
Week 12 new

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Contents

1 The Characteristic Equation

Let A be an $n \times n$ matrix. In the previous section we learned that the eigenvectors and eigenvalues of A are vectors $\mathbf{x}$ and scalars λ that satisfy the equation

$$A\mathbf{x} = \lambda\mathbf{x} \quad (1)$$

We listed a few reasons why we are interested in finding eigenvalues and eigenvectors, but we did not give any process for finding them. In this section we will focus on a process which can be used for small matrices. For larger matrices, the best methods we have are iterative methods, and we will explore some of these in [The Power Method and the Dominant Eigenvalue](#).

For an $n \times n$ matrix, we will see that the eigenvalues are the roots of a polynomial called the *characteristic polynomial*. So finding eigenvalues is equivalent to solving a polynomial equation of degree n . Finding the corresponding eigenvectors turns out to be a matter of computing the null space of a matrix, as the following exploration demonstrates.

Exploration 1 If a vector $\mathbf{x}$ is an eigenvector satisfying Equation (1), then clearly it also satisfies $A\mathbf{x} - \lambda\mathbf{x} = (0/0)$.

It seems natural at this point to try to factor. We would love to “factor out” $\mathbf{x}$. Here is the procedure:

$$\begin{aligned} A\mathbf{x} - \lambda\mathbf{x} &= \mathbf{0} \\ A\mathbf{x} - \lambda I\mathbf{x} &= \mathbf{0} \\ (A - \lambda I)\mathbf{x} &= \mathbf{0} \end{aligned}$$

This shows that any eigenvector $\mathbf{x}$ of A is in the (row space/ column space/ null space) of the related matrix, $A - \lambda I$.

Since eigenvectors are non-zero vectors, this means that A will have eigenvectors if and only if the null space of $A - \lambda I$ is nontrivial. The only way that $\text{null}(A - \lambda I)$ can be nontrivial is if $\text{rank}(A - \lambda I) (= / < / >) n$.

If the rank of an $n \times n$ matrix is less than n , then the matrix is singular. Since $(A - \lambda I)$ must be singular for any eigenvalue λ , we see that λ is an eigenvalue of A if and only if

$$\det(A - \lambda I) = \dots \dots \quad (2)$$

In theory, Exploration ?? offers us a way to find eigenvalues. To find the eigenvalues of A , one can solve Equation (2) for λ .

Eigenvalues

Definition 1. The equation

$$\det(A - \lambda I) = 0$$

is called the *characteristic equation* of A .

Example 1. Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Compute the eigenvalues of this matrix using the characteristic equation.

Explanation.

$$\begin{aligned}\det(A - \lambda I) &= \begin{vmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{vmatrix} = (2 - \lambda)^2 - 1 \\ &= \lambda^2 - 4\lambda + 3 \\ &= (\lambda - 1)(\lambda - 3)\end{aligned}$$

The characteristic equation $(\lambda - 1)(\lambda - 3) = 0$ has solutions $\lambda_1 = 1$ and $\lambda_2 = 3$. These are the eigenvalues of A .

Example 2. Let $B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$. Compute the eigenvalues of B using the characteristic equation. (List your answers in an increasing order.)

$\lambda_1 = \dots$ and $\lambda_2 = \dots$

Example 3. Let $C = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$. Compute the eigenvalues of C using the characteristic equation.

Explanation.

$$\begin{aligned}\det(C - \lambda I) &= \begin{vmatrix} 2 - \lambda & 1 & 1 \\ 1 & 2 - \lambda & 1 \\ 1 & 1 & 2 - \lambda \end{vmatrix} \\ &= (2 - \lambda) \begin{vmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 1 & 2 - \lambda \end{vmatrix} + 1 \begin{vmatrix} 1 & 2 - \lambda \\ 1 & 1 \end{vmatrix} \\ &= (2 - \lambda)^3 - (2 - \lambda) - ((2 - \lambda) - 1) + 1 - (2 - \lambda) \\ &= 8 - 12\lambda + 6\lambda^2 - \lambda^3 - 2 + \lambda - 2 + \lambda + 1 + 1 - 2 + \lambda \\ &= 4 - 9\lambda + 6\lambda^2 - \lambda^3 \\ &= -(\lambda - 4)(\lambda - 1)^2\end{aligned}$$

Matrix C has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 4$.

Remark 1. The factor $(\lambda - 1)$ appears twice. This repeated factor gives rise to the eigenvalue $\lambda_1 = 1$. We say that the eigenvalue $\lambda_1 = 1$ has *algebraic multiplicity* 2.

Definition 2. *Algebraic multiplicity* of an eigenvalue is its multiplicity as a root of the characteristic equation.

It is not always possible to completely factor the characteristic polynomial using only real numbers. However, a fundamental fact from algebra is that every degree n polynomial has n roots (counting multiplicity) provided that we allow complex numbers. This is why sometimes eigenvalues and their corresponding eigenvectors involve complex numbers. The next example illustrates this point.

Example 4. Let $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$. Compute the eigenvalues of this matrix.

Explanation.

$$\begin{aligned}
 \det(D - \lambda I) &= \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & 1 - \lambda & 1 \\ 0 & -1 & 1 - \lambda \end{vmatrix} \\
 &= -\lambda \begin{vmatrix} 1 - \lambda & 1 \\ -1 & 1 - \lambda \end{vmatrix} \\
 &= -\lambda((1 - \lambda)^2 + 1) \\
 &= -\lambda(\lambda^2 - 2\lambda + 2)
 \end{aligned}$$

So one of the eigenvalues of D is $\lambda = 0$. To get the other eigenvalues we must solve $\lambda^2 - 2\lambda + 2 = 0$. Using the quadratic formula, we compute that $\lambda_1 = 1 + i$ and $\lambda_2 = 1 - i$ are also eigenvalues of D .

Exploration 2 Let $T = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{bmatrix}$. Compute the eigenvalues of this matrix. (List your answers in an increasing order.)

$$\lambda_1 = \dots, \quad \lambda_2 = \dots, \quad \text{and} \quad \lambda_3 = \dots$$

What do you observe about the eigenvalues?

What property of the matrix makes this “coincidence” possible?

The matrix in Exploration Problem ?? is a triangular matrix, and the property you observed holds in general.

Theorem 1. Let T be a triangular matrix. Then the eigenvalues of T are the entries on the main diagonal.

Proof See Practice Problem ??. ■

Corollary 1. Let D be a diagonal matrix. Then the eigenvalues of D are the entries on the main diagonal.

One final note about eigenvalues. We began this section with the sentence, “In theory, then, to find the eigenvalues of A , one can solve Equation (??) for λ .” In general, one does not attempt to compute eigenvalues by solving the characteristic equation of a matrix, as there is no simple way to solve this polynomial equation for $n > 4$. Instead, one can often approximate the eigenvalues using *iterative methods*. We will explore some of these techniques in [The Power Method and the Dominant Eigenvalue](#).

Eigenvectors

Once we have computed an eigenvalue λ of an $n \times n$ matrix A , the next step is to compute the associated eigenvectors. In other words, we seek vectors $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$, or equivalently,

$$(A - \lambda I)\mathbf{x} = \mathbf{0} \tag{3}$$

As we have seen in the previous section, a (non-zero) scalar multiple of an eigenvector is an eigenvector. So for any given eigenvalue λ there are infinitely many eigenvectors associated with it. In fact, the eigenvectors associated with λ , together with the zero vector, form a subspace of $\mathbb{R}^n$.

Theorem 2. Let A be an $n \times n$ matrix and let λ be an eigenvalue of A . Then the set of all eigenvectors associated with λ , together with the zero vector, is a subspace of $\mathbb{R}^n$.

Proof See Practice Problems ?? and ??. ■

This motivates the following definition.

Definition 3. The set of all eigenvectors associated with a given eigenvalue of a matrix, together with the zero vector, is known as the *eigenspace* associated with that eigenvalue.

Given an eigenvalue λ , there is an associated eigenspace S_λ . Our goal is to find a basis of S_λ . Then any eigenvector $\mathbf{x}$ associated with λ will be a linear combination of the vectors in that basis. In seeking a basis for S_λ , we are trying to find a basis for the set of vectors that satisfy $(A - \lambda I)\mathbf{x} = \mathbf{0}$, which means we seek a basis for $\text{null}(A - \lambda I)$.

Let's return to the examples we did in the first part of this section.

Example 5. (Finding eigenvectors for Example ??)

Recall that $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 3$. Compute a basis for the eigenspace associated with each of these eigenvalues.

Explanation. Eigenvectors associated with the eigenvalue $\lambda_1 = 1$ are in the null space of $A - I$. So we seek a basis for $\text{null}(A - I)$. We compute:

$$\text{rref}(A - I) = \text{rref} \left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$

From this we see that the eigenspace S_1 associated with $\lambda_1 = 1$ consists of vectors of the form $\begin{bmatrix} -1 \\ 1 \end{bmatrix} t$. This means

that $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ is one possible basis for S_1 .

In a similar way, we compute a basis for S_3 , the subspace of all eigenvectors associated with the eigenvalue $\lambda_2 = 3$. Now we compute:

$$\text{rref}(A - 3I) = \text{rref} \left(\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \right) = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix},$$

Vectors in the null space have the form $\begin{bmatrix} 1 \\ 1 \end{bmatrix} t$. This means that $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is one possible basis for the eigenspace S_3 .

These computations reinforce the geometric insights into this problem that we gained in the previous section.

Example 6. (Finding eigenvectors for Example ??) We know from Example ?? that $B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 4$. Compute a basis for the eigenspace associated with each of these eigenvalues.

Explanation. Let's begin by finding a basis for the eigenspace S_0 , which is the subspace of $\mathbb{R}^n$ consisting of eigenvectors corresponding to the eigenvalue $\lambda_1 = 0$. We need to compute a basis for $\text{null}(B - 0I) = \text{null}(B)$. We compute:

$$\text{rref}(B) = \text{rref}\left(\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix},$$

From this we see that an eigenvector in S_0 has the form $\begin{bmatrix} -1/2 \\ 1 \end{bmatrix} t$. This means that $\left\{ \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \right\}$ is one possible basis for the eigenspace S_0 . By letting $t = -2$, we obtain an arguably nicer-looking basis: $\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$.

To compute a basis for S_4 , the subspace of all eigenvectors associated to the eigenvalue $\lambda_2 = 4$, we compute:

$$\text{rref}(B - 4I) = \text{rref}\left(\begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}\right) = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

From this we find that $\left\{ \begin{bmatrix} 1 \\ \dots \end{bmatrix} \right\}$ is one possible basis for the eigenspace S_4 .

Example 7. (Finding eigenvectors for Example ??) We know from Example ?? that $C = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 4$. Compute a basis for the eigenspace associated to each of these eigenvalues.

Explanation. We first find a basis for the eigenspace S_1 . We need to compute a basis for $\text{null}(C - I)$. We compute:

$$\text{rref}(C - I) = \text{rref}\left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

Notice that there are two free variables. The eigenvectors in S_1 have the form

$$\begin{bmatrix} -s - t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

So one possible basis for the eigenspace S_1 is given by $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Next we find a basis for the eigenspace S_4 . We need to compute a basis for $\text{null}(C - 4I)$. We compute:

$$\text{rref}(C - 4I) = \text{rref}\left(\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

This time there is one free variable. The eigenvectors in $\mathcal{S}_4$ have the form $\begin{bmatrix} t \\ t \\ t \end{bmatrix}$, so a possible basis for the eigenspace

$$\mathcal{S}_4 \text{ is given by } \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Remark 2. Recall that $\lambda_1 = 1$ has algebraic multiplicity 2. Note that $\dim(\mathcal{S}_{\lambda_1}) = 2$. We say that λ_1 has geometric multiplicity 2. We have to be careful, and not draw hasty conclusions. Geometric and algebraic multiplicity are NOT always the same! We will revisit this issue later in the text.

Definition 4. The *geometric multiplicity* of an eigenvalue is the dimension of the eigenspace corresponding to it.

Example 8. (Finding eigenvectors for Example ??) We know from Example ?? that $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$ has eigen-

values $\lambda = 0$, $\lambda_1 = 1 + i$, and $\lambda_2 = 1 - i$. Compute a basis for the eigenspace associated with each eigenvalue.

Explanation. We first find a basis for the eigenspace $\mathcal{S}_0$. We need to compute a basis for $\text{null}(D - 0I) = \text{null}(D)$. We compute:

$$\text{rref}(D) = \text{rref} \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \right) = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

From this we see that for any eigenvector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in $\mathcal{S}_0$ we have $x_2 = 0$ and $x_3 = 0$, but x_1 is a free variable. So one possible basis for the eigenspace $\mathcal{S}_0$ is given by

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Next we find a basis for the eigenspace $\mathcal{S}_{1+i}$. We need to compute a basis for $\text{null}(D - (1 + i)I)$. We compute:

$$\begin{aligned} \text{rref}(D - (1 + i)I) &= \text{rref} \left(\begin{bmatrix} -(1 + i) & 0 & 0 \\ 0 & 1 - (1 + i) & 1 \\ 0 & -1 & 1 - (1 + i) \end{bmatrix} \right) \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & i \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

There is one free variable. Setting $x_3 = t$, we get $x_1 = 0$ and $x_2 = ti$. From this we see that eigenvectors in $\mathcal{S}_{1+i}$ have the form $\begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix} t$, so a possible basis for the eigenspace $\mathcal{S}_{1+i}$ is given by $\left\{ \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix} \right\}$. We ask you in Practice Problem

?? to show that $\left\{ \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathcal{S}_{1-i}$.

We conclude this section by establishing the significance of a matrix having an eigenvalue of zero.

Theorem 3. A square matrix has an eigenvalue of zero if and only if it is singular.

Proof A square matrix A is singular if and only if $\det A = 0$. But $\det A = 0$ if and only if $\det A - 0I = 0$, which is true if and only if zero is an eigenvalue of A . ■

Practice Problems

Problem 3 Compute the eigenvalues of A , and find the corresponding eigenspaces.

$$A = \begin{bmatrix} 4 & 1 \\ 8 & -3 \end{bmatrix}$$

Answer: (List the eigenvalues in an increasing order.)

$$\lambda_1 = \dots, \quad \lambda_2 = \dots$$

A basis for S_{λ_1} is $\left\{ \begin{bmatrix} \dots \\ 1 \end{bmatrix} \right\}$. A basis for S_{λ_2} is $\left\{ \begin{bmatrix} \dots \\ 1 \end{bmatrix} \right\}$.

Problem 4 Compute the eigenvalues of A , and find the corresponding eigenspaces.

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

Answer:

$$\lambda_1 = \dots + \dots i, \quad \lambda_2 = \dots - \dots i$$

A basis for S_{λ_1} is $\left\{ \begin{bmatrix} i \\ \dots \end{bmatrix} \right\}$. A basis for S_{λ_2} is $\left\{ \begin{bmatrix} -i \\ \dots \end{bmatrix} \right\}$.

Problem 5 Let $T = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{bmatrix}$. Compute a basis for each of the eigenspaces of this matrix, S_1 , S_5 , and S_9 .

Problem 6 Let

$$A = \begin{bmatrix} 9 & 2 & 8 \\ 2 & -6 & -2 \\ -8 & 2 & -5 \end{bmatrix}$$

Question 6.1 Compute the eigenvalues of A .

Answer:

(List your answers in an increasing order.)

$$\lambda_1 = \dots, \quad \lambda_2 = \dots, \quad \lambda_3 = \dots$$

Question 6.2 Compute a basis for each of the eigenspaces of A , S_{λ_1} , S_{λ_2} , and S_{λ_3} .

Answer: A basis for S_{λ_1} is $\left\{ \begin{bmatrix} 1 \\ \dots \\ \dots \end{bmatrix} \right\}$, a basis for S_{λ_2} is $\left\{ \begin{bmatrix} -2 \\ \dots \\ \dots \end{bmatrix} \right\}$,
and a basis for S_{λ_3} is $\left\{ \begin{bmatrix} 2 \\ \dots \\ \dots \end{bmatrix} \right\}$.

Problem 7 Complete Example ?? by showing that a basis for S_{1-i} is given by $\left\{ \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} \right\}$, where S_{1-i} is the

eigenspace associated with the eigenvalue $\lambda = 1 - i$ of the matrix $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$.

Problem 8 Prove Theorem ??.

Problem 9 Recall that a vertical stretch/compression of the plane is a linear transformation whose standard matrix is

$$M_v = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

Find the eigenvalues of M_v . Find a basis for the eigenspace corresponding to each eigenvalue.

Answer: A basis for S_1 is $\left\{ \begin{bmatrix} 1 \\ \dots \end{bmatrix} \right\}$ and a basis for S_k is $\left\{ \begin{bmatrix} \dots \\ 1 \end{bmatrix} \right\}$

Sketch several vectors in each eigenspace and use geometry to explain why the eigenvectors you sketched make sense.

Problem 10 Recall that a horizontal shear of the plane is a linear transformation whose standard matrix is

$$M_{hs} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

Find the eigenvalue of M_{hs} .

Answer: $\lambda = \dots$

Find a basis for the eigenspace corresponding to λ .

Answer: A basis for S is $\left\{ \begin{bmatrix} 1 \\ \dots \end{bmatrix} \right\}$

What is the algebraic multiplicity of λ ? ...

What is the geometric multiplicity of λ ? ...

Problem 11 Recall that a counterclockwise rotation of the plane through angle θ is a linear transformation whose standard matrix is

$$M_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

Verify that the eigenvalues of M_θ are

$$\lambda = \cos \theta \pm \sqrt{\cos^2 \theta - 1}$$

Use algebra, then use geometry to explain why λ is a real number if and only if θ is a multiple of π .

Suppose $\theta = \pi$. Then $\lambda = \dots$ has algebraic multiplicity $\dots$. Which of the following describes the corresponding eigenspace?

Multiple Choice:

1. All vectors in $\mathbb{R}^2$.
2. All vectors along the x -axis.
3. All vectors along the y -axis.
4. All vectors along the line $y = x$.

Problem 12 Recall that a reflection of the plane about the line $y = mx$ is a linear transformation whose standard matrix is

$$M_{y=mx} = \frac{1}{1+m^2} \begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}$$

Verify that the eigenvalues of $M_{y=mx}$ are 1 and -1 .

Find a basis for eigenspaces S_1 and S_{-1} . (For simplicity, assume that $m \neq 0$.)

Answer: A basis for S_1 is $\left\{ \begin{bmatrix} 1 \\ \dots \end{bmatrix} \right\}$ and a basis for S_{-1} is $\left\{ \begin{bmatrix} m \\ \dots \end{bmatrix} \right\}$

Choose the best description of S_1 .

Multiple Choice:

1. All vectors in $\mathbb{R}^2$.
2. All vectors with "slope" m .
3. All vectors with "slope" $1/m$.
4. All vectors with "slope" $-1/m$.

Choose the best description of S_{-1} .

Multiple Choice:

1. All vectors along the line $y = mx$.

2. All vectors parallel to the x -axis.
3. All vectors parallel to the y -axis.
4. All vectors perpendicular to the line $y = mx$.

Use geometry to explain why the eigenspaces you found make sense.

Problem 13 Prove that the eigenvectors associated with an eigenvalue λ of an $n \times n$ matrix A , together with the zero vector, form a subspace of $\mathbb{R}^n$.

Exercise Source

Practice Problem ?? is adopted from Problem 7.1.11 of Ken Kuttler's *A First Course in Linear Algebra*. (CC-BY)
Ken Kuttler, *A First Course in Linear Algebra*, Lyryx 2017, Open Edition, p. 361.

2 Similar Matrices

Definition 5. If A and B are $n \times n$ matrices, we say that A and B are *similar*, if $B = P^{-1}AP$ for some invertible matrix P . In this case we write $A \sim B$.

Similar matrices share many properties.

Theorem 4. If A and B are $n \times n$ matrices and $A \sim B$, then

1. $\det(A) = \det(B)$,
2. $\text{rank}(A) = \text{rank}(B)$,
3. A and B have the same characteristic equations, and
4. A and B have the same eigenvalues.

Proof Let $B = P^{-1}AP$ for some invertible matrix P .

For ??, $\det B = \det(P^{-1}) \det A \det P = \det A$ because $\det(P^{-1}) = 1 / \det P$ (Theorem ??).

Similarly, for ?? $\text{rank} B = \text{rank}(P^{-1}AP) = \text{rank} A$, because multiplication by an invertible matrix cannot change the rank. To see this, note that any invertible matrix is a product of elementary matrices. Multiplying by elementary matrices is equivalent to performing elementary row (column) operations on A , which does not change the row (column) space, nor the rank. It follows that similar matrices have the same rank.

As for ??,

$$\begin{aligned} \det(B - \lambda I) &= \det\{P^{-1}AP - \lambda(P^{-1}P)\} \\ &= \det\{P^{-1}(A - \lambda I)P\} \\ &= \det(A - \lambda I), \end{aligned}$$

so A and B have the same characteristic equation. Finally, ?? implies ?? because the eigenvalues of a matrix are the roots of its characteristic polynomial. ■

Remark 3. Sharing the four properties in Theorem ?? does not guarantee that two matrices are similar. The matrices

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ have the same determinant, rank, characteristic polynomial, and eigenvalues, but they are not similar because $P^{-1}IP = I$ for any invertible matrix P .

The next theorem shows that similarity is preserved under inverses, transposes, and powers:

Theorem 5. If A and B are $n \times n$ matrices and $A \sim B$, then

1. $A^{-1} \sim B^{-1}$ (provided that the inverses exist),
2. $A^T \sim B^T$, and
3. $A^k \sim B^k$ for all integers $k \geq 1$.

Proof See Practice Problem ??. ■

Practice Problems

Problem 14 Prove that if A and B are $n \times n$ matrices and $A \sim B$, then

1. $A^{-1} \sim B^{-1}$ (provided that the inverses exist),
2. $A^T \sim B^T$, and
3. $A^k \sim B^k$ for all integers $k \geq 1$.

(See Theorem ??)

Problem 15 Let

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & -2 \\ 0 & -2 & 1 \end{bmatrix}$$

Question 15.1 The eigenvalues of A are known to be $\lambda_1 = -3$, $\lambda_2 = 3$ and $\lambda_3 = 1$. Find the corresponding eigenspaces.

$$S_{-3} = \text{span} \left(\begin{bmatrix} \dots \\ \dots \\ 1 \end{bmatrix} \right)$$

$$S_3 = \text{span} \left(\begin{bmatrix} \dots \\ \dots \\ 1 \end{bmatrix} \right)$$

$$S_1 = \text{span} \left(\begin{bmatrix} \dots \\ \dots \\ 1 \end{bmatrix} \right)$$

We will refer to the eigenvectors you listed above as $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$.

Question 15.2 Form matrix P whose columns are the eigenvectors $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ you found in the previous part. Use technology to find P^{-1} .

$$P^{-1} = \frac{1}{6} \begin{bmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{bmatrix}$$

Question 15.3 Find the product $P^{-1}AP$. What did you get? Which of the following is true?

Multiple Choice:

1. $A \sim P$
2. $A \sim D$ for ANY diagonal matrix D
3. $A \sim D$ for the diagonal matrix D whose diagonal entries are the eigenvalues of A

Problem 16 Let λ be an eigenvalue of A with corresponding eigenvector $\mathbf{x}$. If $B = P^{-1}AP$ is similar to A , show that $P^{-1}\mathbf{x}$ is an eigenvector of B corresponding to λ .

Text Source

The text in this section is a compilation of material from Section 7.2.1 of Ken Kuttler's *A First Course in Linear Algebra* (CC-BY) and Section 5.5 of Keith Nicholson's *Linear Algebra with Applications* (CC-BY-NC-SA).

Ken Kuttler, *A First Course in Linear Algebra*, Lyryx 2017, Open Edition, p. 362-364.

3 Diagonalizable Matrices and Multiplicity

Recall that a *diagonal matrix* D is a matrix containing a zero in every entry except those on the main diagonal. More precisely, if d_{ij} is the ij^{th} entry of a diagonal matrix D , then $d_{ij} = 0$ unless $i = j$. Such matrices look like the following.

$$D = \begin{bmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{bmatrix}$$

where $*$ is a number which might not be zero.

Diagonal matrices have some nice properties, as we demonstrate below.

Exploration 17 Let $M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ and let $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 10 \end{bmatrix}$. Compute MD and DM using Octave.

```
% Define M
M=[1 2 3; 4 5 6; 7 8 9]

% The function diag(vector) creates a square matrix with diagonal
%vector and the remaining entries zero.
D=diag([2 -5 10])

% The function diag() can also be used to return the diagonal of a matrix.
diag_M=diag(M)

% Understanding multiplication by a diagonal matrix
product_MD=M*D
product_DM=D*M

% Raising a diagonal matrix to a power
D5=D^5
```

[Link to code.](#)

Notice the patterns present in the product matrices.

- Each row of DM is the same as its corresponding row of M multiplied by the scalar which is the corresponding diagonal element of D .
- In the product MD , it is the columns of M that have been multiplied by the diagonal elements.
- Raising D to the 5^{th} power amounts to raising each diagonal element to the 5^{th} power.

These patterns hold in general for any diagonal matrix, and they are fundamental to understanding diagonalization and its uses. We discuss diagonalization next.

Definition 6. Let A be an $n \times n$ matrix. Then A is said to be *diagonalizable* if there exists an invertible matrix P such that

$$P^{-1}AP = D$$

where D is a diagonal matrix. In other words, a matrix A is diagonalizable if it is similar to a diagonal matrix, $A \sim D$.

If we are given a matrix A that is diagonalizable, then we can write $P^{-1}AP = D$ for some matrix P , or, equivalently,

$$AP = PD \quad (4)$$

If we pause to examine Equation (??), the work that we did in Exploration ?? can help us to understand how to find P that will diagonalize A . The product PD is formed by multiplying each column of P by a scalar which is the corresponding element on the diagonal of D . To restate this, if $\mathbf{x}_i$ is column i in our matrix P , then Equation (??) tells us that

$$A\mathbf{x}_i = \lambda_i \mathbf{x}_i, \quad (5)$$

where λ_i is the i th diagonal element of D .

Of course, Equation (??) is very familiar! We see that if we are able to diagonalize a matrix A , the columns of matrix P will be the eigenvectors of A , and the corresponding diagonal entries of D will be the corresponding eigenvalues of A . This is summed up in the following theorem.

Theorem 6. An $n \times n$ matrix A is diagonalizable if and only if there is an invertible matrix P given by

$$P = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \\ | & | & \cdots & | \end{bmatrix},$$

where the columns $\mathbf{x}_i$ are eigenvectors of A .

Moreover, if A is diagonalizable, the corresponding eigenvalues of A are the diagonal entries of the diagonal matrix D .

Proof Suppose P is given as above as an invertible matrix whose columns are eigenvectors of A . To show that A is diagonalizable, we will show

$$AP = PD,$$

which is equivalent to $P^{-1}AP = D$. We have

$$AP = \begin{bmatrix} | & | & \cdots & | \\ A\mathbf{x}_1 & A\mathbf{x}_2 & \cdots & A\mathbf{x}_n \\ | & | & \cdots & | \end{bmatrix},$$

while

$$PD = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ \lambda_1 \mathbf{x}_1 & \lambda_2 \mathbf{x}_2 & \cdots & \lambda_n \mathbf{x}_n \\ | & | & \cdots & | \end{bmatrix}$$

We can complete this half of the proof by comparing columns, and noting that

$$A\mathbf{x}_i = \lambda_i \mathbf{x}_i \quad (6)$$

for $i = 1, \dots, n$ since the $\mathbf{x}_i$ are eigenvectors of A and the λ_i are corresponding eigenvalues of A .

Conversely, suppose A is diagonalizable so that $P^{-1}AP = D$. Let

$$P = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \\ | & | & \cdots & | \end{bmatrix}$$

where the columns are the vectors $\mathbf{x}_i$ and

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

Then

$$AP = PD = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

and so

$$\begin{bmatrix} | & | & \cdots & | \\ A\mathbf{x}_1 & A\mathbf{x}_2 & \cdots & A\mathbf{x}_n \\ | & | & \cdots & | \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ \lambda_1\mathbf{x}_1 & \lambda_2\mathbf{x}_2 & \cdots & \lambda_n\mathbf{x}_n \\ | & | & \cdots & | \end{bmatrix}$$

showing the $\mathbf{x}_i$ are eigenvectors of A and the λ_i are eigenvalues. ■

Notice that because the matrix P defined above is invertible it follows that the set of eigenvectors of A , $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$, is a basis of $\mathbb{R}^n$.

We demonstrate the concept given in the above theorem in the next example. Note that not only are the columns of the matrix P formed by eigenvectors, but P must be invertible, and therefore must consist of a linearly independent set of eigenvectors.

Example 9. Let

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{bmatrix}$$

Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

Explanation. We will use eigenvectors of A as the columns of P , and the corresponding eigenvalues of A as the diagonal entries of D . The eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = 2$, and $\lambda_3 = 6$. We leave these computations as exercises, as well as the computations to find a basis for each eigenspace.

One possible basis for $\mathcal{S}_2$, the eigenspace corresponding to 2, is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$, while a basis for $\mathcal{S}_6$ is given by

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \right\}.$$

We construct the matrix P by using these basis elements as columns.

$$P = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

You can verify (See Practice Problem) that

$$P^{-1} = \begin{bmatrix} -1/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & -1/4 \end{bmatrix}$$

Thus,

$$\begin{aligned} P^{-1}AP &= \begin{bmatrix} -1/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & -1/4 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix} \end{aligned}$$

You can see that the result here is a diagonal matrix where the entries on the main diagonal are the eigenvalues of A . Notice that eigenvalues on the main diagonal *must* be in the same order as the corresponding eigenvectors in P .

Exploration 18 We will continue to use matrices from Example ?? to illustrate why diagonalizable matrices are easier to work with. Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{bmatrix}$ and let $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}$. Would it be easier to compute A^5 or D^5 if you had to do so by hand, without a computer? Certainly D^5 is easier, due to the number of zero entries!

Raising a diagonal matrix to a power amounts to simply raising each entry to that same power, whereas computing A^5 requires many more calculations. However, we learned in Example ?? that A is similar to D . We will use this to make our computation easier as follows

$$\begin{aligned} A^5 &= (PDP^{-1})^5 \\ &= (PDP^{-1})(PDP^{-1})(PDP^{-1})(PDP^{-1})(PDP^{-1}) \\ &= PD(P^{-1}P)D(P^{-1}P)D(P^{-1}P)D(P^{-1}P)DP^{-1} \\ &= PD(I)D(I)D(I)D(I)DP^{-1} \\ &= PD^5P^{-1} \end{aligned}$$

With this in mind, it is not as daunting to calculate A^5 by hand.

$$A^5 = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 32 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 7776 \end{bmatrix} \begin{bmatrix} -1/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & -1/4 \end{bmatrix} = \begin{bmatrix} 32 & 0 & 0 \\ 1936 & 3904 & -1936 \\ -3872 & -7744 & 3904 \end{bmatrix}$$

The savings in work would certainly be more pronounced for larger matrices or for powers larger than 5.

In Exploration ??, because matrix A was diagonalizable, we were able to cut down on computations. When we chose to work with D and P instead of A we worked with the eigenvalues and eigenvectors of A . Each column of P is an eigenvector of A , and so we repeatedly made use of the following theorem (with $m = 5$).

Theorem 7. Let A be an $n \times n$ matrix and suppose $A\mathbf{x} = \lambda\mathbf{x}$. Then $A^m\mathbf{x} = \lambda^m\mathbf{x}$

Proof We prove this theorem by induction on m . Clearly $A^m\mathbf{x} = \lambda^m\mathbf{x}$ holds when $m = 1$, as that was given. For the inductive step, suppose that we know $A^{m-1}\mathbf{x} = \lambda^{m-1}\mathbf{x}$. Then

$$\begin{aligned} A^m\mathbf{x} &= (AA^{m-1})\mathbf{x} \\ &= A(A^{m-1}\mathbf{x}) \\ &= A(\lambda^{m-1}\mathbf{x}) \\ &= \lambda^{m-1}A\mathbf{x} \\ &= \lambda^{m-1}\lambda\mathbf{x} \\ &= \lambda^m\mathbf{x} \end{aligned}$$

as desired. ■

Matrix A from the Example ?? and Exploration ?? had a repeated eigenvalue of 2. The next theorem and corollary show that matrices which have *distinct* eigenvalues (where none are repeated) have desirable properties.

Theorem 8. Let A be an $n \times n$ matrix, and suppose that A has distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_m$. For each i , let $\mathbf{x}_i$ be a λ_i -eigenvector of A . Then $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ is linearly independent.

Proof We prove this by induction on m , the number of vectors in the set. If $m = 1$, then $\{\mathbf{x}_1\}$ is a linearly independent set because $\mathbf{x}_1 \neq \mathbf{0}$. In general, suppose we have established that the theorem is true for some $m \geq 1$. Given eigenvectors $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{m+1}\}$, suppose

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \dots + c_{m+1}\mathbf{x}_{m+1} = \mathbf{0}. \quad (7)$$

We must show that each $c_i = 0$. Multiply both sides of (??) on the left by A and use the fact that $A\mathbf{x}_i = \lambda_i\mathbf{x}_i$ to get

$$c_1\lambda_1\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2 + \dots + c_{m+1}\lambda_{m+1}\mathbf{x}_{m+1} = \mathbf{0}. \quad (8)$$

If we multiply (??) by λ_1 and subtract the result from (??), the first terms cancel and we obtain

$$c_2(\lambda_2 - \lambda_1)\mathbf{x}_2 + c_3(\lambda_3 - \lambda_1)\mathbf{x}_3 + \dots + c_{m+1}(\lambda_{m+1} - \lambda_1)\mathbf{x}_{m+1} = \mathbf{0}.$$

Since $\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_{m+1}$ correspond to distinct eigenvalues $\lambda_2, \lambda_3, \dots, \lambda_{m+1}$, the set $\{\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_{m+1}\}$ is linearly independent by the induction hypothesis. Hence,

$$c_2(\lambda_2 - \lambda_1) = 0, \quad c_3(\lambda_3 - \lambda_1) = 0, \quad \dots, \quad c_{m+1}(\lambda_{m+1} - \lambda_1) = 0$$

and so $c_2 = c_3 = \dots = c_{m+1} = 0$ because the λ_i are distinct. It follows that (??) becomes $c_1\mathbf{x}_1 = \mathbf{0}$, which implies that $c_1 = 0$ because $\mathbf{x}_1 \neq \mathbf{0}$, and the proof is complete. ■

The corollary that follows from this theorem gives a useful tool in determining if A is diagonalizable.

Corollary 2. Let A be an $n \times n$ matrix and suppose it has n distinct eigenvalues. Then it follows that A is diagonalizable.

Remark 4. Note that Corollary ?? is NOT an “if and only if statement”. This means that if A has repeated eigenvalues it is still sometimes possible to diagonalize A , as seen in Example ??.

Definition 7. If we are able to diagonalize A , say $A = PDP^{-1}$, we say that PDP^{-1} is an *eigenvalue decomposition* of A .

Not every matrix has an eigenvalue decomposition. Sometimes we cannot find an invertible matrix P such that $P^{-1}AP = D$. Consider the following example.

Example 10. Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

If possible, find an invertible matrix P and a diagonal matrix D so that $P^{-1}AP = D$.

Explanation. We see immediately (how?) that the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 1$. To find P , the next step would be to find a basis for the corresponding eigenspace S_1 . We solve the equation $(A - \lambda I)\mathbf{x} = \mathbf{0}$. Writing this equation as an augmented matrix, we already have a matrix in row echelon form:

$$\left[\begin{array}{cc|c} 0 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

We see that the eigenvectors in S_1 are of the form

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

so a basis for the eigenspace S_1 is given by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. It is easy to see that we cannot form an invertible matrix P , because any two eigenvectors will be of the form $\begin{bmatrix} t \\ 0 \end{bmatrix}$, and so the second row of P would have a row of zeros, and P could not be invertible. Hence A cannot be diagonalized.

We saw earlier in Corollary ?? that an $n \times n$ matrix with n distinct eigenvalues is diagonalizable. It turns out that there are other useful diagonalizability tests.

Recall that the *algebraic multiplicity* of an eigenvalue λ is the number of times that it occurs as a root of the characteristic polynomial.

Definition 8. The *geometric multiplicity* of an eigenvalue λ is the dimension of the corresponding eigenspace S_λ .

Consider now the following lemma.

Lemma 1. Let A be an $n \times n$ matrix, and let S_{λ_1} be the eigenspace corresponding to the eigenvalue λ_1 which has algebraic multiplicity m . Then

$$\dim(S_{\lambda_1}) \leq m$$

In other words, the geometric multiplicity of an eigenvalue is less than or equal to the algebraic multiplicity of that same eigenvalue.

Proof Let k be the geometric multiplicity of λ_1 , i.e., $k = \dim(S_{\lambda_1})$. Suppose $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ is a basis for the eigenspace S_{λ_1} . Let P be any invertible matrix having $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ as its first k columns, say

$$P = \begin{bmatrix} | & | & & | & | & & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_k & \mathbf{x}_{k+1} & \cdots & \mathbf{x}_n \\ | & | & & | & | & & | \end{bmatrix}.$$

In block form we may write

$$P = \begin{bmatrix} B & C \end{bmatrix} \quad \text{and} \quad P^{-1} = \begin{bmatrix} D \\ E \end{bmatrix}$$

where B is $n \times k$, C is $n \times (n - k)$, D is $k \times n$, and E is $(n - k) \times n$. We observe $I_n = P^{-1}P = \left[\begin{array}{c|c} DB & DC \\ \hline EB & EC \end{array} \right]$. This implies

$$DB = I_k, \quad DC = O_{k \times (n-k)}, \quad EB = O_{(n-k) \times k} \quad \text{and} \quad EC = I_{n-k}$$

Therefore,

$$\begin{aligned} P^{-1}AP &= \begin{bmatrix} D \\ E \end{bmatrix} A \begin{bmatrix} B & C \end{bmatrix} = \left[\begin{array}{c|c} DAB & DAC \\ \hline EAB & EAC \end{array} \right] \\ &= \left[\begin{array}{c|c} \lambda_1 DB & DAC \\ \hline \lambda_1 EB & EAC \end{array} \right] = \left[\begin{array}{c|c} \lambda_1 I_k & DAC \\ \hline O & EAC \end{array} \right] \end{aligned}$$

We finish the proof by comparing the characteristic polynomials on both sides of this equation, and making use of the fact that similar matrices have the same characteristic polynomials.

$$\det(A - \lambda I) = \det(P^{-1}AP - \lambda I) = (\lambda_1 - \lambda)^k \det(EAC)$$

We see that the characteristic polynomial of A has $(\lambda_1 - \lambda)^k$ as a factor. This tells us that algebraic multiplicity of λ_1 is at least k , proving the desired inequality. ■

This result tells us that if λ is an eigenvalue of A , then the number of linearly independent λ -eigenvectors is never more than the multiplicity of λ . We now use this fact to provide a useful diagonalizability condition.

Theorem 9. Let A be an $n \times n$ matrix A . Then A is diagonalizable if and only if for each eigenvalue λ of A , the algebraic multiplicity of λ is equal to the geometric multiplicity of λ .

Proof Suppose A is diagonalizable and let $\lambda_1, \dots, \lambda_t$ be the distinct eigenvalues of A , with algebraic multiplicities $m_1, \dots, m_t$, respectively and geometric multiplicities $k_1, \dots, k_t$, respectively. Since A is diagonalizable, Theorem ?? implies that $k_1 + \dots + k_t = n$. By applying Lemma ?? t times, we have

$$n = k_1 + \dots + k_t \leq m_1 + \dots + m_t = n,$$

which is only possible if $k_i = m_i$ for $i = 1, \dots, t$.

Conversely, if the geometric multiplicity equals the algebraic multiplicity of each eigenvalue, then obtaining a basis for each eigenspace yields n eigenvectors. Applying Theorem ??, we know that these n eigenvectors are linearly independent, so Theorem ?? implies that A is diagonalizable. ■

Practice Problems

Problem 19 Fill in the details of Example ??.

Question 19.1 Find the eigenvalues of matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{bmatrix}$$

Question 19.2 Find a basis for each eigenspace of the matrix A .

Question 19.3 Compute the inverse of

$$P = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

Question 19.4 Compute $D = P^{-1}AP$

Question 19.5 Show that computing the inverse of P is not really necessary by comparing the products PD and AP .

Problem 20 Let $A = \begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix}$

Question 20.1 Find the eigenvalues of A and list them in increasing order.

$$\lambda_1 = \dots, \quad \lambda_2 = \dots$$

Question 20.2 Find a diagonalization for A

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = P^{-1}AP = \frac{1}{4} \begin{bmatrix} 1 & \dots \\ \dots & 1 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & \dots \\ \dots & 1 \end{bmatrix}$$

Question 20.3 Find A^6 .

$$A^6 = \begin{bmatrix} \dots & \dots \\ \dots & \dots \end{bmatrix}$$

Problem 21 Explain why the shearing matrix $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$, where $k \neq 0$ is not diagonalizable.

Problem 22 Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Question 22.1 Find eigenvalues of A and list them in increasing order.

$$\lambda_1 = \dots, \quad \lambda_2 = \dots, \quad \lambda_3 = \dots$$

Question 22.2 Find a diagonalization for A .

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = P^{-1}AP = \begin{bmatrix} \dots & \dots & \dots \\ 0 & 0 & 1 \\ \dots & \dots & \dots \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dots & \dots & \dots \\ 1 & \dots & 1 \\ \dots & 1 & \dots \end{bmatrix}$$

Problem 23 Let

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 2 \\ 2 & 3 & 1 \end{bmatrix}$$

$\lambda_1 = 4$ is an eigenvalue of algebraic multiplicity ...

$$\mathcal{S}_{\lambda_1} = \text{span} \left(\begin{bmatrix} \dots \\ \dots \\ 1 \end{bmatrix} \right)$$

Geometric multiplicity of λ_1 is ...

Explain why A is not diagonalizable.

Problem 24 Let A denote an $n \times n$ upper triangular matrix. If all the main diagonal entries of A are distinct, show that A is diagonalizable.

Problem 25 Let A be a diagonalizable $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ (including multiplicities). Show that:

1. $\det A = \lambda_1 \lambda_2 \cdots \lambda_n$
2. $\text{tr} A = \lambda_1 + \lambda_2 + \cdots + \lambda_n$

Text Source

The text in this section is a compilation of material from Section 7.2.1 of Ken Kuttler's *A First Course in Linear Algebra* (CC-BY) and Section 5.5 of Keith Nicholson's *Linear Algebra with Applications* (CC-BY-NC-SA).

Ken Kuttler, *A First Course in Linear Algebra*, Lyryx 2017, Open Edition, p. 362-364.

Many of the Practice Problems are Exercises from W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2018, Open Edition, pp. 298-310.