
Week 11

June 5, 2026

Contents

1 Describing Eigenvalues and Eigenvectors

At several places in this course it has been valuable to restrict ourselves to square matrices, and we do so again when discussing eigenvalues and eigenvectors.

In Theorem ??, we proved that any $n \times n$ matrix induces a linear transformation from $\mathbb{R}^n$ to itself. For our first few examples, let us consider the case $n = 2$.

Exploration 1 Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. The following interactive helps us to visualize the matrix transformation associated with A . Given a vector $\mathbf{x}$ in $\mathbb{R}^2$ (left panel), its image, $A\mathbf{x}$, is also in $\mathbb{R}^2$ (right panel). Drag the head of the vector $\mathbf{x}$ to see its image.

What do you observe about the relationship between vectors along the red line $y = -x$ and their images?

For many vectors $\mathbf{x}$ of $\mathbb{R}^2$, $A\mathbf{x}$ does not point in the same direction as $\mathbf{x}$ but if we look at the vectors along the line $y = -x$, we notice that the transformation does not seem to affect their magnitude and direction. Such vectors are sometimes called *fixed vectors* of A . For example $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is a fixed vector.

What do you observe about the relationship between vectors along the blue line $y = x$ and their images?

We observe that the magnitudes of these vectors are changed by the transformation, but the direction in which the vectors point is unchanged.

In Exploration ?? we found that the images of vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ stay parallel to the original vectors under the linear transformation induced by matrix A . Such vectors are examples of *eigenvectors* of A .

In general, any nonzero vector whose image under a matrix transformation is parallel to the original vector is called an *eigenvector* of the matrix that induced the transformation. The following definition captures this idea algebraically.

Definition 1. Let A be an $n \times n$ matrix. We say that a non-zero vector $\mathbf{x}$ is an *eigenvector* of A if

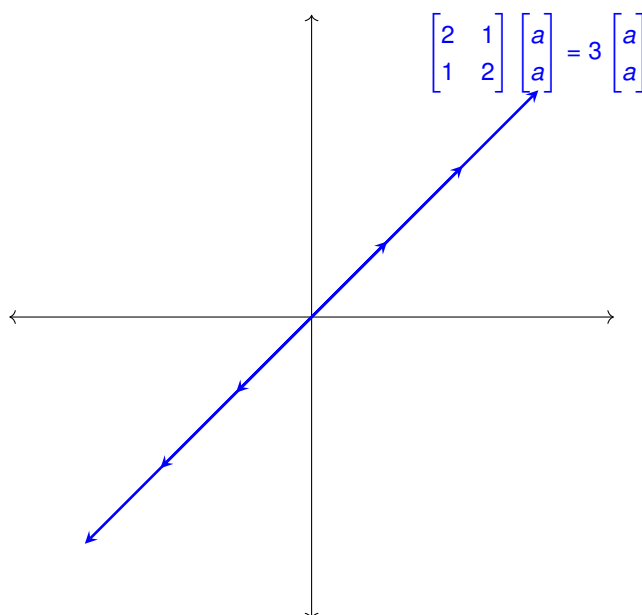
$$A\mathbf{x} = \lambda\mathbf{x}$$

for some scalar λ . We say that λ is an *eigenvalue* of A associated with the eigenvector $\mathbf{x}$.

Let's revisit Exploration ?? in light of Definition ?. In the exploration, we observed visually that vectors parallel to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ were eigenvectors associated with $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, as these vectors changed length but remained parallel to the original vector under the linear transformation induced by A . To verify this algebraically, observe that all vectors parallel to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ can be written in the form $\begin{bmatrix} a \\ a \end{bmatrix}$, ($a \neq 0$). We compute

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ a \end{bmatrix} = \begin{bmatrix} 3a \\ 3a \end{bmatrix} = 3 \begin{bmatrix} a \\ a \end{bmatrix}$$

This shows that any non-zero scalar multiple of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A which has a corresponding eigenvalue of 3.



Fixed vectors of Exploration ?? are also eigenvectors. For example,

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

This shows that $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is a fixed vector and an eigenvector of A which has a corresponding eigenvalue of 1.

The above discussion leads us to the following result.

Theorem 1. If $\mathbf{x}$ is an eigenvector of matrix A and λ is the corresponding eigenvalue, then every non-zero scalar multiple of $\mathbf{x}$ is also an eigenvector of A and λ is the corresponding eigenvalue.

Proof See Practice Problem ??

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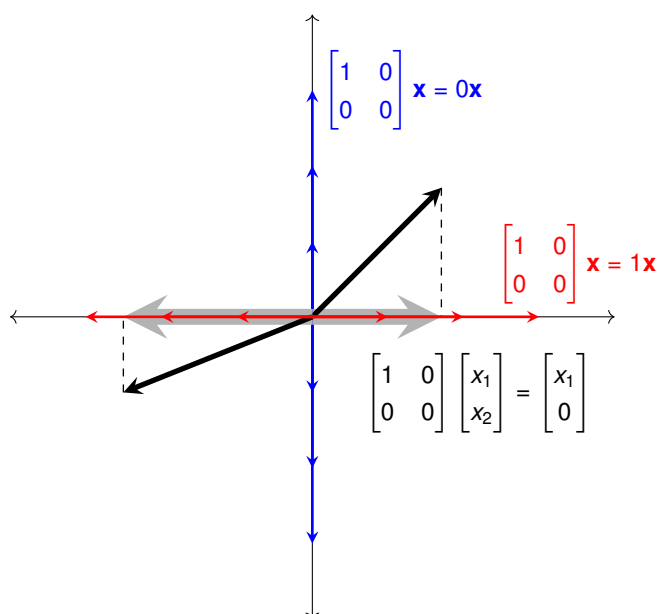
Observation 1. A couple of finer points of Definition ?? require clarification.

- The definition requires that eigenvectors be non-zero. Imagine what would happen if we allowed $\mathbf{x} = \mathbf{0}$ to be an eigenvector of A . Clearly $A\mathbf{0} = \lambda\mathbf{0}$ for all scalars λ . This means that every number would be an eigenvalue of every matrix. Because eigenvalues are supposed to capture certain information about the matrix, allowing every number to be an eigenvalue of every matrix would defeat the purpose.
- Until now, we had talked about eigenvectors as vectors whose images under a matrix transformation are parallel to the original vectors. But the algebraic definition allows non-zero vectors that map to zero to be

considered eigenvectors. (What would an eigenvalue of such an eigenvector be?) The zero vector has no direction, so we cannot say that the image of such an eigenvector is parallel to the original vector. Example ?? will illustrate this point.

Example 1. Let $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Note that P takes a vector in $\mathbb{R}^2$ and projects it onto the x -axis, as we learned in Practice Problem ??. Which vectors in $\mathbb{R}^2$ would be the eigenvectors, and what are the corresponding eigenvalues?

Explanation. Since $P\mathbf{x}$ is the projection of $\mathbf{x}$ onto the x -axis, in many cases $P\mathbf{x}$ and $\mathbf{x}$ are not parallel. Notice, however, that all of the red vectors located along the x -axis in the diagram are fixed by P . So, for any of the red vectors we have $P\mathbf{x} = \mathbf{x} = 1\mathbf{x}$, which means that each of the red vectors is an eigenvector of P with the corresponding eigenvalue of 1.



The blue vectors along the y -axis are also eigenvectors. To see this, note that each of the blue vectors is of the form $\mathbf{x} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$. But then

$$P\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = 0\mathbf{x}$$

So each of the blue vectors is an eigenvector of P with the corresponding eigenvalue of 0.

Exploration 2 Let $A = \begin{bmatrix} 1 & 6 \\ 1 & 0 \end{bmatrix}$. The GeoGebra interactive below shows vector $\mathbf{x}$, and its image, $A\mathbf{x}$ in the same coordinate plane. (Move the head of $\mathbf{x}$ to see the effect of A on different vectors.)

Question 2.1 Note that vectors $\mathbf{x}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ (and their scalar multiples) remain positioned along the

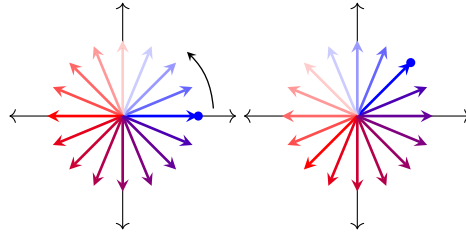
same lines even as they change magnitude and direction. This indicates that $\mathbf{x}_1$ and $\mathbf{x}_2$, along with all of their scalar multiples, are eigenvectors of A . What are the eigenvalues associated with these eigenvectors?

Eigenvalue associated with $\mathbf{x}_1$ is $\lambda_1 = \dots$

Eigenvalue associated with $\mathbf{x}_2$ is $\lambda_2 = \dots$

A natural question is this: does every square matrix have eigenvalues and eigenvectors? We will see later that the answer to this question is “yes”, provided that we permit eigenvalues and entries of eigenvectors to be complex numbers. The next example illustrates this.

Example 2. Let $M = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$. Note that M takes any vector $\mathbf{x}$ in $\mathbb{R}^2$ and rotates it 45° , as we saw in Example ??.



Since M rotates every vector in $\mathbb{R}^2$, every nonzero vector changes direction, so there are no eigenvectors in the plane. It turns out that M does have eigenvectors and eigenvalues, but in order to find them we need to work with vectors whose entries are complex numbers. Since these vectors are not in $\mathbb{R}^2$, we cannot see them.

To follow the computation below, you need to recall that the imaginary unit i is defined by $i^2 = -1$.

Consider the vector $\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2}i \end{bmatrix}$. We compute:

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2}i \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{1}{2}i \\ \frac{1}{2} + \frac{1}{2}i \end{bmatrix} = \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i \right) \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2}i \end{bmatrix},$$

so $\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2}i \end{bmatrix}$ is an eigenvector of M . Its corresponding eigenvalue is $\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$.

We will continue to work with complex numbers as we study eigenvalues and eigenvectors.

Why All the Fuss About Eigenvalues and Eigenvectors?

Remark 1. The first in-depth study of eigenvalues can probably be attributed to Fourier as he studied partial differential equations early in the nineteenth century, and in particular when he studied what is known as the heat equation.

[Trefethen and Embree] By the twentieth century mathematicians understood the connections between differential equations and eigenvalues. Systems of differential equations are often best represented by matrices, especially in the context of using computers to find numerical solutions. Most algorithms to solve these systems work by iterating some process, and eigenvalues along with their corresponding eigenvectors indicate what will happen to such a process after many repetitions.

The most famous modern example of a large-scale eigenvalue problem is the Google PageRank algorithm, which helped set Google apart from its competitors as a search engine. Some of the relevant mathematics can be learned by working through the paper, “The \$25,000,000,000 Eigenvector, The Linear Algebra Behind Google”, by Kurt Bryan and Tanya Leise.

Practice Problems

Problem 3 Let $C = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

Question 3.1 Show that $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of C . What is its corresponding eigenvalue?

Question 3.2 Show that $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ is an eigenvector of C . What is its corresponding eigenvalue?

Question 3.3 Show that $\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ is an eigenvector of C . What is its corresponding eigenvalue?

Problem 4 Let $Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Note that Q takes any vector in $\mathbb{R}^2$ and projects it onto the y -axis. Which vectors in $\mathbb{R}^2$ would be eigenvectors, and what are the corresponding eigenvalues?

Problem 5 Returning to Example ??, let $M = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$. Show that $\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2}i \end{bmatrix}$ is an eigenvector of M . What is its corresponding eigenvalue?

Please enter an exact answer; no decimal approximations. To enter square roots, click inside the answer box, then use the “Math Editor” button that appears at the top of your XIMERA window.

... .. + i

Problem 6 Arguing geometrically, identify the linear transformation whose standard matrix has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$.

Multiple Choice:

1. Vertical Shear
2. Horizontal Shear
3. Counterclockwise Rotation through a 90° angle
4. Reflection About the line $y = mx$
5. Horizontal Stretch
6. Vertical Stretch

Problem 7 Let $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Can you find an eigenvector and its corresponding eigenvalue? Can you find another "eigenpair"? Can you find all of the eigenvectors of A ?

Problem 8 The rotation matrix in Example ?? has complex eigenvectors and eigenvalues. Think geometrically to find an example of a (non-identity) rotation matrix with real eigenvectors and eigenvalues.

Enter degree measure between 0 and 360.

Answer: Rotation through degrees.

Problem 9 Can an eigenvalue have multiple eigenvectors associated with it?

Multiple Choice:

1. Yes
2. No

Can an eigenvector have multiple eigenvalues associated with it?

Multiple Choice:

1. Yes
2. No

Problem 10 Prove that if $\mathbf{x}$ is an eigenvector of matrix A and λ is the corresponding eigenvalue, then every non-zero scalar multiple of $\mathbf{x}$ is also an eigenvector of A and λ is the corresponding eigenvalue. (Theorem ??)

Bibliography

[Trefethen and Embree] Trefethen, Lloyd and Embree, Mark, *Spectra and Pseudospectra*, Princeton University Press, 2005, p. 5-6

2 The Characteristic Equation

Let A be an $n \times n$ matrix. In the previous section we learned that the eigenvectors and eigenvalues of A are vectors $\mathbf{x}$ and scalars λ that satisfy the equation

$$A\mathbf{x} = \lambda\mathbf{x} \quad (1)$$

We listed a few reasons why we are interested in finding eigenvalues and eigenvectors, but we did not give any process for finding them. In this section we will focus on a process which can be used for small matrices. For larger matrices, the best methods we have are iterative methods, and we will explore some of these in [The Power Method and the Dominant Eigenvalue](#).

For an $n \times n$ matrix, we will see that the eigenvalues are the roots of a polynomial called the *characteristic polynomial*. So finding eigenvalues is equivalent to solving a polynomial equation of degree n . Finding the corresponding eigenvectors turns out to be a matter of computing the null space of a matrix, as the following exploration demonstrates.

Exploration 11 If a vector $\mathbf{x}$ is an eigenvector satisfying Equation (??), then clearly it also satisfies $A\mathbf{x} - \lambda\mathbf{x} = (0/0)$.

It seems natural at this point to try to factor. We would love to “factor out” $\mathbf{x}$. Here is the procedure:

$$\begin{aligned} A\mathbf{x} - \lambda\mathbf{x} &= \mathbf{0} \\ A\mathbf{x} - \lambda I\mathbf{x} &= \mathbf{0} \\ (A - \lambda I)\mathbf{x} &= \mathbf{0} \end{aligned}$$

This shows that any eigenvector $\mathbf{x}$ of A is in the (row space/ column space/ null space) of the related matrix, $A - \lambda I$.

Since eigenvectors are non-zero vectors, this means that A will have eigenvectors if and only if the null space of $A - \lambda I$ is nontrivial. The only way that $\text{null}(A - \lambda I)$ can be nontrivial is if $\text{rank}(A - \lambda I) (= / < / >) n$.

If the rank of an $n \times n$ matrix is less than n , then the matrix is singular. Since $(A - \lambda I)$ must be singular for any eigenvalue λ , we see that λ is an eigenvalue of A if and only if

$$\det(A - \lambda I) = \dots \dots \quad (2)$$

In theory, Exploration ?? offers us a way to find eigenvalues. To find the eigenvalues of A , one can solve Equation (??) for λ .

Eigenvalues

Definition 2. The equation

$$\det(A - \lambda I) = 0$$

is called the *characteristic equation* of A .

Example 3. Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Compute the eigenvalues of this matrix using the characteristic equation.

Explanation.

$$\begin{aligned}\det(A - \lambda I) &= \begin{vmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{vmatrix} = (2 - \lambda)^2 - 1 \\ &= \lambda^2 - 4\lambda + 3 \\ &= (\lambda - 1)(\lambda - 3)\end{aligned}$$

The characteristic equation $(\lambda - 1)(\lambda - 3) = 0$ has solutions $\lambda_1 = 1$ and $\lambda_2 = 3$. These are the eigenvalues of A .

Example 4. Let $B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$. Compute the eigenvalues of B using the characteristic equation. (List your answers in an increasing order.)

$\lambda_1 = \dots$ and $\lambda_2 = \dots$

Example 5. Let $C = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$. Compute the eigenvalues of C using the characteristic equation.

Explanation.

$$\begin{aligned}\det(C - \lambda I) &= \begin{vmatrix} 2 - \lambda & 1 & 1 \\ 1 & 2 - \lambda & 1 \\ 1 & 1 & 2 - \lambda \end{vmatrix} \\ &= (2 - \lambda) \begin{vmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 1 & 2 - \lambda \end{vmatrix} + 1 \begin{vmatrix} 1 & 2 - \lambda \\ 1 & 1 \end{vmatrix} \\ &= (2 - \lambda)^3 - (2 - \lambda) - ((2 - \lambda) - 1) + 1 - (2 - \lambda) \\ &= 8 - 12\lambda + 6\lambda^2 - \lambda^3 - 2 + \lambda - 2 + \lambda + 1 + 1 - 2 + \lambda \\ &= 4 - 9\lambda + 6\lambda^2 - \lambda^3 \\ &= -(\lambda - 4)(\lambda - 1)^2\end{aligned}$$

Matrix C has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 4$.

Remark 2. The factor $(\lambda - 1)$ appears twice. This repeated factor gives rise to the eigenvalue $\lambda_1 = 1$. We say that the eigenvalue $\lambda_1 = 1$ has *algebraic multiplicity* 2.

Definition 3. *Algebraic multiplicity* of an eigenvalue is its multiplicity as a root of the characteristic equation.

It is not always possible to completely factor the characteristic polynomial using only real numbers. However, a fundamental fact from algebra is that every degree n polynomial has n roots (counting multiplicity) provided that we allow complex numbers. This is why sometimes eigenvalues and their corresponding eigenvectors involve complex numbers. The next example illustrates this point.

Example 6. Let $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$. Compute the eigenvalues of this matrix.

Explanation.

$$\begin{aligned}
 \det(D - \lambda I) &= \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & 1 - \lambda & 1 \\ 0 & -1 & 1 - \lambda \end{vmatrix} \\
 &= -\lambda \begin{vmatrix} 1 - \lambda & 1 \\ -1 & 1 - \lambda \end{vmatrix} \\
 &= -\lambda((1 - \lambda)^2 + 1) \\
 &= -\lambda(\lambda^2 - 2\lambda + 2)
 \end{aligned}$$

So one of the eigenvalues of D is $\lambda = 0$. To get the other eigenvalues we must solve $\lambda^2 - 2\lambda + 2 = 0$. Using the quadratic formula, we compute that $\lambda_1 = 1 + i$ and $\lambda_2 = 1 - i$ are also eigenvalues of D .

Exploration 12 Let $T = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{bmatrix}$. Compute the eigenvalues of this matrix. (List your answers in an increasing order.)

$$\lambda_1 = \dots, \quad \lambda_2 = \dots, \quad \text{and} \quad \lambda_3 = \dots$$

What do you observe about the eigenvalues?

What property of the matrix makes this “coincidence” possible?

The matrix in Exploration Problem ?? is a triangular matrix, and the property you observed holds in general.

Theorem 2. Let T be a triangular matrix. Then the eigenvalues of T are the entries on the main diagonal.

Proof See Practice Problem ??. ■

Corollary 1. Let D be a diagonal matrix. Then the eigenvalues of D are the entries on the main diagonal.

One final note about eigenvalues. We began this section with the sentence, “In theory, then, to find the eigenvalues of A , one can solve Equation (??) for λ .” In general, one does not attempt to compute eigenvalues by solving the characteristic equation of a matrix, as there is no simple way to solve this polynomial equation for $n > 4$. Instead, one can often approximate the eigenvalues using *iterative methods*. We will explore some of these techniques in [The Power Method and the Dominant Eigenvalue](#).

Eigenvectors

Once we have computed an eigenvalue λ of an $n \times n$ matrix A , the next step is to compute the associated eigenvectors. In other words, we seek vectors $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$, or equivalently,

$$(A - \lambda I)\mathbf{x} = \mathbf{0} \tag{3}$$

As we have seen in the previous section, a (non-zero) scalar multiple of an eigenvector is an eigenvector. So for any given eigenvalue λ there are infinitely many eigenvectors associated with it. In fact, the eigenvectors associated with λ , together with the zero vector, form a subspace of $\mathbb{R}^n$.

Theorem 3. Let A be an $n \times n$ matrix and let λ be an eigenvalue of A . Then the set of all eigenvectors associated with λ , together with the zero vector, is a subspace of $\mathbb{R}^n$.

Proof See Practice Problems ?? and ??. ■

This motivates the following definition.

Definition 4. The set of all eigenvectors associated with a given eigenvalue of a matrix, together with the zero vector, is known as the *eigenspace* associated with that eigenvalue.

Given an eigenvalue λ , there is an associated eigenspace S_λ . Our goal is to find a basis of S_λ . Then any eigenvector $\mathbf{x}$ associated with λ will be a linear combination of the vectors in that basis. In seeking a basis for S_λ , we are trying to find a basis for the set of vectors that satisfy $(A - \lambda I)\mathbf{x} = \mathbf{0}$, which means we seek a basis for $\text{null}(A - \lambda I)$.

Let's return to the examples we did in the first part of this section.

Example 7. (Finding eigenvectors for Example ??)

Recall that $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 3$. Compute a basis for the eigenspace associated with each of these eigenvalues.

Explanation. Eigenvectors associated with the eigenvalue $\lambda_1 = 1$ are in the null space of $A - I$. So we seek a basis for $\text{null}(A - I)$. We compute:

$$\text{rref}(A - I) = \text{rref} \left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$

From this we see that the eigenspace S_1 associated with $\lambda_1 = 1$ consists of vectors of the form $\begin{bmatrix} -1 \\ 1 \end{bmatrix} t$. This means

that $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ is one possible basis for S_1 .

In a similar way, we compute a basis for S_3 , the subspace of all eigenvectors associated with the eigenvalue $\lambda_2 = 3$. Now we compute:

$$\text{rref}(A - 3I) = \text{rref} \left(\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \right) = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix},$$

Vectors in the null space have the form $\begin{bmatrix} 1 \\ 1 \end{bmatrix} t$. This means that $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is one possible basis for the eigenspace S_3 .

These computations reinforce the geometric insights into this problem that we gained in the previous section.

Example 8. (Finding eigenvectors for Example ??) We know from Example ?? that $B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 4$. Compute a basis for the eigenspace associated with each of these eigenvalues.

Explanation. Let's begin by finding a basis for the eigenspace S_0 , which is the subspace of $\mathbb{R}^n$ consisting of eigenvectors corresponding to the eigenvalue $\lambda_1 = 0$. We need to compute a basis for $\text{null}(B - 0I) = \text{null}(B)$. We compute:

$$\text{rref}(B) = \text{rref}\left(\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix},$$

From this we see that an eigenvector in S_0 has the form $\begin{bmatrix} -1/2 \\ 1 \end{bmatrix} t$. This means that $\left\{ \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \right\}$ is one possible basis for the eigenspace S_0 . By letting $t = -2$, we obtain an arguably nicer-looking basis: $\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$.

To compute a basis for S_4 , the subspace of all eigenvectors associated to the eigenvalue $\lambda_2 = 4$, we compute:

$$\text{rref}(B - 4I) = \text{rref}\left(\begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}\right) = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

From this we find that $\left\{ \begin{bmatrix} 1 \\ \dots \end{bmatrix} \right\}$ is one possible basis for the eigenspace S_4 .

Example 9. (Finding eigenvectors for Example ??) We know from Example ?? that $C = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 4$. Compute a basis for the eigenspace associated to each of these eigenvalues.

Explanation. We first find a basis for the eigenspace S_1 . We need to compute a basis for $\text{null}(C - I)$. We compute:

$$\text{rref}(C - I) = \text{rref}\left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

Notice that there are two free variables. The eigenvectors in S_1 have the form

$$\begin{bmatrix} -s - t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

So one possible basis for the eigenspace S_1 is given by $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Next we find a basis for the eigenspace S_4 . We need to compute a basis for $\text{null}(C - 4I)$. We compute:

$$\text{rref}(C - 4I) = \text{rref}\left(\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

This time there is one free variable. The eigenvectors in $\mathcal{S}_4$ have the form $\begin{bmatrix} t \\ t \\ t \end{bmatrix}$, so a possible basis for the eigenspace

$$\mathcal{S}_4 \text{ is given by } \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Remark 3. Recall that $\lambda_1 = 1$ has algebraic multiplicity 2. Note that $\dim(\mathcal{S}_{\lambda_1}) = 2$. We say that λ_1 has geometric multiplicity 2. We have to be careful, and not draw hasty conclusions. Geometric and algebraic multiplicity are NOT always the same! We will revisit this issue later in the text.

Definition 5. The *geometric multiplicity* of an eigenvalue is the dimension of the eigenspace corresponding to it.

Example 10. (Finding eigenvectors for Example ??) We know from Example ?? that $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$ has eigen-

values $\lambda = 0$, $\lambda_1 = 1 + i$, and $\lambda_2 = 1 - i$. Compute a basis for the eigenspace associated with each eigenvalue.

Explanation. We first find a basis for the eigenspace $\mathcal{S}_0$. We need to compute a basis for $\text{null}(D - 0I) = \text{null}(D)$. We compute:

$$\text{rref}(D) = \text{rref} \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \right) = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

From this we see that for any eigenvector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in $\mathcal{S}_0$ we have $x_2 = 0$ and $x_3 = 0$, but x_1 is a free variable. So one possible basis for the eigenspace $\mathcal{S}_0$ is given by

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Next we find a basis for the eigenspace $\mathcal{S}_{1+i}$. We need to compute a basis for $\text{null}(D - (1 + i)I)$. We compute:

$$\begin{aligned} \text{rref}(D - (1 + i)I) &= \text{rref} \left(\begin{bmatrix} -(1 + i) & 0 & 0 \\ 0 & 1 - (1 + i) & 1 \\ 0 & -1 & 1 - (1 + i) \end{bmatrix} \right) \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & i \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

There is one free variable. Setting $x_3 = t$, we get $x_1 = 0$ and $x_2 = ti$. From this we see that eigenvectors in $\mathcal{S}_{1+i}$ have the form $\begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix} t$, so a possible basis for the eigenspace $\mathcal{S}_{1+i}$ is given by $\left\{ \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix} \right\}$. We ask you in Practice Problem

?? to show that $\left\{ \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathcal{S}_{1-i}$.

We conclude this section by establishing the significance of a matrix having an eigenvalue of zero.

Theorem 4. A square matrix has an eigenvalue of zero if and only if it is singular.

Proof A square matrix A is singular if and only if $\det A = 0$. But $\det A = 0$ if and only if $\det A - 0I = 0$, which is true if and only if zero is an eigenvalue of A . ■

Practice Problems

Problem 13 Compute the eigenvalues of A , and find the corresponding eigenspaces.

$$A = \begin{bmatrix} 4 & 1 \\ 8 & -3 \end{bmatrix}$$

Answer: (List the eigenvalues in an increasing order.)

$$\lambda_1 = \dots, \quad \lambda_2 = \dots$$

A basis for S_{λ_1} is $\left\{ \begin{bmatrix} \dots \\ 1 \end{bmatrix} \right\}$. A basis for S_{λ_2} is $\left\{ \begin{bmatrix} \dots \\ 1 \end{bmatrix} \right\}$.

Problem 14 Compute the eigenvalues of A , and find the corresponding eigenspaces.

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

Answer:

$$\lambda_1 = \dots + \dots i, \quad \lambda_2 = \dots - \dots i$$

A basis for S_{λ_1} is $\left\{ \begin{bmatrix} i \\ \dots \end{bmatrix} \right\}$. A basis for S_{λ_2} is $\left\{ \begin{bmatrix} -i \\ \dots \end{bmatrix} \right\}$.

Problem 15 Let $T = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{bmatrix}$. Compute a basis for each of the eigenspaces of this matrix, S_1 , S_5 , and S_9 .

Problem 16 Let

$$A = \begin{bmatrix} 9 & 2 & 8 \\ 2 & -6 & -2 \\ -8 & 2 & -5 \end{bmatrix}$$

Question 16.1 Compute the eigenvalues of A .

Answer:

(List your answers in an increasing order.)

$$\lambda_1 = \dots, \quad \lambda_2 = \dots, \quad \lambda_3 = \dots$$

Question 16.2 Compute a basis for each of the eigenspaces of A , S_{λ_1} , S_{λ_2} , and S_{λ_3} .

Answer: A basis for S_{λ_1} is $\left\{ \begin{bmatrix} 1 \\ \dots \\ \dots \end{bmatrix} \right\}$, a basis for S_{λ_2} is $\left\{ \begin{bmatrix} -2 \\ \dots \\ \dots \end{bmatrix} \right\}$,
and a basis for S_{λ_3} is $\left\{ \begin{bmatrix} 2 \\ \dots \\ \dots \end{bmatrix} \right\}$.

Problem 17 Complete Example ?? by showing that a basis for S_{1-i} is given by $\left\{ \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} \right\}$, where S_{1-i} is the

eigenspace associated with the eigenvalue $\lambda = 1 - i$ of the matrix $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$.

Problem 18 Prove Theorem ??.

Problem 19 Recall that a vertical stretch/compression of the plane is a linear transformation whose standard matrix is

$$M_v = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

Find the eigenvalues of M_v . Find a basis for the eigenspace corresponding to each eigenvalue.

Answer: A basis for S_1 is $\left\{ \begin{bmatrix} 1 \\ \dots \end{bmatrix} \right\}$ and a basis for S_k is $\left\{ \begin{bmatrix} \dots \\ 1 \end{bmatrix} \right\}$

Sketch several vectors in each eigenspace and use geometry to explain why the eigenvectors you sketched make sense.

Problem 20 Recall that a horizontal shear of the plane is a linear transformation whose standard matrix is

$$M_{hs} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

Find the eigenvalue of M_{hs} .

Answer: $\lambda = \dots$

Find a basis for the eigenspace corresponding to λ .

Answer: A basis for S is $\left\{ \begin{bmatrix} 1 \\ \dots \end{bmatrix} \right\}$

What is the algebraic multiplicity of λ ? ...

What is the geometric multiplicity of λ ? ...

Problem 21 Recall that a counterclockwise rotation of the plane through angle θ is a linear transformation whose standard matrix is

$$M_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

Verify that the eigenvalues of M_θ are

$$\lambda = \cos \theta \pm \sqrt{\cos^2 \theta - 1}$$

Use algebra, then use geometry to explain why λ is a real number if and only if θ is a multiple of π .

Suppose $\theta = \pi$. Then $\lambda = \dots$ has algebraic multiplicity $\dots$. Which of the following describes the corresponding eigenspace?

Multiple Choice:

1. All vectors in $\mathbb{R}^2$.
2. All vectors along the x -axis.
3. All vectors along the y -axis.
4. All vectors along the line $y = x$.

Problem 22 Recall that a reflection of the plane about the line $y = mx$ is a linear transformation whose standard matrix is

$$M_{y=mx} = \frac{1}{1+m^2} \begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}$$

Verify that the eigenvalues of $M_{y=mx}$ are 1 and -1 .

Find a basis for eigenspaces S_1 and S_{-1} . (For simplicity, assume that $m \neq 0$.)

Answer: A basis for S_1 is $\left\{ \begin{bmatrix} 1 \\ \dots \end{bmatrix} \right\}$ and a basis for S_{-1} is $\left\{ \begin{bmatrix} m \\ \dots \end{bmatrix} \right\}$

Choose the best description of S_1 .

Multiple Choice:

1. All vectors in $\mathbb{R}^2$.
2. All vectors with "slope" m .
3. All vectors with "slope" $1/m$.
4. All vectors with "slope" $-1/m$.

Choose the best description of S_{-1} .

Multiple Choice:

1. All vectors along the line $y = mx$.

2. All vectors parallel to the x -axis.
3. All vectors parallel to the y -axis.
4. All vectors perpendicular to the line $y = mx$.

Use geometry to explain why the eigenspaces you found make sense.

Problem 23 Prove that the eigenvectors associated with an eigenvalue λ of an $n \times n$ matrix A , together with the zero vector, form a subspace of $\mathbb{R}^n$.

Exercise Source

Practice Problem ?? is adopted from Problem 7.1.11 of Ken Kuttler's *A First Course in Linear Algebra*. (CC-BY)
Ken Kuttler, *A First Course in Linear Algebra*, Lyryx 2017, Open Edition, p. 361.

3 Similar Matrices

Definition 6. If A and B are $n \times n$ matrices, we say that A and B are *similar*, if $B = P^{-1}AP$ for some invertible matrix P . In this case we write $A \sim B$.

Similar matrices share many properties.

Theorem 5. If A and B are $n \times n$ matrices and $A \sim B$, then

1. $\det(A) = \det(B)$,
2. $\text{rank}(A) = \text{rank}(B)$,
3. A and B have the same characteristic equations, and
4. A and B have the same eigenvalues.

Proof Let $B = P^{-1}AP$ for some invertible matrix P .

For ??, $\det B = \det(P^{-1}) \det A \det P = \det A$ because $\det(P^{-1}) = 1 / \det P$ (Theorem ??).

Similarly, for ?? $\text{rank} B = \text{rank}(P^{-1}AP) = \text{rank} A$, because multiplication by an invertible matrix cannot change the rank. To see this, note that any invertible matrix is a product of elementary matrices. Multiplying by elementary matrices is equivalent to performing elementary row (column) operations on A , which does not change the row (column) space, nor the rank. It follows that similar matrices have the same rank.

As for ??,

$$\begin{aligned} \det(B - \lambda I) &= \det\{P^{-1}AP - \lambda(P^{-1}P)\} \\ &= \det\{P^{-1}(A - \lambda I)P\} \\ &= \det(A - \lambda I), \end{aligned}$$

so A and B have the same characteristic equation. Finally, ?? implies ?? because the eigenvalues of a matrix are the roots of its characteristic polynomial. ■

Remark 4. Sharing the four properties in Theorem ?? does not guarantee that two matrices are similar. The matrices

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ have the same determinant, rank, characteristic polynomial, and eigenvalues, but they are not similar because $P^{-1}IP = I$ for any invertible matrix P .

The next theorem shows that similarity is preserved under inverses, transposes, and powers:

Theorem 6. If A and B are $n \times n$ matrices and $A \sim B$, then

1. $A^{-1} \sim B^{-1}$ (provided that the inverses exist),
2. $A^T \sim B^T$, and
3. $A^k \sim B^k$ for all integers $k \geq 1$.

Proof See Practice Problem ??. ■

Practice Problems

Problem 24 Prove that if A and B are $n \times n$ matrices and $A \sim B$, then

1. $A^{-1} \sim B^{-1}$ (provided that the inverses exist),
2. $A^T \sim B^T$, and
3. $A^k \sim B^k$ for all integers $k \geq 1$.

(See Theorem ??)

Problem 25 Let

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & -2 \\ 0 & -2 & 1 \end{bmatrix}$$

Question 25.1 The eigenvalues of A are known to be $\lambda_1 = -3$, $\lambda_2 = 3$ and $\lambda_3 = 1$. Find the corresponding eigenspaces.

$$S_{-3} = \text{span} \left(\begin{bmatrix} \dots \\ \dots \\ 1 \end{bmatrix} \right)$$

$$S_3 = \text{span} \left(\begin{bmatrix} \dots \\ \dots \\ 1 \end{bmatrix} \right)$$

$$S_1 = \text{span} \left(\begin{bmatrix} \dots \\ \dots \\ 1 \end{bmatrix} \right)$$

We will refer to the eigenvectors you listed above as $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$.

Question 25.2 Form matrix P whose columns are the eigenvectors $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ you found in the previous part. Use technology to find P^{-1} .

$$P^{-1} = \frac{1}{6} \begin{bmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{bmatrix}$$

Question 25.3 Find the product $P^{-1}AP$. What did you get? Which of the following is true?

Multiple Choice:

1. $A \sim P$
2. $A \sim D$ for ANY diagonal matrix D
3. $A \sim D$ for the diagonal matrix D whose diagonal entries are the eigenvalues of A

Problem 26 Let λ be an eigenvalue of A with corresponding eigenvector $\mathbf{x}$. If $B = P^{-1}AP$ is similar to A , show that $P^{-1}\mathbf{x}$ is an eigenvector of B corresponding to λ .

Text Source

The text in this section is a compilation of material from Section 7.2.1 of Ken Kuttler's *A First Course in Linear Algebra* (CC-BY) and Section 5.5 of Keith Nicholson's *Linear Algebra with Applications* (CC-BY-NC-SA).

Ken Kuttler, *A First Course in Linear Algebra*, Lyryx 2017, Open Edition, p. 362-364.