
Week 13v1

June 5, 2026

Contents

1 Orthogonal Matrices and Symmetric Matrices

Recall that an $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors. (see [Diagonalizable Matrices and Multiplicity](#)) Moreover, the matrix P with these eigenvectors as columns is a diagonalizing matrix for A , that is

$$P^{-1}AP \text{ is diagonal.}$$

As we have seen, the nice bases of $\mathbb{R}^n$ are the orthogonal ones, so a natural question is: which $n \times n$ matrices have n orthogonal eigenvectors, so that columns of P form an orthogonal basis for $\mathbb{R}^n$? These turn out to be precisely the *symmetric matrices* (matrices for which $A = A^T$), and this is the main result of this section.

Orthogonal Matrices

A collection of non-zero, pairwise orthogonal vectors in $\mathbb{R}^n$ is called an *orthogonal* set of vectors. An orthogonal set of vectors is called *orthonormal* if $\|\mathbf{q}\| = 1$ for each vector $\mathbf{q}$ in the set. A set of orthogonal vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ can be “normalized”, i.e. converted into an orthonormal set $\left\{ \frac{1}{\|\mathbf{v}_1\|} \mathbf{v}_1, \frac{1}{\|\mathbf{v}_2\|} \mathbf{v}_2, \dots, \frac{1}{\|\mathbf{v}_k\|} \mathbf{v}_k \right\}$. In particular, if a matrix A has n orthogonal eigenvectors, they can (by normalizing) be taken to be orthonormal. The corresponding diagonalizing matrix (we will use Q instead of P) has orthonormal columns, and such matrices are very easy to invert.

Theorem 1. The following conditions are equivalent for an $n \times n$ matrix Q .

1. Q is invertible and $Q^{-1} = Q^T$.
2. The rows of Q are orthonormal.
3. The columns of Q are orthonormal.

Proof First note that condition ?? is equivalent to $Q^T Q = I$. Let $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n$ denote the columns of Q . Then $\mathbf{q}_i^T$ is the i th row of Q^T , so the (i, j) -entry of $Q^T Q$ is $\mathbf{q}_i \cdot \mathbf{q}_j$. Thus $Q^T Q = I$ means that $\mathbf{q}_i \cdot \mathbf{q}_j = 0$ if $i \neq j$ and $\mathbf{q}_i \cdot \mathbf{q}_i = 1$ if $i = j$. Hence condition ?? is equivalent to ??. The proof of the equivalence of ?? and ?? is similar. ■

Definition 1 (Orthogonal Matrices). An $n \times n$ matrix Q is called an *orthogonal matrix* if it satisfies one (and hence all) of the conditions in Theorem ??.

Example 1. The rotation matrix $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ is orthogonal for any angle θ .

Explanation. See Practice Problem ??.

Remark 1. In view of ?? and ?? of Theorem ??, *orthonormal matrix* might be a better name. But *orthogonal matrix* is standard.

It is not enough that the rows of a matrix A are merely orthogonal for A to be an orthogonal matrix. Here is an example.

Exploration 1 Let $A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$

1. Check that matrix A has rows that are orthogonal.
2. Check that matrix A has columns that are NOT orthogonal.
3. Check that matrix A has rows that are NOT orthonormal.
4. Create a matrix Q by normalizing each of the rows of A .
5. Check that Q is an orthogonal matrix.

Click the arrow to see the answer. You should get $Q = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$, and one can check that this is orthogonal in a number of ways.

This exploration can certainly be done by hand (although it takes some time), but it also makes for a very nice Octave exercise.

```
%Exploration from Section 9.4 Orthogonal Matrices and Symmetric Matrices
A=[2 1 1; -1 1 1; 0 -1 1]
%Check that matrix A has rows that are orthogonal.
A(1,:)*transpose(A(2,:))
A(2,:)*transpose(A(3,:))
A(1,:)*transpose(A(3,:))
%Check that matrix A has columns that are NOT orthogonal.
transpose(A(:,1))*A(:,2)
%(This is 1 of 3 calculations to do.)
%Check that matrix A in the Octave window has rows that are NOT orthonormal.
%(See the results from the first question.)
%Create a matrix Q by normalizing each of the rows of A.
q1=A(1,:)/norm(A(1,:));
q2=A(2,:)/norm(A(2,:));
q3=A(3,:)/norm(A(3,:));
Q = [q1;q2;q3]
%Check that Q is an orthogonal matrix.
Q*transpose(Q)
%(You may get numbers close to zero in some places you expect
%to get zero due to rounding error)
```

[Link to code.](#)

For more information about using Octave, please see [Octave for Chapter 9](#).

We studied the idea of closure when we studied [Subspaces of \$\mathbb{R}^n\$](#) . The next theorem tells us that orthogonal matrices are closed under matrix multiplication.

- Theorem 2.** 1. If P and Q are orthogonal matrices, then PQ is also orthogonal. (We say that the set of orthogonal matrices is closed under matrix multiplication.)
2. If P is an orthogonal matrix, then so is $P^{-1} = P^T$.

Proof For ??, P and Q are invertible, so PQ is also invertible and

$$(PQ)^{-1} = Q^{-1}P^{-1} = Q^T P^T = (PQ)^T$$

Hence PQ is orthogonal. For ??,

$$(P^{-1})^{-1} = P = (P^T)^T = (P^{-1})^T$$

shows that P^{-1} is orthogonal. ■

Symmetric Matrices

We now shift our focus from orthogonal matrices to another important class of $n \times n$ matrices called *symmetric* matrices. A *symmetric matrix* is a matrix which is equal to its transpose. We saw a few examples of such matrices in [Transpose of a Matrix](#).

When we began our study of eigenvalues and eigenvectors, we saw numerous examples of matrices with entries that were real numbers with eigenvalues that were complex numbers. It can be shown that symmetric matrices only have real eigenvalues. We also learned that some matrices are diagonalizable while other matrices are not. It turns out that every symmetric matrix is diagonalizable. In fact, we can say more, but first we need the following definition.

Definition 2. An $n \times n$ matrix A is said to be *orthogonally diagonalizable* if an orthogonal matrix Q can be found such that $Q^{-1}AQ = Q^T A Q$ is diagonal.

We have learned earlier that when we diagonalize a matrix A , we write $P^{-1}AP = D$ for some matrix P where D is diagonal, and the diagonal entries are the eigenvalues of A . We have also learned that the columns of the matrix P are the corresponding eigenvectors of A . So when a matrix is orthogonally diagonalizable, we are able to accomplish the diagonalization using a matrix Q consisting of n eigenvectors that form an orthonormal basis for $\mathbb{R}^n$. The following remarkable theorem shows that the matrices that have this property are precisely the symmetric matrices.

Theorem 3 (Real Spectral Theorem). Let A be an $n \times n$ matrix. Then A is symmetric if and only if A is orthogonally diagonalizable.

Proof If A is orthogonally diagonalizable, then it is an easy exercise to prove that it is symmetric. You are asked to do this in Practice Problem ??.

To prove the “only if” part of this theorem, we assume A is symmetric, and we need to show it is orthogonally diagonalizable. We proceed by induction on n , the size of the symmetric matrix. If $n = 1$, A is already diagonal. If $n > 1$, assume that we know the “only if” statement holds for $(n - 1) \times (n - 1)$ symmetric matrices. Let λ_1 be an eigenvalue of A , and let $A\mathbf{x}_1 = \lambda_1\mathbf{x}_1$, where $\|\mathbf{x}_1\| = 1$. Next, set $\mathbf{q}_1 = \mathbf{x}_1$, and use the Gram-Schmidt algorithm to find

an orthonormal basis $\{\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n\}$ for $\mathbb{R}^n$. Let $Q_1 = \begin{bmatrix} | & | & & | \\ \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_n \\ | & | & & | \end{bmatrix}$, so that Q_1 is an orthogonal matrix. We

have

$$\begin{aligned}
 Q_1^T A Q_1 &= \begin{bmatrix} \text{---} & \mathbf{q}_1^T & \text{---} \\ \text{---} & \mathbf{q}_2^T & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{q}_n^T & \text{---} \end{bmatrix} A \begin{bmatrix} | & | & & | \\ \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_n \\ | & | & & | \end{bmatrix} \\
 &= \begin{bmatrix} \text{---} & \mathbf{q}_1^T & \text{---} \\ \text{---} & \mathbf{q}_2^T & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{q}_n^T & \text{---} \end{bmatrix} \begin{bmatrix} | & | & & | \\ A\mathbf{q}_1 & A\mathbf{q}_2 & \cdots & A\mathbf{q}_n \\ | & | & & | \end{bmatrix} \\
 &= \begin{bmatrix} \lambda_1 & B \\ \mathbf{0} & A_1 \end{bmatrix},
 \end{aligned}$$

where the block B has dimensions $1 \times (n-1)$, and the block under λ_1 is a $(n-1) \times 1$ zero matrix, because of the orthogonality of the basis vectors.

Next, using the fact that A is symmetric, we notice that

$$(Q_1^T A Q_1)^T = Q_1^T A^T (Q_1^T)^T = Q_1^T A Q_1,$$

so $Q_1^T A Q_1$ is symmetric. It follows that B is also a zero matrix and that A_1 is symmetric. Since A_1 is an $(n-1) \times (n-1)$ symmetric matrix, we may apply the inductive hypothesis, so there exists an $(n-1) \times (n-1)$ orthogonal matrix Q such that $Q^T A_1 Q = D_1$ is diagonal. We observe that $Q_2 = \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix}$ is orthogonal, and we compute:

$$\begin{aligned}
 (Q_1 Q_2)^T A (Q_1 Q_2) &= Q_2^T (Q_1^T A Q_1) Q_2 \\
 &= \begin{bmatrix} 1 & 0 \\ 0 & Q^T \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & A_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix} \\
 &= \begin{bmatrix} \lambda_1 & 0 \\ 0 & D_1 \end{bmatrix}
 \end{aligned}$$

is diagonal. Because $Q_1 Q_2$ is orthogonal by Theorem 11.1, this completes the proof. ■

Because the eigenvalues of a real symmetric matrix are real, Theorem 11.1 is called the *Real Spectral Theorem*, and the set of distinct eigenvalues is called the *spectrum* of the matrix. A similar result holds for matrices with complex entries (Theorem 11.2).

Example 2. Find an orthogonal matrix Q such that $Q^{-1} A Q$ is diagonal, where $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ -1 & 2 & 5 \end{bmatrix}$.

Explanation. The characteristic equation of A is:

$$\det \begin{bmatrix} 1-\lambda & 0 & -1 \\ 0 & 1-\lambda & 2 \\ -1 & 2 & 5-\lambda \end{bmatrix} = -\lambda(\lambda-1)(\lambda-6) = 0$$

Thus the eigenvalues are $\lambda = 0, 1$, and 6 , and corresponding eigenvectors are

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{x}_3 = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$$

respectively. Moreover, by what at first appears to be remarkably good luck, these eigenvectors are *orthogonal*. We have $\|\mathbf{x}_1\|^2 = 6$, $\|\mathbf{x}_2\|^2 = 5$, and $\|\mathbf{x}_3\|^2 = 30$, so

$$P = \begin{bmatrix} \frac{1}{\sqrt{6}}\mathbf{x}_1 & \frac{1}{\sqrt{5}}\mathbf{x}_2 & \frac{1}{\sqrt{30}}\mathbf{x}_3 \end{bmatrix} = \frac{1}{\sqrt{30}} \begin{bmatrix} \sqrt{5} & 2\sqrt{6} & -1 \\ -2\sqrt{5} & \sqrt{6} & 2 \\ \sqrt{5} & 0 & 5 \end{bmatrix}$$

is an orthogonal matrix. Thus $P^{-1} = P^T$ and

$$P^T A P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$

Actually, the fact that the eigenvectors in Example ?? are orthogonal is no coincidence. These vectors certainly must be linearly independent (they correspond to distinct eigenvalues). We will see that the fact that the matrix is *symmetric* implies that the eigenvectors are orthogonal. To prove this we need the following useful fact about symmetric matrices.

Theorem 4. If A is an $n \times n$ symmetric matrix, then

$$(A\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot (A\mathbf{y})$$

for all columns $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$.

Remark 2. The converse also holds (see Practice Problem ??).

Proof Recall that $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T \mathbf{y}$ for all columns $\mathbf{x}$ and $\mathbf{y}$. Because $A^T = A$, we get

$$(A\mathbf{x}) \cdot \mathbf{y} = (A\mathbf{x})^T \mathbf{y} = \mathbf{x}^T A^T \mathbf{y} = \mathbf{x}^T A \mathbf{y} = \mathbf{x} \cdot (A\mathbf{y})$$

■

Theorem 5. If A is a symmetric matrix, then eigenvectors of A corresponding to distinct eigenvalues are orthogonal.

Proof Let $A\mathbf{x} = \lambda\mathbf{x}$ and $A\mathbf{y} = \mu\mathbf{y}$, where $\lambda \neq \mu$. We compute

$$\lambda(\mathbf{x} \cdot \mathbf{y}) = (\lambda\mathbf{x}) \cdot \mathbf{y} = (A\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot (A\mathbf{y}) = \mathbf{x} \cdot (\mu\mathbf{y}) = \mu(\mathbf{x} \cdot \mathbf{y})$$

Hence $(\lambda - \mu)(\mathbf{x} \cdot \mathbf{y}) = 0$, and so $\mathbf{x} \cdot \mathbf{y} = 0$ because $\lambda \neq \mu$.

■

Now the procedure for diagonalizing a symmetric $n \times n$ matrix is clear. Find the distinct eigenvalues and find orthonormal bases for each eigenspace (the Gram-Schmidt algorithm may be needed when there is a repeated eigenvalue). Then the set of all these basis vectors is orthonormal (by Theorem ??) and contains n vectors. Here is an example.

Example 3. Orthogonally diagonalize the symmetric matrix $A = \begin{bmatrix} 8 & -2 & 2 \\ -2 & 5 & 4 \\ 2 & 4 & 5 \end{bmatrix}$.

Explanation. The characteristic equation is

$$\det \begin{bmatrix} 8 - \lambda & -2 & 2 \\ -2 & 5 - \lambda & 4 \\ 2 & 4 & 5 - \lambda \end{bmatrix} = -\lambda(\lambda - 9)^2 = 0$$

Hence the distinct eigenvalues are 0 and 9 are of algebraic multiplicity 1 and 2, respectively. The geometric multiplicities must be the same, for A is diagonalizable, being symmetric. It follows that $\dim(S_0) = 1$ and $\dim(S_9) = 2$. Gaussian elimination gives

$$S_0(A) = \text{span}\{\mathbf{x}_1\}, \quad \mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \quad \text{and} \quad S_9(A) = \text{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The eigenvectors in S_9 are both orthogonal to $\mathbf{x}_1$ as Theorem ?? guarantees, but not to each other. However, an orthogonal basis can be found using projections. (See [Gram-Schmidt Orthogonalization](#))

$$\{\mathbf{f}_2, \mathbf{f}_3\} \text{ of } S_9(A) \quad \text{where} \quad \mathbf{f}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{f}_3 = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$$

Normalizing gives orthonormal vectors $\{\frac{1}{3}\mathbf{x}_1, \frac{1}{\sqrt{5}}\mathbf{f}_2, \frac{1}{3\sqrt{5}}\mathbf{f}_3\}$, so

$$Q = \begin{bmatrix} | & | & | \\ \frac{1}{3}\mathbf{x}_1 & \frac{1}{\sqrt{5}}\mathbf{f}_2 & \frac{1}{3\sqrt{5}}\mathbf{f}_3 \\ | & | & | \end{bmatrix} = \frac{1}{3\sqrt{5}} \begin{bmatrix} \sqrt{5} & -6 & 2 \\ 2\sqrt{5} & 3 & 4 \\ -2\sqrt{5} & 0 & 5 \end{bmatrix}$$

is an orthogonal matrix such that $Q^{-1}AQ$ is diagonal.

It is worth noting that other, more convenient, diagonalizing matrices Q exist. For example, $\mathbf{y}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ and $\mathbf{y}_3 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$ lie in $S_9(A)$ and they are orthogonal. Moreover, they both have norm 3 (as does $\mathbf{x}_1$), so

$$\hat{Q} = \begin{bmatrix} | & | & | \\ \frac{1}{3}\mathbf{x}_1 & \frac{1}{3}\mathbf{y}_2 & \frac{1}{3}\mathbf{y}_3 \\ | & | & | \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix}$$

is a nicer orthogonal matrix with the property that $\hat{Q}^{-1}A\hat{Q}$ is diagonal.

Theorem 6. Let A be an $n \times n$ matrix. A has an orthonormal set of n eigenvectors if and only if A is orthogonally diagonalizable.

Proof Let $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n$ be orthonormal eigenvectors of A with corresponding eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. We must

show A is orthogonally diagonalizable. Let $Q = \begin{bmatrix} | & | & & | \\ \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_n \\ | & | & & | \end{bmatrix}$ so that Q is orthogonal. We have

$$AQ = \begin{bmatrix} | & | & & | \\ A\mathbf{q}_1 & A\mathbf{q}_2 & \cdots & A\mathbf{q}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \lambda_1\mathbf{q}_1 & \lambda_2\mathbf{q}_2 & \cdots & \lambda_n\mathbf{q}_n \\ | & | & & | \end{bmatrix} = QD,$$

where D is the diagonal matrix with diagonal entries $\lambda_1, \lambda_2, \dots, \lambda_n$. But then $Q^T AQ = D$, proving this half of the theorem.

For the converse, if A is orthogonally diagonalizable, then by Theorem ?? it is symmetric. But then Theorem ?? tells us that eigenvectors corresponding to distinct eigenvalues are orthogonal. Because A is (orthogonally) diagonalizable, we know the geometric multiplicity of each eigenvalue is equal to its algebraic multiplicity. This implies that we can use Gram-Schmidt on each eigenspace of dimension > 1 to get a full set of n orthogonal eigenvectors. ■

If we are willing to replace “diagonal” by “upper triangular” in the real spectral theorem, we can weaken the requirement that A is symmetric to insisting only that A has real eigenvalues.

Theorem 7 (Schur Triangularization Theorem). If A is an $n \times n$ matrix with n real eigenvalues, an orthogonal matrix Q exists such that $Q^T AQ$ is upper triangular.

Remark 3. There is also a lower triangular version of this theorem.

Proof See Practice Problem ?? ■

The eigenvalues of an upper triangular matrix are displayed along the main diagonal. Because A and $Q^T AQ$ have the same determinant and trace whenever Q is orthogonal (for they are similar matrices), Theorem ?? gives:

Corollary 1. If A is an $n \times n$ matrix with real eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ (possibly not all distinct), then $\det A = \lambda_1 \lambda_2 \cdots \lambda_n$ and $\text{tr} A = \lambda_1 + \lambda_2 + \cdots + \lambda_n$.

Remark 4. This corollary remains true even if the eigenvalues are not real.

Practice Problems

Problem 2 Normalize the columns to make the following matrix orthogonal. $A = \begin{bmatrix} 1 & -4 \\ 2 & 2 \end{bmatrix}$

$$\frac{1}{\sqrt{\dots}} \begin{bmatrix} \dots & \dots \\ \dots & \dots \end{bmatrix}$$

Problem 3 Normalize the columns to make the following matrix orthogonal. $A = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

Problem 4 For each matrix A , find an orthogonal matrix Q such that $Q^{-1}AQ$ is diagonal.

Question 4.1

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$Q = \frac{1}{\sqrt{\dots\dots}} \begin{bmatrix} 1 & -1 \\ \dots\dots & \dots\dots \end{bmatrix}$$

Question 4.2

$$A = \begin{bmatrix} 3 & 0 & 7 \\ 0 & 5 & 0 \\ 7 & 0 & 3 \end{bmatrix}$$

Problem 5 Suppose A is orthogonally diagonalizable. Prove that A is symmetric. (This is the easy direction of the "if and only if" in Theorem ??.)

Problem 6 Prove the converse of Theorem ??:

If $(A\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot (A\mathbf{y})$ for all n -columns $\mathbf{x}$ and $\mathbf{y}$, then A is symmetric.

Problem 7 Let Q be an orthogonal matrix.

1. Show that $\det Q = 1$ or $\det Q = -1$.
2. Give 2×2 examples of Q such that $\det Q = 1$ and $\det Q = -1$.

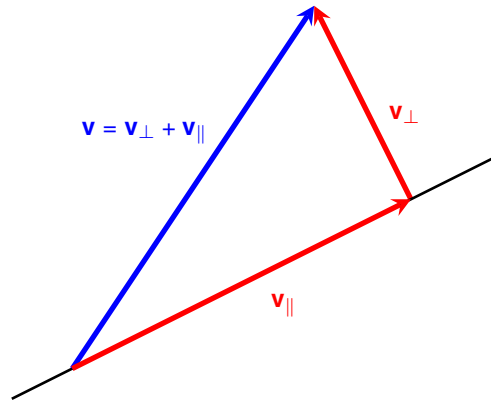
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This section was adapted from Section 8.2 of Keith Nicholson's *Linear Algebra with Applications*. (CC-BY-NC-SA)

W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2018, Open Edition, p. 424

2 Orthogonal Projections

Given a line l and a vector $\mathbf{v}$ emanating from a point on l , it is sometimes convenient to express $\mathbf{v}$ as the sum of a vector $\mathbf{v}_{\parallel}$, parallel to l , and a vector $\mathbf{v}_{\perp}$, perpendicular to l . If you have taken a physics course, you may have seen a force vector decomposed into the sum of two components: one parallel and one perpendicular to the direction of motion.



Suppose $\mathbf{d}$ is a direction vector for l . Then $\mathbf{v}_{\parallel} = k\mathbf{d}$ for some scalar k . Our goal is to find k .

$$\begin{aligned}\mathbf{v} \cdot \mathbf{d} &= (\mathbf{v}_{\parallel} + \mathbf{v}_{\perp}) \cdot \mathbf{d} \\ &= (k\mathbf{d} + \mathbf{v}_{\perp}) \cdot \mathbf{d} \\ &= k\mathbf{d} \cdot \mathbf{d} + \mathbf{v}_{\perp} \cdot \mathbf{d} \\ &= k\|\mathbf{d}\|^2 + 0 \\ &= k\|\mathbf{d}\|^2\end{aligned}$$

We conclude that

$$k = \frac{\mathbf{v} \cdot \mathbf{d}}{\|\mathbf{d}\|^2}$$

and

$$\mathbf{v}_{\parallel} = k\mathbf{d} = \left(\frac{\mathbf{v} \cdot \mathbf{d}}{\|\mathbf{d}\|^2} \right) \mathbf{d}$$

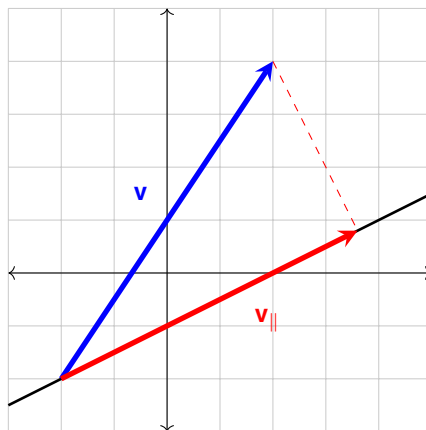
The vector $\mathbf{v}_{\parallel} = \left(\frac{\mathbf{v} \cdot \mathbf{d}}{\|\mathbf{d}\|^2} \right) \mathbf{d}$ is called the *projection of $\mathbf{v}$ onto $\mathbf{d}$* . In our discussion, $\mathbf{d}$ is a direction vector for line l . So, we can also say that $\mathbf{v}_{\parallel}$ is the *projection of $\mathbf{v}$ onto l* .

To find $\mathbf{v}_{\perp}$, observe that $\mathbf{v}_{\perp} = \mathbf{v} - \mathbf{v}_{\parallel}$.

Definition 3. Let $\mathbf{v}$ be a vector, and let $\mathbf{d}$ be a non-zero vector. The *projection of $\mathbf{v}$ onto $\mathbf{d}$* is given by

$$\text{proj}_{\mathbf{d}} \mathbf{v} = \left(\frac{\mathbf{v} \cdot \mathbf{d}}{\|\mathbf{d}\|^2} \right) \mathbf{d}$$

Example 4. Find the projection of $\mathbf{v}$, shown below, onto the line given by $y = \frac{1}{2}x - 1$.



Explanation. We begin by finding vectors $\mathbf{v}$ and $\mathbf{d}$. The tail of $\mathbf{v}$ is located at $(-2, -2)$, and the head of $\mathbf{v}$ is at $(2, 4)$. Using the “head-tail” formula we get

$$\mathbf{v} = \begin{bmatrix} 2 - (-2) \\ 4 - (-2) \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

The direction vector for the line $y = \frac{1}{2}x - 1$ is

$$\mathbf{d} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

We find that $\mathbf{v} \cdot \mathbf{d} = 14$ and $\|\mathbf{d}\|^2 = 5$. Thus

$$\text{proj}_{\mathbf{d}} \mathbf{v} = \left(\frac{\mathbf{v} \cdot \mathbf{d}}{\|\mathbf{d}\|^2} \right) \mathbf{d} = \frac{14}{5} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 28/5 \\ 14/5 \end{bmatrix}$$

Practice Problems

Problem 8 Find $\text{proj}_{\mathbf{d}} \mathbf{v}$ if $\mathbf{d} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

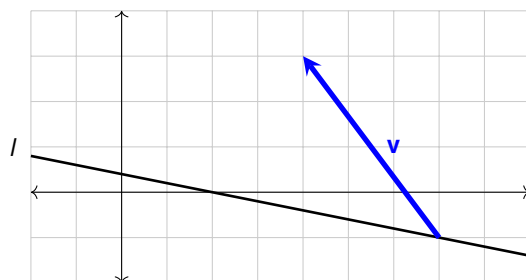
$$\text{proj}_{\mathbf{d}} \mathbf{v} = \begin{bmatrix} \dots \dots \\ \dots \dots \end{bmatrix}$$

Problem 9 Find $\text{proj}_{\mathbf{d}} \mathbf{v}$ if $\mathbf{d} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ -4 \\ 2 \end{bmatrix}$.

$$\text{proj}_{\mathbf{d}} \mathbf{v} = \begin{bmatrix} \dots \dots \dots \\ \dots \dots \dots \\ \dots \dots \dots \end{bmatrix}$$

Problem 10 Show that $\text{proj}_{\mathbf{d}} \mathbf{v}$ does not depend on the length of $\mathbf{d}$ by proving that $\text{proj}_{\mathbf{d}} \mathbf{v} = \text{proj}_{k\mathbf{d}} \mathbf{v}$ for $k \neq 0$. What does this result mean geometrically? Illustrate your response with a diagram.

Problem 11 Find the projection of vector $\mathbf{v}$ onto line l . (If entering answers in decimal form, round to the nearest one hundredth.)



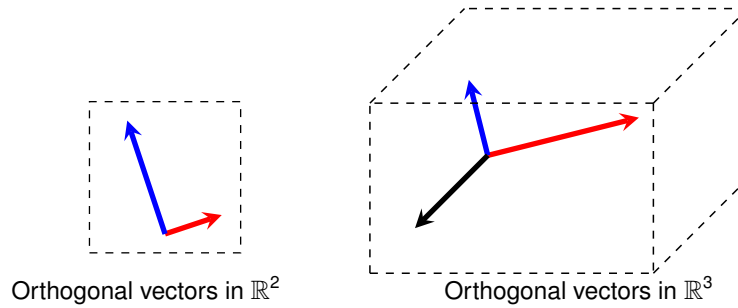
Answer:

$$\begin{bmatrix} \dots \\ \dots \end{bmatrix}$$

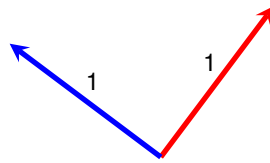
3 Orthogonality and Projections

Orthogonal and Orthonormal Sets

In this section, we examine what it means for vectors (and sets of vectors) to be orthogonal and orthonormal. Recall that two non-zero vectors are orthogonal if their dot product is zero. A collection of non-zero vectors in $\mathbb{R}^n$ is called *orthogonal* if the vectors are pair-wise orthogonal. The diagram below shows two orthogonal vectors in $\mathbb{R}^2$ and three orthogonal vectors in $\mathbb{R}^3$.



If every vector in an orthogonal set of vectors is also a unit vector, then we say that the given set of vectors is *orthonormal*.



An orthonormal set of two vectors

Formally, we can define orthogonal and orthonormal vectors as follows.

Definition 4. Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ be a set of nonzero vectors in $\mathbb{R}^n$. Then this set is called an *orthogonal set* if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for all $i \neq j$. Moreover, if $\|\mathbf{v}_i\| = 1$ for $i = 1, \dots, m$ (i.e. each vector in the set is a unit vector), we say the set of vectors is an *orthonormal set*.

An orthogonal set of vectors may not be orthonormal. To convert an orthogonal set to an orthonormal set, we need to divide each vector by its own length.

Definition 5. *Normalizing* an orthogonal set is the process of turning an orthogonal set into an orthonormal set. If $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is an orthogonal subset of $\mathbb{R}^n$, then

$$\left\{ \frac{1}{\|\mathbf{v}_1\|} \mathbf{v}_1, \frac{1}{\|\mathbf{v}_2\|} \mathbf{v}_2, \dots, \frac{1}{\|\mathbf{v}_k\|} \mathbf{v}_k \right\}$$

is an orthonormal set.

We illustrate this concept in the following example.

Example 5. Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Show that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal set of vectors but not an orthonormal one. Find the corresponding orthonormal set.

Explanation. One easily verifies that $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$ and $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal set of vectors. On the other hand one can compute that $\|\mathbf{v}_1\| = \|\mathbf{v}_2\| = \sqrt{2} \neq 1$ and so the set is not orthonormal.

To find a corresponding orthonormal set, we need to normalize each vector.

$$\begin{aligned} \mathbf{q}_1 &= \frac{1}{\|\mathbf{v}_1\|} \mathbf{v}_1 \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \end{aligned}$$

Similarly,

$$\begin{aligned} \mathbf{q}_2 &= \frac{1}{\|\mathbf{v}_2\|} \mathbf{v}_2 \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \end{aligned}$$

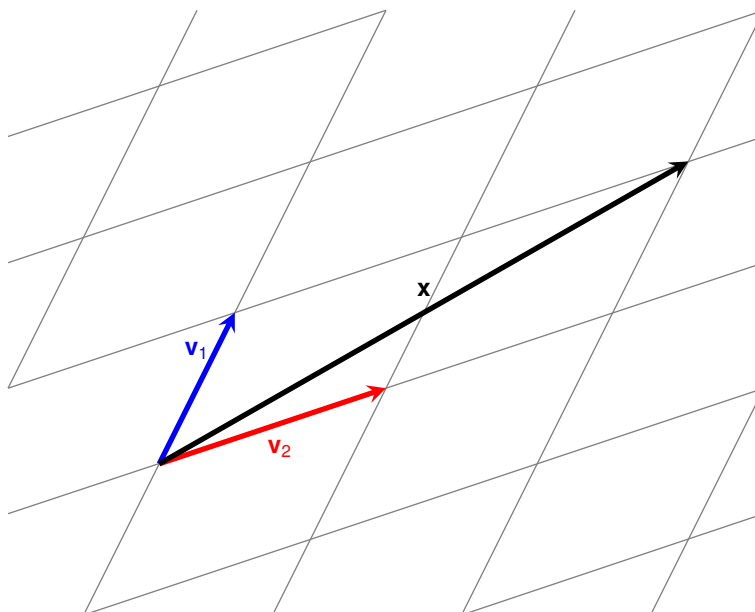
Therefore the corresponding orthonormal set is

$$\{\mathbf{q}_1, \mathbf{q}_2\} = \left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\}$$

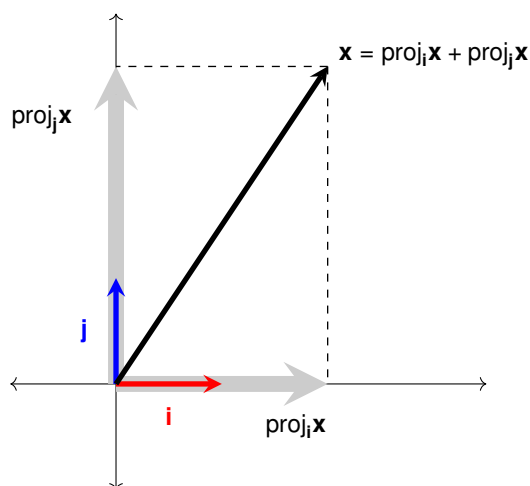
You can verify that this set is orthonormal.

Orthogonal and Orthonormal Bases

Recall that every basis of $\mathbb{R}^n$ (or a subspace W of $\mathbb{R}^n$) imposes a coordinate system on $\mathbb{R}^n$ (or W) that can be used to express any vector of $\mathbb{R}^n$ (or W) as a linear combination of the elements of the basis. For example, vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ impose a coordinate system onto the plane, as shown in the figure below. We readily see that $\mathbf{x}$, contained in the plane, can be written as $\mathbf{x} = \mathbf{v}_1 + 2\mathbf{v}_2$.



Vector $\mathbf{x}$ is visually easy to work with. In general, one way to express an arbitrary vector as a linear combination of the basis vectors is to solve a system of linear equations, which can be costly. One reason we like $\{\mathbf{i}, \mathbf{j}\}$ as a basis of $\mathbb{R}^2$ is because any vector $\mathbf{x}$ of $\mathbb{R}^2$ can be easily expressed as the sum of the orthogonal projections of $\mathbf{x}$ onto the basis vectors $\mathbf{i}$ and $\mathbf{j}$, as shown below.



We can see why an “upright” coordinate system with basis $\{\mathbf{i}, \mathbf{j}\}$ works well. What if we tilt this coordinate system while preserving the orthogonal relationship between the basis vectors? The following exploration allows you to investigate the consequences.

Exploration 12 In the following GeoGebra interactive, vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal (slopes of the lines containing them are negative reciprocals of each other). These vectors are clearly linearly independent and span $\mathbb{R}^2$. Therefore $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis of $\mathbb{R}^2$.

Let $\mathbf{x}$ be an arbitrary vector. Orthogonal projections of $\mathbf{x}$ onto $\mathbf{v}_1$ and $\mathbf{v}_2$ are depicted in light grey.

- Use the tip of vector $\mathbf{x}$ to manipulate the vector and convince yourself that $\mathbf{x}$ is always the diagonal of the parallelogram (a rectangle!) determined by the projections.
- Use the tips of $\mathbf{v}_1$ and $\mathbf{v}_2$ to change the basis vectors. What happens when $\mathbf{v}_1$ and $\mathbf{v}_2$ are no longer orthogonal?
- Pick another pair of orthogonal vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Verify that $\mathbf{x}$ is the sum of its projections.

Access GeoGebra interactives through the online version of this text at

<https://ximera.osu.edu/linearalgebradv3/LinearAlgebraInteractiveIntro>.

As you have just discovered in Exploration ??, we can express an arbitrary vector of $\mathbb{R}^2$ as the sum of its projections onto the basis vectors, provided that the basis is orthogonal. It turns out that this result holds for any subspace of $\mathbb{R}^n$, making a basis consisting of orthogonal vectors especially useful.

If an orthogonal set is a basis, we call it an *orthogonal basis*. Similarly, if an orthonormal set is a basis, we call it an *orthonormal basis*.

The following theorem generalizes our observation in Exploration ?? . As you read the statement of the theorem, it will be helpful to recall that the orthogonal projection of vector $\mathbf{x}$ onto a non-zero vector $\mathbf{d}$ is given by

$$\text{proj}_{\mathbf{d}} \mathbf{x} = \left(\frac{\mathbf{x} \cdot \mathbf{d}}{\|\mathbf{d}\|^2} \right) \mathbf{d} \quad (1)$$

(See [Orthogonal Projections](#).)

Theorem 8. Let W be a subspace of $\mathbb{R}^n$ and suppose $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m\}$ is an orthogonal basis of W . Then for every $\mathbf{x}$ in W ,

$$\mathbf{x} = \left(\frac{\mathbf{x} \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2} \right) \mathbf{f}_1 + \left(\frac{\mathbf{x} \cdot \mathbf{f}_2}{\|\mathbf{f}_2\|^2} \right) \mathbf{f}_2 + \dots + \left(\frac{\mathbf{x} \cdot \mathbf{f}_m}{\|\mathbf{f}_m\|^2} \right) \mathbf{f}_m. \quad (2)$$

Proof We may express $\mathbf{x}$ as a linear combination of the basis elements:

$$\mathbf{x} = c_1 \mathbf{f}_1 + c_2 \mathbf{f}_2 + \dots + c_m \mathbf{f}_m.$$

We claim that $c_i = \frac{\mathbf{x} \cdot \mathbf{f}_i}{\|\mathbf{f}_i\|^2}$ for $i = 1, \dots, m$. To see this, we take the dot product of each side with the vector $\mathbf{f}_i$ and obtain the following.

$$\mathbf{x} \cdot \mathbf{f}_i = (c_1 \mathbf{f}_1 + c_2 \mathbf{f}_2 + \dots + c_m \mathbf{f}_m) \cdot \mathbf{f}_i$$

Our basis is orthogonal, so $\mathbf{f}_j \cdot \mathbf{f}_i = 0$ for all $j \neq i$, which means after we distribute the dot product, only one term will remain on the right-hand side. We have

$$\mathbf{x} \cdot \mathbf{f}_i = c_i \mathbf{f}_i \cdot \mathbf{f}_i$$

We now divide both sides by $\mathbf{f}_i \cdot \mathbf{f}_i = \|\mathbf{f}_i\|^2$, and since our claim holds for $i = 1, \dots, m$, the proof is complete. ■

Theorem ?? shows one important benefit of a basis being orthogonal. With an orthogonal basis it is easy to represent any vector in terms of the basis vectors.

Example 6. Let $\mathbf{f}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, $\mathbf{f}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{f}_3 = \begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix}$, and let $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Notice that $\mathcal{B} = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ is an orthogonal set of vectors, and $\mathcal{B}$ spans $\mathbb{R}^3$. Use this fact to write $\mathbf{x}$ as a linear combination of the vectors of $\mathcal{B}$.

Explanation. We first observe that $\mathcal{B}$ is a linearly independent set of vectors, and so $\mathcal{B}$ is a basis for $\mathbb{R}^3$. Next we apply Theorem ?? to express $\mathbf{x}$ as a linear combination of the vectors of $\mathcal{B}$. We wish to write:

$$\mathbf{x} = \left(\frac{\mathbf{x} \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2} \right) \mathbf{f}_1 + \left(\frac{\mathbf{x} \cdot \mathbf{f}_2}{\|\mathbf{f}_2\|^2} \right) \mathbf{f}_2 + \left(\frac{\mathbf{x} \cdot \mathbf{f}_3}{\|\mathbf{f}_3\|^2} \right) \mathbf{f}_3.$$

We readily compute:

$$\frac{\mathbf{x} \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2} = \frac{2}{6}, \quad \frac{\mathbf{x} \cdot \mathbf{f}_2}{\|\mathbf{f}_2\|^2} = \frac{3}{5}, \quad \text{and} \quad \frac{\mathbf{x} \cdot \mathbf{f}_3}{\|\mathbf{f}_3\|^2} = \frac{4}{30}.$$

Therefore,

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + \frac{3}{5} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + \frac{2}{15} \begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix}.$$

The formula from Theorem ?? is easy to use, and it becomes even easier when our basis is *orthonormal*.

Corollary 2. Let W be a subspace of $\mathbb{R}^n$ and suppose $\{\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_m\}$ is an orthonormal basis of W . Then for any $\mathbf{x}$ in W ,

$$\mathbf{x} = (\mathbf{x} \cdot \mathbf{q}_1) \mathbf{q}_1 + (\mathbf{x} \cdot \mathbf{q}_2) \mathbf{q}_2 + \dots + (\mathbf{x} \cdot \mathbf{q}_m) \mathbf{q}_m.$$

Proof This is a special case of Theorem ??. Because $\|\mathbf{u}_i\| = 1$ for $i = 1, \dots, m$, the terms are given by

$$\left(\frac{\mathbf{x} \cdot \mathbf{q}_i}{\|\mathbf{q}_i\|^2} \right) \mathbf{q}_i = (\mathbf{x} \cdot \mathbf{q}_i) \mathbf{q}_i.$$

■

Orthogonal Projection onto a Subspace

In the previous section we found that given a subspace W of $\mathbb{R}^n$ with an orthogonal basis $\mathcal{B}$, every vector $\mathbf{x}$ in W can be expressed as the sum of the orthogonal projections of $\mathbf{x}$ onto the elements of $\mathcal{B}$. Note that our premise was that $\mathbf{x}$ is in W . In this section, we look into the meaning of the sum of orthogonal projections of $\mathbf{x}$ onto the elements of an orthogonal basis of W for those vectors $\mathbf{x}$ of $\mathbb{R}^n$ that are *not* in W .

Exploration 13 In the GeoGebra interactive below, W is a plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$, in $\mathbb{R}^3$. W is subspace of $\mathbb{R}^3$. In the initial set up, $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal. Vector $\mathbf{x}$ is not in W .

Use check-boxes to construct the sum of orthogonal projections of $\mathbf{x}$ onto $\mathbf{v}_1$ and $\mathbf{v}_2$. RIGHT-CLICK and DRAG to rotate the image.

Access GeoGebra interactives through the online version of this text at

<https://ximera.osu.edu/linearalgebradv3/LinearAlgebraInteractiveIntro>.

Question 13.1 If moved, return the basis vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ to their default position (set $s_1 = s_2 = 0$) to ensure that they are orthogonal.

- Rotate the image to convince yourself that the perpendiculars dropped from the tip of $\mathbf{x}$ to $\mathbf{v}_1$ and $\mathbf{v}_2$ are indeed perpendicular to $\mathbf{v}_1$ and $\mathbf{v}_2$ in the diagram. (You'll have to look at it just right to convince yourself of this.) Are both of these perpendiculars also necessarily perpendicular to the plane? (Yes, / No)
- Use sliders x_1 , x_2 and x_3 to manipulate $\mathbf{x}$. Rotate the figure for a better view. What is true about vector $\mathbf{p}$?

Multiple Choice:

1. $\mathbf{p} = \mathbf{x} - (\text{proj}_{\mathbf{v}_1} \mathbf{x} + \text{proj}_{\mathbf{v}_2} \mathbf{x})$.
 2. Vector $\mathbf{p}$ is orthogonal to W .
 3. All of the above.
- Rotate the figure so that you're looking directly down at the plane. If you're looking at it correctly, you will notice that (1) the parallelogram determined by the projections of $\mathbf{x}$ onto $\mathbf{v}_1$ and $\mathbf{v}_2$ is a rectangle; (2) the sum of projections, $\text{proj}_{\mathbf{v}_1} \mathbf{x} + \text{proj}_{\mathbf{v}_2} \mathbf{x}$, is located directly underneath $\mathbf{x}$, like a shadow at midday.

Question 13.2 Use sliders s_1 and s_2 to manipulate the basis vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ so that they are no longer orthogonal.

- Rotate the figure for a better view. Which of the following is true?

Multiple Choice:

1. $\mathbf{p} = \mathbf{x} - (\text{proj}_{\mathbf{v}_1} \mathbf{x} + \text{proj}_{\mathbf{v}_2} \mathbf{x})$.
 2. Vector $\mathbf{p}$ is orthogonal to W .
 3. All of the above.
- Rotate your figure so that you're looking directly down at the plane. Which of the following is true?

Multiple Choice:

1. Parallelogram determined by $\mathbf{v}_1$ and $\mathbf{v}_2$ is a rectangle.
2. $\text{proj}_{\mathbf{v}_1} \mathbf{x} + \text{proj}_{\mathbf{v}_2} \mathbf{x}$ is located directly underneath $\mathbf{x}$.
3. None of the above.

In Exploration ??, you discovered that given a plane, spanned by orthogonal vectors $\mathbf{v}_1, \mathbf{v}_2$, in $\mathbb{R}^3$, and a vector $\mathbf{x}$, not in the plane, we can interpret the sum of orthogonal projections of $\mathbf{x}$ onto $\mathbf{v}_1$ and $\mathbf{v}_2$ as a “shadow” of $\mathbf{x}$ that lies in the plane directly underneath the vector $\mathbf{x}$. We say that this “shadow” is an *orthogonal projection of $\mathbf{x}$ onto W* . You have also found that if $\mathbf{v}_1, \mathbf{v}_2$ are not orthogonal, the parallelogram representing the sum of the orthogonal projections of $\mathbf{x}$ onto $\mathbf{v}_1$ and $\mathbf{v}_2$ will not be a rectangle. In this case, $\mathbf{x}$ minus this sum will NOT be orthogonal to the plane. It is essential that $\mathbf{v}_1, \mathbf{v}_2$ are orthogonal for $\text{proj}_{\mathbf{v}_1} \mathbf{x} + \text{proj}_{\mathbf{v}_2} \mathbf{x}$ to be considered an orthogonal projection.

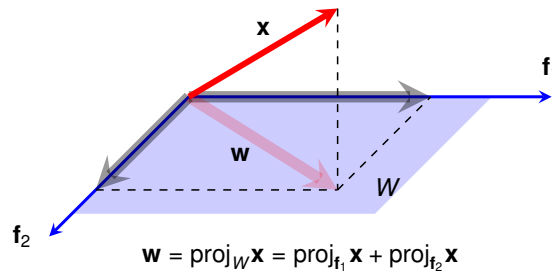
In general, we can define an orthogonal projection of $\mathbf{x}$ in $\mathbb{R}^n$ onto a subspace W of $\mathbb{R}^n$ as the sum of the orthogonal projections of $\mathbf{x}$ onto the elements of an orthogonal basis of W . Definition ?? and the subsequent diagram summarize this discussion.

Definition 6 (Projection onto a Subspace of $\mathbb{R}^n$). Let W be a subspace of $\mathbb{R}^n$ with orthogonal basis $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m\}$. If $\mathbf{x}$ is in $\mathbb{R}^n$, the vector

$$\mathbf{w} = \text{proj}_W(\mathbf{x}) = \text{proj}_{\mathbf{f}_1} \mathbf{x} + \text{proj}_{\mathbf{f}_2} \mathbf{x} + \dots + \text{proj}_{\mathbf{f}_m} \mathbf{x} \quad (3)$$

is called the *orthogonal projection* of $\mathbf{x}$ onto W .

An illustration of Definition ?? for a two-dimensional subspace W with orthogonal basis $\{\mathbf{f}_1, \mathbf{f}_2\}$ is shown below.



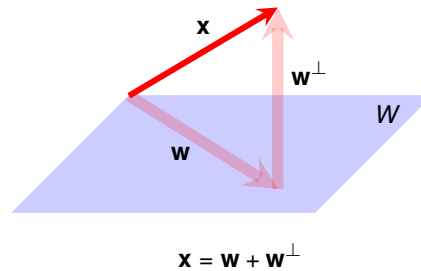
Using equation (??) multiple times, we can also express $\mathbf{w}$ in Definition ?? using the following formula.

Formula 1.

$$\mathbf{w} = \text{proj}_W(\mathbf{x}) = \frac{\mathbf{x} \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2} \mathbf{f}_1 + \frac{\mathbf{x} \cdot \mathbf{f}_2}{\|\mathbf{f}_2\|^2} \mathbf{f}_2 + \dots + \frac{\mathbf{x} \cdot \mathbf{f}_m}{\|\mathbf{f}_m\|^2} \mathbf{f}_m \quad (4)$$

Orthogonal Decomposition of $\mathbf{x}$

Definition ?? allows us to express $\mathbf{x}$ as the sum of its orthogonal projection, $\mathbf{w} = \text{proj}_W \mathbf{x}$, located in W , and a vector we will call $\mathbf{w}^\perp$ (pronounced “W-perp”), given by $\mathbf{w}^\perp = \mathbf{x} - \mathbf{w}$. This decomposition of $\mathbf{x}$ is shown in the diagram below.



You have already met $\mathbf{w}^\perp$, under the name of $\mathbf{p}$ in Exploration ??, and observed that this vector is orthogonal to W . We will now prove that $\mathbf{w}^\perp$ is orthogonal to every vector in W . This will be accomplished in two steps. First, in Theorem ?? we will prove that $\mathbf{w}^\perp$ is orthogonal to all of the basis elements of W . Next, you will use this result to demonstrate that $\mathbf{w}^\perp$ is orthogonal to every vector in W .

Theorem 9. Let W be a subspace of $\mathbb{R}^n$ with orthogonal basis $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m\}$. Let $\mathbf{x}$ be in $\mathbb{R}^n$, and define $\mathbf{w}^\perp$ as

$$\mathbf{w}^\perp = \mathbf{x} - \text{proj}_W \mathbf{x} = \mathbf{x} - (\text{proj}_{\mathbf{f}_1} \mathbf{x} + \text{proj}_{\mathbf{f}_2} \mathbf{x} + \dots + \text{proj}_{\mathbf{f}_m} \mathbf{x})$$

Then $\mathbf{w}^\perp$ is orthogonal to $\mathbf{f}_i$ for $1 \leq i \leq m$.

Proof We will use Formula ?? to show that $\mathbf{w}^\perp \cdot \mathbf{f}_i = 0$. Recall that $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m\}$ is an orthogonal basis. Therefore $\mathbf{f}_j \cdot \mathbf{f}_i = 0$ for $i \neq j$. This observation enables us to compute as follows.

$$\begin{aligned} \mathbf{w}^\perp \cdot \mathbf{f}_i &= \left[\mathbf{x} - \left(\frac{\mathbf{x} \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2} \mathbf{f}_1 + \dots + \frac{\mathbf{x} \cdot \mathbf{f}_i}{\|\mathbf{f}_i\|^2} \mathbf{f}_i + \dots + \frac{\mathbf{x} \cdot \mathbf{f}_m}{\|\mathbf{f}_m\|^2} \mathbf{f}_m \right) \right] \cdot \mathbf{f}_i \\ &= \mathbf{x} \cdot \mathbf{f}_i - \frac{\mathbf{x} \cdot \mathbf{f}_i}{\|\mathbf{f}_i\|^2} (\mathbf{f}_i \cdot \mathbf{f}_i) \\ &= \mathbf{x} \cdot \mathbf{f}_i - \frac{\mathbf{x} \cdot \mathbf{f}_i}{\|\mathbf{f}_i\|^2} \|\mathbf{f}_i\|^2 = \mathbf{x} \cdot \mathbf{f}_i - \mathbf{x} \cdot \mathbf{f}_i = 0 \end{aligned}$$

■

We leave the proof of the following Corollary as Practice Problem ??

Corollary 3. Let W be a subspace of $\mathbb{R}^n$ with orthogonal basis $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m\}$. Let $\mathbf{x}$ be in $\mathbb{R}^n$, and define $\mathbf{w}^\perp$ as

$$\mathbf{w}^\perp = \mathbf{x} - \text{proj}_W \mathbf{x} = \mathbf{x} - (\text{proj}_{\mathbf{f}_1} \mathbf{x} + \text{proj}_{\mathbf{f}_2} \mathbf{x} + \dots + \text{proj}_{\mathbf{f}_m} \mathbf{x})$$

Then $\mathbf{w}^\perp$ is orthogonal to every vector in W .

The fact that the decomposition of $\mathbf{x}$ into the sum of $\mathbf{w}$ and $\mathbf{w}^\perp$ is unique is the subject of the Orthogonal Decomposition Theorem which we will prove in [Orthogonal Complements and Decompositions](#).

Throughout this section we have worked with orthogonal bases of subspaces. Does every subspace of $\mathbb{R}^n$ have an orthogonal basis? If so, how do we find one? These questions will be addressed in [Gram-Schmidt Orthogonalization](#).

Practice Problems

Problem 14 *Retry Example ?? using Gaussian elimination.*

Problem 15 Let $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k \in \mathbb{R}^n$ and suppose $\text{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\} = \mathbb{R}^n$. Furthermore, suppose that there exists a vector $\mathbf{v} \in \mathbb{R}^n$ for which $\mathbf{v} \cdot \mathbf{x}_j = 0$ for all j , $1 \leq j \leq k$. Show that $\mathbf{v} = \mathbf{0}$.

Problem 16 Let $\mathbf{x} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 6 \end{bmatrix}$ in $\mathbb{R}^4$, and let $W = \text{span} \left(\begin{bmatrix} 2 \\ 1 \\ 3 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right)$.

Question 16.1 Compute $\text{proj}_W(\mathbf{x})$.

Answer:

$$\frac{1}{10} \begin{bmatrix} \dots \\ \dots \\ \dots \\ \dots \end{bmatrix}$$

Question 16.2 Show that $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ 7 \\ 1 \\ 2 \end{bmatrix} \right\}$ is another orthogonal basis of W .

Question 16.3 Use the basis you found in the previous part to compute $\text{proj}_W(\mathbf{x})$.

Answer:

$$\frac{1}{70} \begin{bmatrix} \dots \\ \dots \\ \dots \\ \dots \end{bmatrix}$$

Problem 17 Prove Corollary ??

Text Source

A portion of the text in this section is an adaptation of Section 4.11.1 of Ken Kuttler's [A First Course in Linear Algebra](#). (CC-BY)

Ken Kuttler, *A First Course in Linear Algebra*, Lyryx 2017, Open Edition, p. 233-238.

4 Least-Squares Approximation

Often an exact solution to a problem in applied mathematics is difficult or impossible to obtain. However, it is usually just as useful to find an approximation to a solution. In particular, finding “linear approximations” is a powerful technique in applied mathematics. One basic case is the situation where a system of linear equations has no solution, and it is desirable to find a “best approximation” to a solution to the system.

We begin by defining the “best approximation” in a natural way, and showing that computing the best approximation reduces to solving a related system of linear equations called the *normal equations*. Next, we demonstrate a common application where a collection of data points is approximated by a line (or a curve). We conclude this section by showing that *QR*-factorization provides us with a more efficient way to solve the normal equations and compute the best approximation.

Best Approximate Solutions

Exploration 18 Let

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Consider the matrix equation $A\mathbf{x} = \mathbf{b}$. A quick examination of the last two rows should convince you that this equation has no solutions. In other words, $\mathbf{b}$ is not in the span of the columns of A .

If $\mathbf{z}$ were an exact solution to $A\mathbf{x} = \mathbf{b}$, then $\mathbf{b} - A\mathbf{z}$ would be $\mathbf{0}$. Since the equation does not have a solution, we will attempt to find the next best thing to a solution by finding $\mathbf{z}$ such that $\|\mathbf{b} - A\mathbf{z}\|$ is as small as possible. The quantity $\|\mathbf{b} - A\mathbf{z}\|$ is called the error.

The following GeoGebra interactive will help you understand the geometry behind finding $\mathbf{z}$.

RIGHT-CLICK and DRAG to rotate the image for a better view.

Access GeoGebra interactives through the online version of this text at

<https://ximera.osu.edu/linearalgebradv3/LinearAlgebraInteractiveIntro>.

Record your best guess for $\mathbf{z}$ – you will have a chance to check your answer in Example ??.

What did you discover about the geometry of minimizing $\|\mathbf{b} - A\mathbf{z}\|$? Select all that apply.

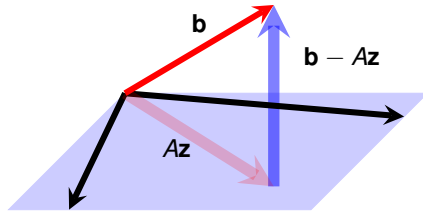
Select All Correct Answers:

1. $\mathbf{z}$ is orthogonal to the plane spanned by the columns of A .
2. $\|\mathbf{b} - A\mathbf{z}\|$ is orthogonal to $\text{col}(A)$.
3. $\mathbf{b} - A\mathbf{z}$ is orthogonal to $\text{col}(A)$.
4. $A\mathbf{z}$ is orthogonal to $\text{col}(A)$.
5. $A\mathbf{z}$ is an orthogonal projection of $\mathbf{b}$ onto $\text{col}(A)$.

Our geometric observations will help us develop a method for finding $\mathbf{z}$.

Suppose A is an $m \times n$ matrix, and $\mathbf{b}$ is a column vector in $\mathbb{R}^m$. Consider the matrix equation $A\mathbf{x} = \mathbf{b}$. If this equation does not have a solution, we can attempt to find a best approximation by finding $\mathbf{z}$ which minimizes the error, $\|\mathbf{b} - A\mathbf{z}\|$. The expression $\|\mathbf{b} - A\mathbf{z}\|$ is also sometimes called the *residual*. The error (or the residual) is given in terms of a vector norm. Recall that our definition of the norm involves the sum of squares of the vector components. When we minimize the norm, we minimize the sum of squares. This is why the method we are describing is often referred to as *least squares*. We will explore this idea further later in this section.

In the case when $\text{col}(A)$ is a subspace of $\mathbb{R}^3$, we can see geometrically that $\mathbf{z}$ is the best approximation if and only if $A\mathbf{z}$ is an orthogonal projection of $\mathbf{b}$ onto $\text{col}(A)$, and the error is the magnitude of $\mathbf{b} - A\mathbf{z}$, as shown below.



What we observed above, holds in general. We will use this fact to find $\mathbf{z}$.

Every vector in $\text{col}(A)$ can be written in the form $A\mathbf{x}$ for some $\mathbf{x}$ in $\mathbb{R}^n$. Our goal is to find $\mathbf{z}$ such that $A\mathbf{z}$ is the orthogonal projection of $\mathbf{b}$ onto $\text{col}(A)$. By Corollary ??, every vector $A\mathbf{x}$ in $\text{col}(A)$ is orthogonal to $\mathbf{b} - A\mathbf{z}$.

Therefore, we have

$$\begin{aligned} A^T(\mathbf{b} - A\mathbf{z}) &= \mathbf{0} \\ A^T\mathbf{b} - A^TA\mathbf{z} &= \mathbf{0} \\ A^TA\mathbf{z} &= A^T\mathbf{b} \end{aligned} \tag{5}$$

Since $\mathbf{b} - A\mathbf{z}$ is normal to the subspace $\text{col}(A)$, we call the system of linear equations in (??) the *normal equations for $\mathbf{z}$* . If A^TA is invertible, then we can write

$$\mathbf{z} = (A^TA)^{-1}A^T\mathbf{b} \tag{6}$$

We will return to the question of invertibility of A^TA in Theorem ??. For now, let's revisit the problem posed in Exploration ??.

Example 7. We now return to the matrix equation $A\mathbf{x} = \mathbf{b}$ of Exploration ?? to find $\mathbf{z}$ that best approximates a solution.

Explanation. Recall that

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

In this case, $(A^T A)^{-1}$ exists. Applying equation (??), we compute

$$\begin{aligned} \mathbf{z} &= \left(\begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 2 \\ 1 & 2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 11 & 7 \\ 7 & 9 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 0.18 & -0.14 \\ -0.14 & 0.22 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \end{bmatrix} \\ &= \begin{bmatrix} 0.4 \\ 0.8 \end{bmatrix} \end{aligned}$$

Compare this answer to your guess in Exploration ???. If your guess was correct, nice job! If your guess was different, try setting $\mathbf{z}$ to the correct answer and use the GeoGebra interactive in Exploration ?? to examine the geometry of the problem.

We now come back to the question of when $A^T A$ is invertible.

Theorem 10. If columns of matrix A are linearly independent, then $A^T A$ is invertible.

Proof Let A be a matrix with linearly independent columns. We will show that $(A^T A) \mathbf{x} = \mathbf{0}$ has only the trivial solution. For $\mathbf{x}$, a solution of $A^T A \mathbf{x} = \mathbf{0}$, we have

$$\begin{aligned} \|\mathbf{Ax}\|^2 &= (\mathbf{Ax}) \cdot (\mathbf{Ax}) \\ &= (\mathbf{Ax})^T \mathbf{Ax} \\ &= \mathbf{x}^T A^T \mathbf{Ax} \\ &= \mathbf{x}^T \cdot \mathbf{0} = 0 \end{aligned}$$

Therefore $\mathbf{Ax} = \mathbf{0}$. By linear independence of the columns of A we conclude that $\mathbf{x} = \mathbf{0}$. ■

We summarize our findings in the following theorem.

Theorem 11. Let A be an $m \times n$ matrix, let $\mathbf{b}$ be a column vector in $\mathbb{R}^m$. Consider the matrix equation

$$\mathbf{Ax} = \mathbf{b}$$

- Any solution $\mathbf{z}$ to the normal equations

$$(A^T A) \mathbf{z} = A^T \mathbf{b}$$

is a best approximation to a solution to $\mathbf{Ax} = \mathbf{b}$ in the sense that $\|\mathbf{b} - \mathbf{Az}\|$ is minimized.

- If the columns of A are linearly independent, then $A^T A$ is invertible and $\mathbf{z}$ is given uniquely by

$$\mathbf{z} = (A^T A)^{-1} A^T \mathbf{b}$$

Example 8. The system of linear equations

$$\begin{aligned} 3x - y &= 4 \\ x + 2y &= 0 \\ 2x + y &= 1 \end{aligned}$$

has no solution. Find the vector $\mathbf{z}$ that best approximates a solution.

Explanation. We have

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}.$$

The normal equations are $(A^T A) \mathbf{z} = A^T \mathbf{b}$. We compute

$$\begin{bmatrix} 3 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{z} = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}.$$

$$\begin{bmatrix} 14 & 1 \\ 1 & 6 \end{bmatrix} \mathbf{z} = \begin{bmatrix} 14 \\ -3 \end{bmatrix}$$

We observe that $A^T A$ is invertible. Multiplying on the left by $(A^T A)^{-1}$ yields

$$\mathbf{z} = \frac{1}{83} \begin{bmatrix} 87 \\ -56 \end{bmatrix}$$

With this vector $\mathbf{z}$, the left sides of the equations become

$$\begin{aligned} 3(87/83) - (-56/83) &\approx 3.82 \\ (87/83) + 2(-56/83) &\approx -0.30 \\ 2(87/83) + (-56/83) &\approx 1.42 \end{aligned}$$

This is as close as possible to a solution.

Example 9. The average number g of goals per game scored by a hockey player seems to be related linearly to two factors: the number x_1 of years of experience and the number x_2 of goals in the preceding 10 games. The data on the following page were collected on four players. Find the linear function $g = a_0 + a_1 x_1 + a_2 x_2$ that best fits the data.

g	x_1	x_2
0.8	5	3
0.8	3	4
0.6	1	5
0.4	2	1

Explanation. If the relationship is given by $g = a_0 + a_1 x_1 + a_2 x_2$, then the data can be described as follows:

$$\begin{bmatrix} 1 & 5 & 3 \\ 1 & 3 & 4 \\ 1 & 1 & 5 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.8 \\ 0.6 \\ 0.4 \end{bmatrix}$$

Using Theorem ??, we get

$$\mathbf{z} = \frac{1}{42} \underbrace{\begin{bmatrix} 119 & -17 & -19 \\ -17 & 5 & 1 \\ -19 & 1 & 5 \end{bmatrix}}_{(A^T A)^{-1}} \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 5 & 3 & 1 & 2 \\ 3 & 4 & 5 & 1 \end{bmatrix}}_{A^T} \begin{bmatrix} 0.8 \\ 0.8 \\ 0.6 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.14 \\ 0.09 \\ 0.08 \end{bmatrix}$$

Hence the best-fitting function is $g = 0.14 + 0.09x_1 + 0.08x_2$.

Application of Least Squares to Curve Fitting

In [Curve Fitting](#), we discussed how to fit a function to a set of data points, so that the graph of the function passes through each of the points. We also discussed why doing so is sometimes impossible (two points lie one above the other), and may not even be desirable (overfitting). In this section we will learn how to approximate a collection of data points with a line (or a curve) that fits the “trend” of the points. We will start with data that fit a linear pattern.

Exploration 19 Consider the points $(1, 1)$, $(2, 3)$ and $(4, 4)$. These points do not lie on a straight line, but they have a general upward linear trend. (Typically there would be many more points to consider, but we will limit our exploration to what we can do by hand.) Our goal is to find a line that fits these points as closely as possible.

We are looking for a function f of the form $f(x) = ax + b$ such that the following infeasible system is satisfied as closely as possible

$$\begin{aligned} a(1) + b &= 1 \\ a(2) + b &= 3 \\ a(4) + b &= 4 \end{aligned}$$

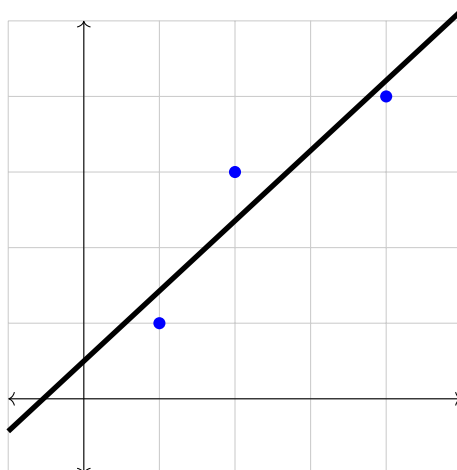
From the first part of this section we know how to find a best approximation. By Theorem ??, we have

$$\begin{aligned} \mathbf{z} = \begin{bmatrix} a \\ b \end{bmatrix} &= (A^T A)^{-1} A^T \mathbf{b} \\ &= \left(\begin{bmatrix} 1 & 2 & 4 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 2 & 4 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 13/14 \\ 1/2 \end{bmatrix} \end{aligned}$$

According to our computations, the line that best fits the data is given by

$$f(x) = \frac{13}{14}x + \frac{1}{2}$$

Let's take a look.



We found this fit by minimizing $\|\mathbf{b} - \mathbf{Az}\|$. We will now investigate the meaning of this expression in relation to the line and the data points.

$$\mathbf{b} - \mathbf{Az} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} (13/14)(1) + 1/2 \\ (13/14)(2) + 1/2 \\ (13/14)(4) + 1/2 \end{bmatrix} \approx \begin{bmatrix} -0.43 \\ 0.64 \\ -0.21 \end{bmatrix} \quad (7)$$

Observe that each entry of $\mathbf{b} - \mathbf{Az}$ is the signed vertical distance between a particular point and the line.

Instead of computing the error, $\|\mathbf{b} - \mathbf{Az}\|$, we will compute $\|\mathbf{b} - \mathbf{Az}\|^2$ to avoid the square root.

$$\|\mathbf{b} - \mathbf{Az}\|^2 = (-0.43)^2 + 0.64^2 + (-0.21)^2 \approx 0.64 \quad (8)$$

Minimizing $\|\mathbf{b} - \mathbf{Az}\|$ also minimizes $\|\mathbf{b} - \mathbf{Az}\|^2$. Therefore, what we have minimized is the sum of squares of the vertical distances between the data points and the line. The following GeoGebra interactive will help you explore this idea.

Access GeoGebra interactives through the online version of this text at

<https://ximera.osu.edu/linearalgebradv3/LinearAlgebraInteractiveIntro>.

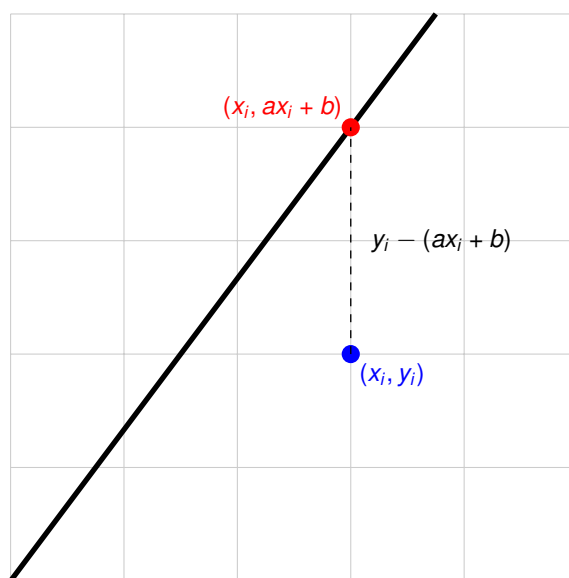
In Exploration ?? we discovered that $\|\mathbf{b} - \mathbf{Az}\|^2$ is the sum of squares of vertical distances between the given data points and the proposed line. By minimizing $\|\mathbf{b} - \mathbf{Az}\|$, we minimize the sum of squares of vertical distances. This observation holds in general. Given a collection of points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$, finding a linear function of the form $f(x) = ax + b$ that best fits the points we would find a best solution to the system

$$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

by minimizing

$$\left\| \begin{bmatrix} y_1 - (ax_1 + b) \\ y_2 - (ax_2 + b) \\ \vdots \\ y_n - (ax_n + b) \end{bmatrix} \right\|^2 = (y_1 - (ax_1 + b))^2 + (y_2 - (ax_2 + b))^2 + \cdots + (y_n - (ax_n + b))^2$$

A geometric interpretation of $y_i - (ax_i + b)$ is shown below.



The line we obtain in this fashion is called a *line of best fit* or a *trendline*, and the method we used is referred to as the *method of least squares*.

We can apply the method of least squares to find best fitting non-linear functions.

Example 10. Find the least squares approximating quadratic polynomial of the form $f(x) = ax^2 + bx + c$ for the following points.

$$(-3, 3), (-1, 1), (0, 1), (1, 2), (3, 4)$$

Explanation. We are looking for an approximate solution to the system of equations

$$\begin{aligned} a(-3)^2 + b(-3) + c &= 3 \\ a(-1)^2 + b(-1) + c &= 1 \\ a(0)^2 + b(0) + c &= 1 \\ a(1)^2 + b(1) + c &= 2 \\ a(3)^2 + b(3) + c &= 4 \end{aligned}$$

This corresponds to the matrix equation

$$\begin{bmatrix} 9 & -3 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 9 & 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 2 \\ 4 \end{bmatrix}$$

Multiplying on the left by A^T gives us the normal equations.

$$A^T A \mathbf{z} = A^T \mathbf{b}$$

$$\begin{bmatrix} 164 & 0 & 20 \\ 0 & 20 & 0 \\ 20 & 0 & 5 \end{bmatrix} \mathbf{z} = \begin{bmatrix} 66 \\ 11 \\ 4 \end{bmatrix}$$

It turns out that $A^T A$ is invertible, so it is easy to solve for $\mathbf{z}$. You can use technology to accomplish this.

We arrive at the solution

$$\mathbf{z} = \begin{bmatrix} 0.26 \\ 0.20 \\ 1.15 \end{bmatrix}$$

Therefore, the quadratic function of best fit is given by $f(x) = 0.26x^2 + 0.2x + 1.15$. You can see the graph and the points shown below.

For more information about doing least squares with Octave, please see [Octave for Chapter 9](#).

Example 11. Given the data points $(-1, 0)$, $(0, 1)$, and $(1, 4)$, find the least squares approximating function of the form $f(x) = ax + b2^x$.

Explanation. We are looking for an approximate solution to the system of equations

$$\begin{aligned} a(-1) + b(2^{-1}) &= 0 \\ a(0) + b(2^0) &= 1 \\ a(1) + b(2^1) &= 4 \end{aligned}$$

This corresponds to the matrix equation

$$\begin{bmatrix} -1 & 1/2 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$$

Using the normal equations, we obtain

$$A^T A \mathbf{z} = A^T \mathbf{b}$$

$$\frac{1}{4} \begin{bmatrix} 8 & 6 \\ 6 & 21 \end{bmatrix} \mathbf{z} = \begin{bmatrix} 4 \\ 9 \end{bmatrix}$$

Solving for $\mathbf{z}$ yields

$$\mathbf{z} = \begin{bmatrix} 10/11 \\ 16/11 \end{bmatrix}$$

Therefore, the function of best fit (of the given form) is given by

$$f(x) = \frac{10}{11}x + \frac{16}{11}(2^x)$$

Practice Problems

Problem 20 Find the best approximation to a solution to the system of equations.

$$\begin{array}{rrcr} 3x & + & y & + & z & = & 6 \\ 2x & + & 3y & - & z & = & 1 \\ 2x & - & y & + & z & = & 0 \\ 3x & - & 3y & + & 3z & = & 8 \end{array}$$

Enter answers in fraction form below.

$$x = \dots\dots, \quad y = \dots\dots, \quad z = \dots\dots$$

Problem 21 Find a linear function of best fit for the given set of data points. Examine how well your line fits the points by typing the equation of the line into the Desmos window.

$$(2, 4), (4, 3), (7, 2), (8, 1)$$

Enter your answers in fraction form.

$$f(x) = \dots\dots x + \dots\dots$$

Problem 22 Find a linear function of best fit for the given set of data points. Examine how well your line fits the points by typing the equation of the line into the Desmos window.

$$(-2, 3), (-1, 1), (0, 0), (1, -2), (2, -4)$$

$$f(x) = \dots\dots x + \dots\dots$$

Problem 23 Use Octave to find the least squares approximating quadratic function for the following data points.

$$(-2, 1), (0, 0), (3, 2), (4, 3)$$

Round your answers to three decimal places.

$$f(x) = \dots\dots x^2 + \dots\dots x + \dots\dots$$

For more information about doing least squares with Octave, please see [Octave for Chapter 9](#).

Text Source

A portion of this section has been adapted from W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2021-A, Open Edition, pp. 308-319.