replica of the above diagram in GeoGebra. RIGHT-CLICK and DRAG to rotate the image.

Access GeoGebra interactives through the online version of this text at

<https://ximera.osu.edu/linearalgebradv3>.

This gives us the following formula for the volume of the parallelepiped

$$\text{Volume} = \|\mathbf{u} \times \mathbf{v}\| \|\mathbf{w}\| |\cos \theta| = |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$$

We have established the following formula.

Formula 3. The volume of a parallelepiped determined by vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$ is given by

$$\text{Volume} = |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$$

Our next goal is to show that this expression for the volume is equal to the determinant of a 3×3 matrix whose rows are the vectors that determine the parallelepiped.

Let

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

then

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \quad (4)$$

The expression in (4) is sometimes referred to as the *box product* or the *scalar triple product*.

Recall that $\det(A) = \det(A^T)$ (Theorem ??). Therefore, the three vectors that determine the parallelepiped can be used to form rows or columns of the determinant on the right side of (4). This gives us the following formula.

Formula 4. Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ be vectors in $\mathbb{R}^3$. Then the volume of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is given by

$$\text{Volume} = \left| \det \begin{bmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} \right| = \left| \det \begin{bmatrix} w_1 & u_1 & v_1 \\ w_2 & u_2 & v_2 \\ w_3 & u_3 & v_3 \end{bmatrix} \right|$$

Determinants and Linear Transformations

We will now turn our attention to the determinant of a matrix of a linear transformation.

Exploration 15 The following GeoGebra interactive shows a polygon P located in the domain of a linear transformation T induced by the matrix M . The right-hand side shows the image of P under T . The number inside each polygon indicates its area.

Access GeoGebra interactives through the online version of this text at

<https://ximera.osu.edu/linearalgebradv3>.

Question 15.1 Let $M = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$. Find the determinant of M .

$\det M = \dots$

Drag the vertices of P to change the polygon. Make a note of how the area of P and the area of the image change. How are the areas related to each other?

$\text{Area}(T(P)) = \dots \text{Area}(P)$

Question 15.2 Change the matrix M to a matrix whose determinant is 1. Compare the areas of P and $T(P)$. Try matrices whose determinant is 0 or negative. What do you observe about the areas?

Formulate a conjecture about the relationship between the area of the polygon and the area of its image under a linear transformation.

We will not prove your conjecture in Exploration ?? for arbitrary figures as it is beyond the scope of this text. However, we can tackle the problem of how linear transformations affect areas of parallelograms. This is the topic of our next example.

Example 16. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation induced by matrix M . Suppose $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^2$. Let P be a parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$. Show that

$$\text{Area}(T(P)) = |\det M| \text{Area}(P)$$

Explanation. Let $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} c \\ d \end{bmatrix}$, and let $M = \begin{bmatrix} m & n \\ p & q \end{bmatrix}$. By Formula ??, $\text{Area}(P) = ad - bc$. Applying M to $\mathbf{u}$ and $\mathbf{v}$ we get

$$T(\mathbf{u}) = \begin{bmatrix} am + bn \\ ap + bq \end{bmatrix} \quad \text{and} \quad T(\mathbf{v}) = \begin{bmatrix} cm + dn \\ cp + dq \end{bmatrix}$$

Using Formula ??, we compute

$$\begin{aligned} \text{Area}(T(P)) &= |(am + bn)(cp + dq) - (ap + bq)(cm + dn)| \\ &= |(mq - np)(ad - bc)| \\ &= |\det M| \text{Area}(P) \end{aligned}$$

Practice Problems

Problem 16 Let S be a square determined by $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$. Let P be a parallelogram determined by vectors $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

1. Sketch both figures in the same coordinate plane, and use geometry to explain why S and P have the same area. Compute the area of P using Formula ??.

$$\text{Area of } P = \dots$$

2. Suppose M is the standard matrix of a linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $T(S) = P$. Find $\det M$.

$$\det M = \dots$$

Problem 17 Find the volume of a parallelepiped determined by

$$\begin{bmatrix} 0 \\ 5 \\ 4 \end{bmatrix} \quad \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix}$$

Answer: Volume = ...

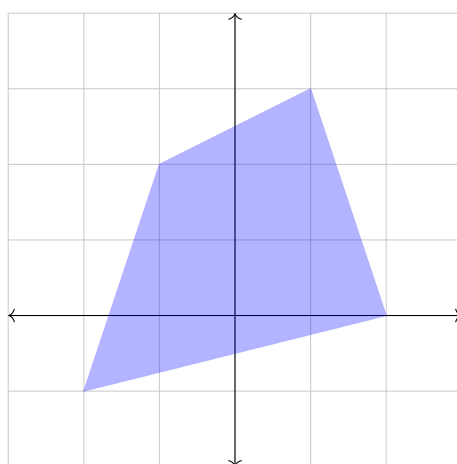
Problem 18 Find the volume of a parallelepiped determined by

$$\begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} \quad \begin{bmatrix} 5 \\ 6 \\ 2 \end{bmatrix}$$

Explain your result geometrically.

Answer: Volume =

Problem 19 Find the area of the figure using determinants.



Answer: Area =

4 Elementary Row Operations and the Determinant

When we first introduced the determinant we motivated its definition for a 2×2 matrix by the fact that the value of the determinant is zero if and only if the matrix is singular. We will soon be able to generalize this result to larger matrices, and will eventually establish a formula for the inverse of a nonsingular matrix in terms of determinants.

Recall that we can find the inverse of a matrix or establish that the inverse does not exist by using elementary row operations to carry the given matrix to its reduced row-echelon form. In order to start relating determinants to inverses we need to find out what elementary row operations do to the determinant of a matrix.

The Effects of Elementary Row Operations on the Determinant

Recall that there are three elementary row operations:

1. Switching the order of two rows
2. Multiplying a row by a non-zero constant
3. Adding a multiple of one row to another

Elementary row operations are used to carry a matrix to its reduced row-echelon form. In Practice Problem ?? we established that elementary row operations are reversible. In other words, if we know what elementary row operations carried A to $\text{rref}(A)$, we can undo each operation with another elementary row operation to carry $\text{rref}(A)$ back to A . This will prove useful for computing the determinant. Computing the determinant of $\text{rref}(A)$ is easy. (Why?) If we know what elementary row operations carry $\text{rref}(A)$ back to A , and what effect each of these operations has on the determinant of $\text{rref}(A)$, we could find the determinant of A .

Exploration 20 Let

$$A = \begin{bmatrix} 3 & -1 & -1 \\ 3 & 1 & -2 \\ -1 & 4 & 2 \end{bmatrix}$$

Find $\det A$.

$$\det A = \dots$$

Construct matrix B by switching the first and the third rows of A .

$$B = \begin{bmatrix} -1 & 4 & 2 \\ 3 & 1 & -2 \\ 3 & -1 & -1 \end{bmatrix}$$

Find $\det B$.

$$\det B = \dots$$

Next, try switching consecutive rows. Construct matrix B' by interchanging the second and third rows of A .

$$B' = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 4 & 2 \\ 3 & 1 & -2 \end{bmatrix}$$

Find $\det B'$.

$$\det B' = \dots$$

It appears that switching any two rows of a matrix produces a determinant that is negative of the determinant of the original matrix.

Next, construct matrix C by multiplying the last row of A by k .

$$C = \begin{bmatrix} 3 & -1 & -1 \\ 3 & 1 & -2 \\ -k & 4k & 2k \end{bmatrix}$$

Find $\det C$.

$$\det C = \dots k$$

It turns out that multiplying the first or the second row of A by k yields exactly the same result as this.

Finally, construct matrix D by adding twice row 3 to row 1.

$$D = \begin{bmatrix} 1 & 7 & 3 \\ 3 & 1 & -2 \\ -1 & 4 & 2 \end{bmatrix}$$

Find $\det D$.

$$\det D = \dots$$

This result is particularly surprising. Try a few more variations of this example to convince yourself that adding a multiple of one row to another row does not appear to affect the determinant.

The following theorem generalizes our observations.

Theorem 8. Let $A = [a_{ij}]$ be an $n \times n$ matrix.

1. If B is obtained from A by interchanging two different rows, then

$$\det B = -\det A$$

2. If B is obtained from A by multiplying one of the rows of A by a non-zero constant k . Then

$$\det B = k \det A$$

3. If B is obtained from A by adding a multiple of one row of A to another row, then

$$\det B = \det A$$

The proof of this theorem is relegated to [Tedious Proofs Concerning Determinants](#). For a sketch of the proof, you can watch this video: YouTube link: <https://www.youtube.com/watch?v=y0XVK-ey30c>

The following lemma is a useful consequence of parts ?? and ?? of Theorem ??.

Lemma 2. Let A be an $n \times n$ matrix.

1. If A has a row of zeros, then $\det A = 0$.
2. If two rows of A are the same, then $\det A = 0$.
3. If one row of A is a scalar multiple of another row, then $\det A = 0$.

We will prove Part ???. Parts ?? and ?? are left as exercises.

Proof of Part ??? Suppose rows p and q of A are the same. Let B be a matrix obtained from A by switching p and q . By Theorem ???? we know that $\det B = -\det A$. But p and q are the same, so $A = B$. But then $\det A = -\det A$. We conclude that $\det A = 0$. ■

Because $\det A = \det A^T$, we have the following counterpart of Theorem ??? for columns.

Theorem 9 (Elementary Column Operations and the Determinant). Let A be an $n \times n$ matrix.

1. If B is obtained from A by interchanging two different columns, then

$$\det B = -\det A$$
2. If B is obtained from A by multiplying one of the columns of A by a non-zero constant k . Then

$$\det B = k \det A$$
3. If B is obtained from A by adding a multiple of one column of A to another column, then

$$\det B = \det A$$

Computing the Determinant Using Elementary Row Operations

What we discovered about the effects of elementary row operations on the determinant will allow us to compute determinants without using the cumbersome process of cofactor expansion.

Example 17. Suppose that a 6×6 matrix A is carried to the identity matrix by a sequence of elementary row operations listed below. Find $\det A$.

$$A \xrightarrow{R_2 - 2R_4} A_1 \xrightarrow{R_1 \leftrightarrow R_3} A_2 \xrightarrow{\frac{1}{2}R_6} A_3 \xrightarrow{R_5 + R_6} I$$

Explanation. Let's take a look at what happens to the determinant of A one step at a time.

$$\begin{aligned} \det A_1 &= \det A && \text{by Theorem ????} \\ \det A_2 &= -\det A_1 = -\det A && \text{by Theorem ????} \\ \det A_3 &= \frac{1}{2} \det A_2 = -\frac{1}{2} \det A && \text{by Theorem ????} \\ \det I &= \det A_3 = -\frac{1}{2} \det A && \text{by Theorem ????} \end{aligned}$$

Recall that $\det I = 1$ (Lemma ??). This gives us

$$-\frac{1}{2} \det A = 1$$

Therefore $\det A = -2$.

Example 18. Let

$$A = \begin{bmatrix} 3 & 0 & -9 \\ 10 & 5 & 2 \\ 8 & 4 & 2 \end{bmatrix}$$

Find $\det A$ by applying elementary row operations to reduce A to its row-echelon form.

Explanation.

$$A = \begin{bmatrix} 3 & 0 & -9 \\ 10 & 5 & 2 \\ 8 & 4 & 2 \end{bmatrix} \xrightarrow{(1/3)R_1} \begin{bmatrix} 1 & 0 & -3 \\ 10 & 5 & 2 \\ 8 & 4 & 2 \end{bmatrix} \quad (5)$$

$$\xrightarrow{(1/2)R_3} \begin{bmatrix} 1 & 0 & -3 \\ 10 & 5 & 2 \\ 4 & 2 & 1 \end{bmatrix} \quad (6)$$

$$\xrightarrow{\begin{matrix} R_2 - 10R_1 \\ R_3 - 4R_1 \end{matrix}} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 5 & 32 \\ 0 & 2 & 13 \end{bmatrix} \quad (7)$$

$$\xrightarrow{(1/5)R_2} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 32/5 \\ 0 & 2 & 13 \end{bmatrix} \quad (8)$$

$$\xrightarrow{R_3 - 2R_2} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 32/5 \\ 0 & 0 & 1/5 \end{bmatrix} \quad (9)$$

We stop when we get to a row-echelon form of A because we can see that its determinant is $\frac{1}{5}$ (Theorem ??).

The following table summarizes the effect of each elementary row operation on the determinant.

Matrix	Determinant
A	$\det A$
(1)	$\frac{1}{3} \det A$
(2)	$\frac{1}{2} \cdot \frac{1}{3} \det A$
(3)	$\frac{1}{2} \cdot \frac{1}{3} \det A$
(4)	$\frac{1}{5} \cdot \frac{1}{2} \cdot \frac{1}{3} \det A$
(5)	$\frac{1}{5} \cdot \frac{1}{2} \cdot \frac{1}{3} \det A$

Since the determinant of the row-echelon form of A in (??) is $\frac{1}{5}$, we have

$$\frac{1}{5} = \frac{1}{5} \cdot \frac{1}{2} \cdot \frac{1}{3} \det A$$

Therefore

$$\det A = 6$$

You should verify this result by direct computation using cofactors.

Practice Problems

Problem 21 Let A be an $n \times n$ matrix. Show that

$$\det kA = k^n \det A$$

Problem 22 Prove that if one row of A is a scalar multiple of another row, then $\det A = 0$. (Lemma ????)

Problem 23 Prove that if one row of a matrix is a linear combination of two other rows of the matrix, then the determinant of the matrix is 0.

Problem 24 Find $\det A$ using elementary row operations.

$$A = \begin{bmatrix} 3 & 1 & -3 \\ 12 & 5 & -5 \\ 4 & 2 & 1 \end{bmatrix}$$

Answer:

$$\det A = \dots$$

Problem 25 Find $\det A$ using elementary row operations.

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

Answer:

$$\det A = \dots \dots$$

Problem 26 The matrix below is an elementary matrix.

1. What elementary row operation does this matrix perform?
2. Compute the determinant of the matrix in two different ways:
 - (i) By cofactor expansion.
 - (ii) By thinking about how the given matrix was obtained from the identity matrix.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Answer:

$$\det E_1 = \dots \dots$$

Problem 27 The matrix below is an elementary matrix.

1. What elementary row operation does this matrix perform?
2. Compute the determinant of the matrix in two different ways:
 - (i) By cofactor expansion.
 - (ii) By thinking about how the given matrix was obtained from the identity matrix.

$$E_2 = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Answer:

$$\det E_2 = \dots \dots$$

Problem 28 The matrix below is an elementary matrix.

1. What elementary row operation does this matrix perform?
2. Compute the determinant of the matrix in two different ways:
 - (i) By cofactor expansion.
 - (ii) By thinking about how the given matrix was obtained from the identity matrix.

$$E_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Answer:

$$\det E_3 = \dots \dots$$

5 Properties of the Determinant

We begin by summarizing the properties of determinants we introduced in previous sections.

Summary 1. • The determinant of the identity matrix is equal to 1. (Lemma ??)

- The determinant of a triangular matrix is equal to the product of the main diagonal entries. (Lemma ??)
- The determinant of the transpose is equal to the determinant of the matrix. (Theorem ??)
- If a matrix contains a row of zeros, then its determinant is equal to 0. (Lemma ??)
- If two rows of a matrix are the same, then the determinant of the matrix is equal to 0. (Lemma ??)
- If one row of a matrix is a scalar multiple of another row, then the determinant of the matrix is equal to 0. (Lemma ??)
- If B is obtained from A by interchanging two different rows, then

$$\det B = -\det A$$

- If B is obtained from A by multiplying one of the rows of A by a non-zero constant k . Then

$$\det B = k \det A$$

- If B is obtained from A by adding a multiple of one row of A to another row, then

$$\det B = \det A$$

(The last three properties comprise Theorem ??)

In this section we will prove the following important results:

1. A square matrix is singular if and only if its determinant is equal to 0.
2. The determinant of a product is the product of the determinants.

To get us started, we need the following lemma.

Lemma 3. Let A be a square matrix, and let E be an elementary matrix, then

$$\det EA = \det E \det A$$

Proof Recall that if E is obtained from I using an elementary row operation, then the same elementary row operation carries A to EA . There are three types of elementary row operations and three types of elementary matrices, so we will have to consider three cases.

Case 1. Suppose E is obtained from I by interchanging two rows, then

$$\det E = -1 \quad \text{and} \quad \det EA = -\det A$$

so

$$\det EA = \det E \det A$$

Case 2. Suppose E is obtained from I by multiplying one of the rows of I by a non-zero constant k , then

$$\det E = k \quad \text{and} \quad \det EA = k \det A$$

so

$$\det EA = \det E \det A$$

Case 3. Suppose E is obtained from I by adding a scalar multiple of one row to another row, then

$$\det E = 1 \quad \text{and} \quad \det EA = \det A$$

so

$$\det EA = \det E \det A$$

■

Invertibility and the Determinant

Recall that we first introduced determinants in the context of invertibility of 2×2 matrices. Specifically, we found that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $\det A \neq 0$. (A logically equivalent statement is: A is singular if and only if $\det A = 0$.) We are now in the position to prove this result for all square matrices.

Theorem 10. A square matrix A is singular if and only if $\det A = 0$.

Proof Let A be a square matrix. To determine whether A is singular we need to find $\text{rref}(A)$. In [Elementary Matrices](#) we found that there exist elementary matrices $E_1, \dots, E_k$ such that

$$E_k \dots E_2 E_1 A = \text{rref}(A)$$

so

$$\det(E_k \dots E_2 E_1 A) = \det(\text{rref}(A))$$

By repeated application of Lemma ??, we find that

$$\det E_k \dots \det E_2 \det E_1 \det A = \det(\text{rref}(A))$$

Suppose that A is singular, then $\text{rref}(A) \neq I$. But then $\text{rref}(A)$ contains a row of zeros, and $\det(\text{rref}(A)) = 0$. (Lemma ??) Since determinants of elementary matrices are non-zero, we conclude that $\det A = 0$.

Conversely, suppose $\det A = 0$, then

$$\det(\text{rref}(A)) = \det E_k \dots \det E_2 \det E_1 \det A = 0$$

But then $\text{rref}(A) \neq I$, so A is singular. ■

Example 19. Determine whether A is an invertible matrix without using elementary row operations.

$$A = \begin{bmatrix} 3 & 4 & -8 \\ 1 & 8 & -10 \\ 1 & -2 & 1 \end{bmatrix}$$

Explanation. Compute the determinant of A . You will find that $\det A = 0$. By Theorem ?? we conclude that A is not invertible.

Determinant of a Product

Theorem 11. Let A and B be square matrices, then

$$\det AB = \det A \det B$$

Proof Suppose A is invertible, then A can be written as a product of elementary matrices. (Theorem ??)

$$A = E_1 E_2 \dots E_k$$

Then, by repeated application of Lemma ??, we get

$$\begin{aligned} \det AB &= \det (E_1 E_2 \dots E_k B) \\ &= \det E_1 \det E_2 \dots \det E_k \det B \\ &= \det (E_1 E_2 \dots E_k) \det B \\ &= \det A \det B \end{aligned}$$

Now suppose that A is not invertible. Then AB is also not invertible. So, $\det A = 0$ and $\det AB = 0$. Thus $\det AB = 0 = \det A \det B$. ■

The following theorem is a nice consequence of Theorem ??. We leave the proof to the reader. (Practice Problem ??)

Theorem 12. Let A be a nonsingular matrix, then

$$\det A^{-1} = \frac{1}{\det A}$$

Practice Problems

Problem 29 Without doing written computations, determine whether matrix A is singular.

$$A = \begin{bmatrix} 0 & 0 & -3 \\ 2 & 3 & -1 \\ 0 & 0 & 10 \end{bmatrix}$$

Multiple Choice:

1. A is singular
2. A is nonsingular

Problem 30 Without doing written computations, determine whether matrix A is singular.

$$A = \begin{bmatrix} 2 & -3 & -1 \\ 0 & 4 & 5 \\ 0 & 0 & -3 \end{bmatrix}$$

Multiple Choice:

1. A is singular
2. A is nonsingular

Problem 31 Show that all matrices of the form

$$\begin{bmatrix} x & x+1 & x+2 \\ x+3 & x+4 & x+5 \\ x+6 & x+7 & x+8 \end{bmatrix}$$

are singular.

Problem 32 Find values of x for which the given matrix is singular.

$$\begin{bmatrix} 2-x & 1 \\ 5 & 6-x \end{bmatrix}$$

List values of x in an increasing order.

Answer:

$$x = \dots \dots \quad x = \dots \dots$$

Problem 33 Suppose A and B are 2×2 matrices such that $\det A = 2$ and $\det B = \frac{1}{3}$. Then, $\det AB^{-1} = \dots \dots$

Problem 34 Suppose A and B are 2×2 matrices such that $\det A = 2$ and $\det B = \frac{1}{3}$. Then, $\det (AB)^{-1} = \dots \dots$

Problem 35 Suppose A and B are 2×2 matrices such that $\det A = 2$ and $\det B = \frac{1}{3}$. Then, $\det 2AB = \dots \dots$

Problem 36 Prove or give a counterexample.

$$\det(A+B) = \det(A^T + B^T)$$

Problem 37 Prove that for a nonsingular matrix A

$$\det A^{-1} = \frac{1}{\det A}$$

(Theorem ??)

Problem 38 Suppose A is an invertible matrix such that

$$A^{-1} = A^T$$

What are the possible values for $\det A$? List your answers in increasing order.

Answer: $\det A = \dots \dots, \dots \dots$