
Week 15

June 5, 2026

Contents

1 Bases and Dimension of Abstract Vector Spaces

When working with $\mathbb{R}^n$ and subspaces of $\mathbb{R}^n$ we developed several fundamental ideas including *span*, *linear independence*, *bases* and *dimension*. We will find that these concepts generalize easily to abstract vector spaces and that analogous results hold in these new settings.

Linear Independence

Definition 1 (Linear Independence). Let V be a vector space. Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ be vectors of V . We say that the set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ is *linearly independent* if the only solution to

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_p\mathbf{v}_p = \mathbf{0}$$

is the *trivial solution* $a_1 = a_2 = \dots = a_p = 0$.

If, in addition to the trivial solution, a *non-trivial solution* (not all $a_1, a_2, \dots, a_p$ are zero) exists, then we say that the set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ is *linearly dependent*.

Example 1. Show that $P = \{1 + x, 3x + x^2, 2 + x - x^2\}$ is linearly independent in $\mathbb{P}^2$.

Explanation. Consider the linear combination equation

$$\begin{aligned} a(1 + x) + b(3x + x^2) + c(2 + x - x^2) &= 0 \\ a + ax + 3bx + bx^2 + 2c + cx - cx^2 &= 0 \\ (a + 2c) + (a + 3b + c)x + (b - c)x^2 &= 0 \end{aligned}$$

The constant term, as well as the coefficients in front of x and x^2 , must be equal to 0. This gives us the following system of equations.

$$\begin{array}{ccccccc} a & + & & & + & 2c & = & 0 \\ a & + & 3b & + & & c & = & 0 \\ & & b & - & c & & = & 0 \end{array}$$

The only solution is $a = b = c = 0$. We conclude that P is linearly independent in $\mathbb{P}^2$.

Bases and Dimension

Recall that our motivation for defining a basis of a subspace of $\mathbb{R}^n$ was to have a collection of vectors such that every vector of the subspace can be expressed as a unique linear combination of the vectors in that collection. Definition of a basis (??) generalizes to abstract vector spaces as follows.

Definition 2. Let V be a vector space. A set $\mathcal{B}$ of vectors of V is called a *basis* of V provided that

1. $\text{span}(\mathcal{B}) = V$
2. $\mathcal{B}$ is linearly independent.

Theorem 1. Let V be a vector space, and let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a basis for V . Then every element $\mathbf{v}$ of V has a unique representation as linear combination of the elements of $\mathcal{B}$.

Proof By the definition of a basis, we know that $\mathbf{v}$ can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$. Suppose there are two such representations. Then,

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_n\mathbf{v}_n$$

$$\mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \dots + b_n\mathbf{v}_n$$

But then we have:

$$\begin{aligned} a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_n\mathbf{v}_n &= b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \dots + b_n\mathbf{v}_n \\ a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_n\mathbf{v}_n - (b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \dots + b_n\mathbf{v}_n) &= \mathbf{0} \\ (a_1 - b_1)\mathbf{v}_1 + (a_2 - b_2)\mathbf{v}_2 + \dots + (a_n - b_n)\mathbf{v}_n &= \mathbf{0} \end{aligned}$$

Because $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are linearly independent, we have $a_i - b_i = 0$ for $1 \leq i \leq n$. Consequently $a_i = b_i$ for $1 \leq i \leq n$. ■

In [Bases and Dimension](#) we defined the dimension of a subspace of $\mathbb{R}^n$ to be the number of elements in a basis. (Definition ??) We will adopt this definition for abstract vector spaces. As before, to ensure that *dimension* is well-defined we need to establish that this definition is independent of our choice of a basis. The proof of the following theorem is identical to the proof of its counterpart in $\mathbb{R}^n$. (Theorem ??)

Theorem 2. Let V be a vector space. Suppose $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_t\}$ and $\mathcal{C} = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_s\}$ be two bases of V . Then $s = t$.

Now we can state the definition.

Definition 3. Let V be a subspace of a vector space W . The *dimension* of V is the number, m , of elements in any basis of V . We write

$$\dim(V) = m$$

In our discussions up to this point, we have always assumed that a basis is nonempty and hence that the dimension of the space is at least 1. However, the zero space $\{\mathbf{0}\}$ has *no* basis. To accommodate for this, we will say that the zero vector space $\{\mathbf{0}\}$ is defined to have dimension 0:

$$\dim\{\mathbf{0}\} = 0$$

Our insistence that $\dim\{\mathbf{0}\} = 0$ amounts to saying that the empty set of vectors is a basis of $\{\mathbf{0}\}$. Thus the statement that “the dimension of a vector space is the number of vectors in any basis” holds even for the zero space.

Example 2. Recall that the vector space $\mathbb{M}_{m,n}$ consists of all $m \times n$ matrices. (See Example ?? of [Abstract Vector Spaces](#)). Find a basis and the dimension of $\mathbb{M}_{m,n}$.

Explanation. Let $\mathcal{B}$ consist of $m \times n$ matrices with exactly one entry equal to 1 and all other entries equal to 0. It is clear that every $m \times n$ matrix can be written as a linear combination of elements of $\mathcal{B}$. It is also easy to see that the elements of $\mathcal{B}$ are linearly independent. Thus $\mathcal{B}$ is a basis for $\mathbb{M}_{m,n}$. The set $\mathcal{B}$ contains mn elements, so $\dim(\mathbb{M}_{m,n}) = mn$.

Example 3. Recall that $\mathbb{P}^n$ is the set of all polynomials of degree n or less. (See Example ?? of [Abstract Vector Spaces](#)) Show that $\dim(\mathbb{P}^n) = n + 1$ and that $\{1, x, x^2, \dots, x^n\}$ is a basis of $\mathbb{P}^n$.

Explanation. Each polynomial $p(x) = a_0 + a_1x + \dots + a_nx^n$ in $\mathbb{P}^n$ is clearly a linear combination of $1, x, \dots, x^n$, so $\mathbb{P}^n = \text{span}\{1, x, \dots, x^n\}$.

Suppose $a_0 \cdot 1 + a_1x + \dots + a_nx^n = 0$, then $a_0 = a_1 = \dots = a_n = 0$. So $\{1, x, \dots, x^n\}$ is linearly independent and hence is a basis containing $n + 1$ vectors. Thus, $\dim(\mathbb{P}^n) = n + 1$.

Example 4. Consider the subset

$$C_A = \{X \in \mathbb{M}_{2,2} : AX = XA\}$$

of $\mathbb{M}_{2,2}$.

It was shown in Example ?? of [Abstract Vector Spaces](#) that C_A is a subspace for any choice of the matrix A .

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Show that $\dim(C_A) = 2$ and find a basis of C_A .

Explanation. Suppose $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is in C_A . Then

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a+c & b+d \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & a \\ c & c \end{bmatrix}$$

This gives us two relationships:

$$b + d = a \quad \text{and} \quad c = 0$$

We can now express a generic element X of C_A as

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} b+d & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} b & b \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix} = b \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + d \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

The set $\mathcal{B}$ is linearly independent. (See Practice Problem ??) Every element X of C_A can be written as a linear combination of elements of $\mathcal{B}$. Thus $C_A = \text{span}\mathcal{B}$. Therefore $\mathcal{B}$ is a basis of C_A , and $\dim(C_A) = 2$.

Example 5. In Practice Problem ?? of [Abstract Vector Spaces](#) you demonstrated that the set of all symmetric $n \times n$ matrices is a subspace of $\mathbb{M}_{n,n}$.

Let V be a subspace of $\mathbb{M}_{2,2}$ consisting of all 2×2 symmetric matrices. Find the dimension of V .

Explanation. A matrix A is symmetric if $A^T = A$. In other words, a matrix A is symmetric when entries directly across the main diagonal are equal, so each 2×2 symmetric matrix has the form

$$\begin{bmatrix} a & c \\ c & b \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + c \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Hence the set $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ spans V . The reader can verify that $\mathcal{B}$ is linearly independent.

Thus $\mathcal{B}$ is a basis of V , so $\dim(V) = 3$.

Finite-Dimensional Vector Spaces

Our definition of dimension of a vector space depends on the vector space having a basis. In this section we will establish that any vector space spanned by finitely many vectors has a basis.

Definition 4. A vector space is said to be *finite-dimensional* if it is spanned by finitely many vectors.

Given a finite-dimensional vector space V we will find a basis for V by starting with a linearly independent subset of V and expanding it to a basis. The following results are more general versions of Lemmas ?? and ??, and Theorem ?? of [Bases and Dimension](#). The proofs are identical and we will omit them.

Lemma 1. Let V be a vector space spanned by n vectors. If a linearly independent subset S of V contains m vectors, then $m \leq n$.

Lemma 2. Let V be a vector space. Let $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ be a linearly independent subset of V . If $\mathbf{u}$ is not in $\text{span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$, then $\{\mathbf{u}, \mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly independent.

Theorem 3. Let V be a finite-dimensional vector space. Any linearly independent subset of V can be expanded to a basis of V .

Coordinate Vectors

Recall that in the context of $\mathbb{R}^n$ (and subspaces of $\mathbb{R}^n$) the requirement that elements of a basis be linearly independent guarantees that every element of the vector space has a *unique* representation in terms of the elements of the basis. (See Theorem ?? of [Introduction to Bases](#)) We proved the same property for abstract vector spaces in Theorem ??.

Uniqueness of representation in terms of the elements of a basis allows us to associate every element of a vector space V with a unique *coordinate vector* with respect to a given basis. Coordinate vectors were first introduced in [Introduction to Bases](#). We now give a formal definition.

Definition 5. Let V be a vector space, and let $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis for V . If $\mathbf{v} = a_1\mathbf{v}_1 + \dots + a_n\mathbf{v}_n$, then the vector in $\mathbb{R}^n$ whose components are the coefficients $a_1, \dots, a_n$ is said to be the *coordinate vector for $\mathbf{v}$ with respect to $\mathcal{B}$* . We denote the coordinate vector by $[\mathbf{v}]_{\mathcal{B}}$ and write:

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

Remark 1. The order of in which vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ appear in $\mathcal{B}$ of Definition ?? is important. Switching the order of these vectors would switch the order of the coordinate vector components. For this reason, we will often use the term *ordered basis* to describe $\mathcal{B}$ in the context of coordinate vectors.

Example 6. The coordinate vector for $p(x) = 4 - 3x^2 + 5x^3$ in $\mathbb{P}^4$ with respect to the ordered basis $\mathcal{B}_1 = \{1, x, x^2, x^3, x^4\}$ is

$$[p(x)]_{\mathcal{B}_1} = \begin{bmatrix} 4 \\ 0 \\ -3 \\ 5 \\ 0 \end{bmatrix}$$

Now let's change the order of the elements in $\mathcal{B}_1$. The coordinate vector for $p(x) = 4 - 3x^2 + 5x^3$ with respect to the ordered basis $\mathcal{B}_2 = \{x^4, x^3, x^2, x, 1\}$ is

$$[p(x)]_{\mathcal{B}_2} = \begin{bmatrix} 0 \\ 5 \\ -3 \\ 0 \\ 4 \end{bmatrix}$$

Example 7. Show that the set $\mathcal{B} = \{x, 1 + x, x + x^2\}$ is a basis for $\mathbb{P}^2$. Keep the order of elements in $\mathcal{B}$ and find the coordinate vector for $p(x) = 4 - x + 3x^2$ with respect to the ordered basis $\mathcal{B}$.

Explanation. We will begin by showing that the elements of $\mathcal{B}$ are linearly independent. Suppose

$$ax + b(1 + x) + c(x + x^2) = 0$$

then

$$b + (a + b + c)x + cx^2 = 0$$

This gives us the following system of equations:

$$\begin{array}{rcl} & b & = 0 \\ a + b + c & = & 0 \\ & c & = 0 \end{array}$$

The solution $a = b = c = 0$ is unique. We conclude that $\mathcal{B}$ is linearly independent.

Next, we need to show that $\mathcal{B}$ spans $\mathbb{P}^2$. To this end, we will consider a generic element $p(x) = \alpha + \beta x + \gamma x^2$ of $\mathbb{P}^2$ and attempt to express it as a linear combination of the elements of $\mathcal{B}$.

$$ax + b(1 + x) + c(x + x^2) = \alpha + \beta x + \gamma x^2$$

then

$$b + (a + b + c)x + cx^2 = \alpha + \beta x + \gamma x^2$$

Setting the coefficients of like terms equal to each other gives us

$$\begin{array}{rcl} & b & = \alpha \\ a + b + c & = \beta \\ & c & = \gamma \end{array}$$

Solving this linear system of a , b and c gives us

$$a = \beta - \alpha - \gamma, \quad b = \alpha, \quad c = \gamma$$

(You should verify this.) This shows that every element of $\mathbb{P}^2$ can be written as a linear combination of elements of $\mathcal{B}$. Therefore $\mathcal{B}$ is a basis for $\mathbb{P}^2$.

To find the coordinate vector for $p(x) = 4 - x + 3x^2$ with respect to $\mathcal{B}$ we need to express $p(x)$ as a linear combination of the elements of $\mathcal{B}$. Fortunately, we have already done all the necessary work. For $p(x)$, $\alpha = 4$, $\beta = -1$ and $\gamma = 3$. This gives us the coefficients of the linear combination: $a = \beta - \alpha - \gamma = -8$, $b = \alpha = 4$, $c = \gamma = 3$. We now write $p(x)$ as a linear combination

$$p(x) = -8(x) + 4(1 + x) + 3(x + x^2)$$

The coordinate vector for $p(x)$ with respect to $\mathcal{B}$ is

$$[p(x)]_{\mathcal{B}} = \begin{bmatrix} -8 \\ 4 \\ 3 \end{bmatrix}$$

Example 8. Recall that the set V of all symmetric 2×2 matrices is a subspace of $\mathbb{M}_{2,2}$. In Example ?? we demonstrated that $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ is a basis for V . Let $A = \begin{bmatrix} 2 & -3 \\ -3 & 1 \end{bmatrix}$. Observe that A is an element of V .

1. Find the coordinate vector with respect to the ordered basis $\mathcal{B}$ for A .
2. Let $\mathcal{B}' = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ be another ordered basis for V . Find the coordinate vector for A with respect to $\mathcal{B}'$.

Explanation. We write A as a linear combination of the elements of $\mathcal{B}$.

$$A = \begin{bmatrix} 2 & -3 \\ -3 & 1 \end{bmatrix} = 2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - 3 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Thus, the coordinate vector with respect to $\mathcal{B}$ is

$$[A]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$$

The coordinate vector with respect to $\mathcal{B}'$ is

$$[A]_{\mathcal{B}'} = \begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix}$$

Coordinate vectors will play a vital role in establishing one of the most fundamental results in linear algebra, that all n -dimensional vector spaces have the same structure as $\mathbb{R}^n$. In Example ?? of [Isomorphic Vector Spaces](#), for instance, we will show that $\mathbb{P}^2$ is essentially the same as $\mathbb{R}^3$.

Practice Problems

Problem 1 Consider the subset

$$C_A = \{X \in \mathbb{M}_{2,2} : AX = XA\}$$

of $\mathbb{M}_{2,2}$. (See Example ??)

Prove that set $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a linearly independent subset of C_A .

Problem 2 Show that the following subset of $\mathbb{P}^2$ is linearly independent, and is a basis of $\mathbb{P}^2$.

$$\{1 + x, 1 - x, x + x^2\}$$

Problem 3 Show that the following subset of $\mathbb{P}^2$ is linearly independent, and is a basis of $\mathbb{P}^2$.

$$\{x^2, x + 1, 1 - x - x^2\}$$

Problem 4 Show that the following subset of $\mathbb{M}_{2,2}$ is linearly independent, and is a basis of $\mathbb{M}_{2,2}$.

$$\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$$

Problem 5 Find the coordinate vector for $p(x) = 6 - 2x + 4x^2$ in $\mathbb{P}^2$ with respect to

$$\mathcal{B} = \{1 + x, 1 - x, x + x^2\}$$

Answer:

$$[p(x)]_{\mathcal{B}} = \begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix}$$

Problem 6 Find the coordinate vector for $p(x) = 6 - 2x + 4x^2$ in $\mathbb{P}^2$ with respect to

$$\mathcal{B} = \{x^2, x + 1, 1 - x - x^2\}$$

Answer:

$$[p(x)]_{\mathcal{B}} = \begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix}$$

Problem 7 Find the coordinate vector for $A = \begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix}$ in $\mathbb{M}_{2,2}$ with respect to the ordered basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$$

Answer:

$$[A]_{\mathcal{B}} = \begin{bmatrix} \dots \\ \dots \\ \dots \\ \dots \end{bmatrix}$$

Text Source

The discussion of the zero space was adapted from Section 6.3 of Keith Nicholson's [Linear Algebra with Applications](#). (CC-BY-NC-SA)

W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2018, Open Edition, p. 349

Example Source

Examples ?? and ?? were adapted from Examples 6.3.1 and 6.3.10 of Keith Nicholson's *Linear Algebra with Applications*. (CC-BY-NC-SA)

W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2018, Open Edition, p. 346, 350

Exercise Source

Practice Problems ??, ?? and ?? are Exercises 6.3(a)(b)(c) from Keith Nicholson's *Linear Algebra with Applications*. (CC-BY-NC-SA)

W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2018, Open Edition, p. 351

2 Coordinate Mappings

Recall that a transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is called a *linear transformation* if the following are true for all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, and scalars k .

$$T(k\mathbf{u}) = kT(\mathbf{u})$$

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$

We generalize this definition as follows.

Definition 6. Let V and W be vector spaces. A transformation $T : V \rightarrow W$ is called a *linear transformation* if the following are true for all vectors $\mathbf{u}$ and $\mathbf{v}$ in V , and scalars k .

$$T(k\mathbf{u}) = kT(\mathbf{u})$$

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$

Linearity of Coordinate Mappings

Transformations that map vectors to their coordinate vectors with respect to some ordered basis will prove to be of great importance. We will start by showing that such transformations are linear.

If V is a vector space, and $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is an ordered basis for V then any vector $\mathbf{v}$ of V can be uniquely expressed as $\mathbf{v} = a_1\mathbf{v}_1 + \dots + a_n\mathbf{v}_n$ for some scalars $a_1, \dots, a_n$. Vector $[\mathbf{v}]_{\mathcal{B}}$ in $\mathbb{R}^n$ given by

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

is said to be the *coordinate vector for $\mathbf{v}$ with respect to the ordered basis $\mathcal{B}$* . (See Definition ??.)

It turns out that the transformation $T : V \rightarrow \mathbb{R}^n$ defined by $T(\mathbf{v}) = [\mathbf{v}]_{\mathcal{B}}$ is linear. Before we prove linearity of T , consider the following example.

Example 9. Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ be an ordered basis for $\mathbb{M}_{2,2}$. (You should do a quick

mental check that $\mathcal{B}$ is a legitimate basis.) Define $T : \mathbb{M}_{2,2} \rightarrow \mathbb{R}^4$ by $T(A) = [A]_{\mathcal{B}}$. Find $T \left(\begin{bmatrix} -2 & 3 \\ 1 & -5 \end{bmatrix} \right)$.

Explanation. We need to find the coordinate vector for $\begin{bmatrix} -2 & 3 \\ 1 & -5 \end{bmatrix}$ with respect to $\mathcal{B}$.

$$\begin{bmatrix} -2 & 3 \\ 1 & -5 \end{bmatrix} = -2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 3 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + (-5) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

This gives us:

$$T\left(\begin{bmatrix} -2 & 3 \\ 1 & -5 \end{bmatrix}\right) = \left[\begin{bmatrix} -2 & 3 \\ 1 & -5 \end{bmatrix}\right]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \\ 1 \\ -5 \end{bmatrix}$$

Theorem 4. Let V be an n -dimensional vector space, and let $\mathcal{B}$ be an ordered basis for V . Then $T : V \rightarrow \mathbb{R}^n$ given by $T(\mathbf{v}) = [\mathbf{v}]_{\mathcal{B}}$ is a linear transformation.

Proof First observe that Theorem ?? of [Bases and Dimension of Abstract Vector Spaces](#) guarantees that there is only one way to represent each element of V as a linear combination of elements of $\mathcal{B}$. Thus each element of V maps to exactly one element of $\mathbb{R}^n$, as long as the order in which elements of $\mathcal{B}$ appear is taken into account. This proves that T is a function, or a transformation. We will now prove that T is linear.

Let $\mathbf{v}$ be an element of V . We will first show that $T(k\mathbf{v}) = kT(\mathbf{v})$. Suppose $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$, then $\mathbf{v}$ can be written as a unique linear combination:

$$\mathbf{v} = a_1\mathbf{v}_1 + \dots + a_n\mathbf{v}_n$$

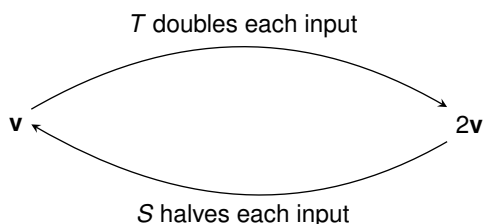
We have:

$$\begin{aligned} T(k\mathbf{v}) &= T(k(a_1\mathbf{v}_1 + \dots + a_n\mathbf{v}_n)) \\ &= T((ka_1)\mathbf{v}_1 + \dots + (ka_n)\mathbf{v}_n) \\ &= \begin{bmatrix} ka_1 \\ \vdots \\ ka_n \end{bmatrix} = k \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = kT(\mathbf{v}) \end{aligned}$$

We leave it to the reader to verify that $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$. (See Practice Problem ??.) ■

Invertibility of Coordinate Mappings

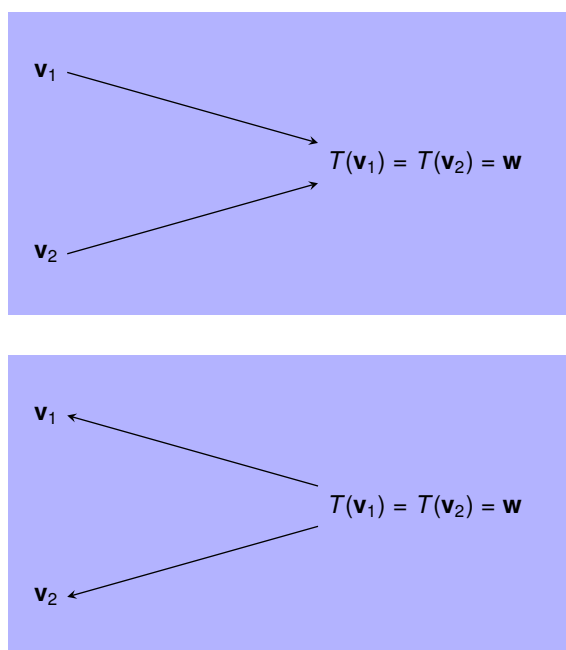
Consider a linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that scales all input vectors by a factor of two, and a linear transformation $S : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that scales all input vectors by a factor of one half. The composite functions $S \circ T$ and $T \circ S$ are both identity transformations. S and T are clearly inverses of each other. Diagrammatically, we can represent T and S as follows:



This gives us a way of thinking about an inverse of T as a transformation that “undoes” the action of T by “reversing” the mapping arrows. We will now use these intuitive ideas to understand which linear transformations are invertible and which are not.

Given an arbitrary linear transformation $T : V \rightarrow W$, “reversing the arrows” may not always result in a transformation. Recall that transformations are functions. The figures below show two ways in which our attempt to “reverse” T may fail to produce a function.

First, if two distinct vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ map to the same vector $\mathbf{w}$ in W , then reversing the arrows gives us a mapping that is clearly not a function.

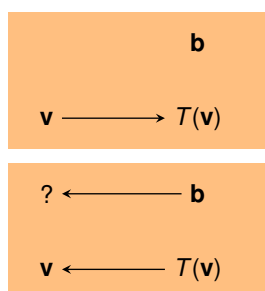


Based on this diagram, it is reasonable to conjecture that for a transformation to be invertible, the transformation must be such that each output is the image of exactly one input. Such transformations are called *one-to-one*.

Definition 7 (One-to-One). A linear transformation $T : V \rightarrow W$ is *one-to-one* if

$$T(\mathbf{v}_1) = T(\mathbf{v}_2) \text{ implies that } \mathbf{v}_1 = \mathbf{v}_2$$

Second, observe that our definition of an inverse of $T : V \rightarrow W$ requires that the domain of the inverse transformation be W . (Definition ??, [Composition and Inverses of Linear Transformations](#)) If there is a vector $\mathbf{b}$ in W that is not an image of any vector in V , then $\mathbf{b}$ cannot be in the domain of an inverse transformation.



The above figure makes a convincing case that for a transformation to be invertible every element of the codomain must have something mapping to it. Transformations such that every element of the codomain is an image of some element of the domain are called *onto*.

Definition 8 (Onto). A linear transformation $T : V \rightarrow W$ is *onto* if for every element $\mathbf{w}$ of W , there exists an element $\mathbf{v}$ of V such that $T(\mathbf{v}) = \mathbf{w}$.

Theorem 5. Let V and W be vector spaces, and let $T : V \rightarrow W$ be a linear transformation. Then T has an inverse if and only if T is one-to-one and onto.

Proof We will first assume that T is one-to-one and onto, and show that there exists a transformation $S : W \rightarrow V$ such that $S \circ T = \text{id}_V$ and $T \circ S = \text{id}_W$. Because T is onto, for every $\mathbf{w}$ in W , there exists $\mathbf{v}$ in V such that $T(\mathbf{v}) = \mathbf{w}$. Moreover, because T is one-to-one, vector $\mathbf{v}$ is the only vector that maps to $\mathbf{w}$. To stress this, we will say that for every $\mathbf{w}$, there exists $\mathbf{v}_\mathbf{w}$ such that $T(\mathbf{v}_\mathbf{w}) = \mathbf{w}$. (Since every $\mathbf{v}$ maps to exactly one $\mathbf{w}$, this notation makes sense for elements of V as well.) We can now define $S : W \rightarrow V$ by $S(\mathbf{w}) = \mathbf{v}_\mathbf{w}$. Then

$$(S \circ T)(\mathbf{v}_\mathbf{w}) = S(T(\mathbf{v}_\mathbf{w})) = S(\mathbf{w}) = \mathbf{v}_\mathbf{w}$$

$$(T \circ S)(\mathbf{w}) = T(S(\mathbf{w})) = T(\mathbf{v}_\mathbf{w}) = \mathbf{w}$$

We conclude that $S \circ T = \text{id}_V$ and $T \circ S = \text{id}_W$. Therefore S is an inverse of T .

We will now assume that T has an inverse S and show that T must be one-to-one and onto. Suppose

$$T(\mathbf{v}_1) = T(\mathbf{v}_2)$$

then

$$S(T(\mathbf{v}_1)) = S(T(\mathbf{v}_2))$$

but then

$$\mathbf{v}_1 = \mathbf{v}_2$$

We conclude that T is one-to-one.

Now suppose that $\mathbf{w}$ is in W . We need to show that some element of V maps to $\mathbf{w}$. Let $\mathbf{v} = S(\mathbf{w})$. Then

$$T(\mathbf{v}) = T(S(\mathbf{w})) = (T \circ S)(\mathbf{w}) = \text{id}_W(\mathbf{w}) = \mathbf{w}$$

We conclude that T is onto. ■

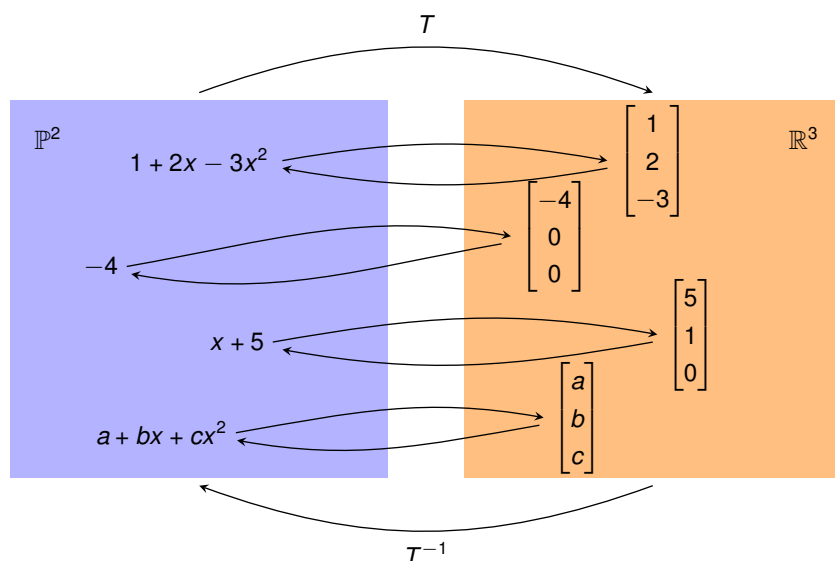
Theorem 6. Let V be a vector space, and let $\mathcal{B}$ be an ordered basis for V . Then the linear transformation $T : V \rightarrow \mathbb{R}^n$ given by $T(\mathbf{v}) = [\mathbf{v}]_{\mathcal{B}}$ is invertible.

Proof We leave the proof of this result to the reader. ■

Example 10. Recall that the set of all polynomials of degree 2 or less, together with polynomial addition and scalar multiplication, is a vector space, denoted by $\mathbb{P}^2$. Let $\mathcal{B} = \{1, x, x^2\}$. You should do a quick mental check that $\mathcal{B}$ is a basis of $\mathbb{P}^2$.

Define a transformation $T : \mathbb{P}^2 \rightarrow \mathbb{R}^3$ by $T(a + bx + cx^2) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. In other words, T maps each element of $\mathbb{P}^2$ to its coordinate vector with respect to the ordered basis $\mathcal{B}$.

The diagram below illustrates the actions of T and T^{-1} on several elements.



Coordinate Isomorphisms

Invertible linear transformations, such as the coordinate mapping, are useful because they preserve the structure of interactions between elements as we move back and forth between two vector spaces, allowing us to answer questions about one vector space in a different vector space. In particular, any question related to linear combinations can be addressed in this fashion. This includes questions concerning linear independence, span, basis and dimension. Specifically, for coordinate mappings, it is easy to see that the following property holds.

Theorem 7. Let V be an n -dimensional vector space, and let $T : V \rightarrow \mathbb{R}^n$ be the coordinate mapping with respect to some ordered basis $\mathcal{B}$ of V . Then the set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ of V is linearly independent if and only if the set $\{T(\mathbf{x}_1), T(\mathbf{x}_2), \dots, T(\mathbf{x}_k)\}$ is linearly independent in $\mathbb{R}^n$.

Definition 9. Let V and W be vector spaces. If there exists an invertible linear transformation $T : V \rightarrow W$ we say that V and W are *isomorphic* and write $V \cong W$. The invertible linear transformation T is called an *isomorphism*.

It is worth pointing out that if $T : V \rightarrow W$ is an isomorphism, then $T^{-1} : W \rightarrow V$, being linear and invertible, is also an isomorphism.

Because coordinate mappings are isomorphisms, we have the following fundamental result.

Theorem 8. Every n -dimensional vector space is isomorphic to $\mathbb{R}^n$.

Practice Problems

Problem 8 Recall that the set V of all symmetric 2×2 matrices is a subspace of $\mathbb{M}_{2,2}$. In Example ?? of *Bases and Dimension of Abstract Vector Spaces* we demonstrated that $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ is a basis for V .

Define $T : V \rightarrow \mathbb{R}^3$ by $T(A) = [A]_{\mathcal{B}}$. Find $T(I_2)$ and $T\left(\begin{bmatrix} 2 & -3 \\ -3 & 1 \end{bmatrix}\right)$.

Answer:

$$T(I_2) = \begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix}$$

$$T\left(\begin{bmatrix} 2 & -3 \\ -3 & 1 \end{bmatrix}\right) = \begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix}$$

Problem 9 Let V be a subspace of $\mathbb{R}^3$ with a basis $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} \right\}$. Find the coordinate vector, $[\mathbf{v}]_{\mathcal{B}}$, for

$$\mathbf{v} = \begin{bmatrix} 4 \\ -1 \\ -4 \end{bmatrix}.$$

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix}$$

If the order of the basis elements in $\mathcal{B}$ were switched to form a new basis

$$\mathcal{B}' = \left\{ \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right\}$$

How would this affect the coordinate vector for $\mathbf{v}$?

$$[\mathbf{v}]_{\mathcal{B}'} = \begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix}$$

Problem 10 Verify that $\mathcal{B} = \{x^2, x + 1, 1 - x - x^2\}$ is a basis for $\mathbb{P}^2$. Define $T : \mathbb{P}^2 \rightarrow \mathbb{R}^3$ by $T(p(x)) = [p(x)]_{\mathcal{B}}$. Find $T(0)$, $T(x + 1)$ and $T(x^2 - 3x + 1)$.

Answer:

$$T(0) = \begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix}$$

$$T(x + 1) = \begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix}$$

$$T(x^2 - 3x + 1) = \begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix}$$

Problem 11 Let V and W be vector spaces, and let $\mathcal{B}_V = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ and $\mathcal{B}_W = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ be ordered bases of V and W , respectively. Suppose $T : V \rightarrow W$ is a linear transformation such that:

$$T(\mathbf{v}_1) = \mathbf{w}_2$$

$$T(\mathbf{v}_2) = 2\mathbf{w}_1 - 3\mathbf{w}_2$$

$$T(\mathbf{v}_3) = \mathbf{w}_2 + \mathbf{w}_3$$

$$T(\mathbf{v}_4) = -\mathbf{w}_1$$

If $\mathbf{v} = -2\mathbf{v}_1 + 3\mathbf{v}_2 - \mathbf{v}_4$, express $T(\mathbf{v})$ as a linear combination of vectors of $\mathcal{B}_W$.

$$T(\mathbf{v}) = \dots \mathbf{w}_1 - \dots \mathbf{w}_2 + \dots \mathbf{w}_3$$

Find $[\mathbf{v}]_{\mathcal{B}_V}$ and $[T(\mathbf{v})]_{\mathcal{B}_W}$

$$[\mathbf{v}]_{\mathcal{B}_V} = \begin{bmatrix} \dots \\ \dots \\ \dots \\ \dots \end{bmatrix}, \quad [T(\mathbf{v})]_{\mathcal{B}_W} = \begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix}$$

Problem 12 Let

$$S = \left\{ \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}, \begin{bmatrix} 4 & -2 \\ 1 & 5 \end{bmatrix}, \begin{bmatrix} 5 & 5 \\ 8 & 4 \end{bmatrix} \right\}$$

Is S linearly independent in $\mathbb{M}_{2,2}$? (Yes/ No)

Problem 13 Let

$$S = \left\{ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}, \begin{bmatrix} 9 & 10 \\ 11 & 12 \end{bmatrix}, \begin{bmatrix} 13 & 14 \\ 15 & 16 \end{bmatrix} \right\}$$

Is S a basis for $\mathbb{M}_{2,2}$? (Yes/ No)

3 Matrices of Linear Transformations with Respect to Arbitrary Bases

We know that every linear transformation from $\mathbb{R}^n$ into $\mathbb{R}^m$ is a matrix transformation (Theorem ??). However, the matrix that represents a linear transformation is tied to the basis we choose. So far we have only used the standard basis of $\mathbb{R}^n$. What about linear transformations between vector spaces other than $\mathbb{R}^n$? In this section we will learn to represent linear transformations between arbitrary finite-dimensional vector spaces using matrices. To do so, we will use the fact that every n -dimensional vector space is isomorphic to $\mathbb{R}^n$. (Theorem ??). What we do here will serve as yet another example of how isomorphisms can be used to translate problems in one vector space to another, more convenient, vector space.

Exploration 14 Define a transformation

$$\tau : \mathbb{M}_{2,2} \rightarrow \mathbb{P}^3$$

by

$$\tau \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = a + (a + c)x + (b - c)x^2 + dx^3$$

You should verify that τ is linear. (See Practice Problem ??.)

We will examine τ in an effort to find a way to represent it with a matrix. The matrix we find will depend on our choice of bases. We will choose bases that will make computations easier.

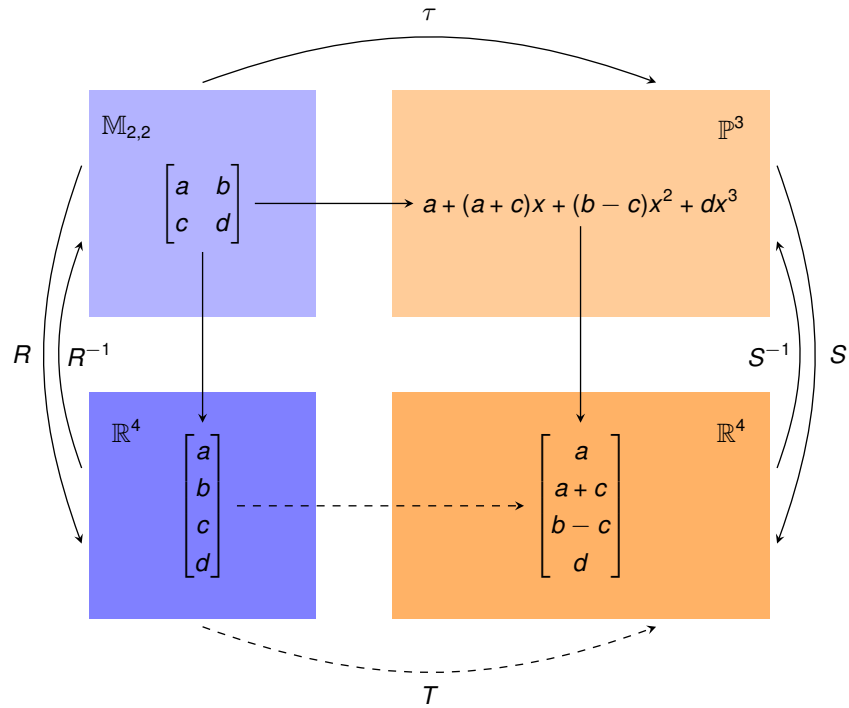
Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$\mathcal{C} = \{1, x, x^2, x^3\}$$

be our ordered bases of choice for $\mathbb{M}_{2,2}$ and $\mathbb{P}^3$, respectively.

Recall that a coordinate isomorphism maps a vector to its coordinate vector with respect to the given ordered basis (Theorem ??). In the diagram below, let R and S be coordinate isomorphisms with respect to $\mathcal{B}$ and $\mathcal{C}$, respectively.



Define $T : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ by

$$T \left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \right) = \begin{bmatrix} a \\ a+c \\ b-c \\ d \end{bmatrix}$$

Observe that $T = S \circ \tau \circ R^{-1}$. Because T is a composition of linear transformations, T itself is linear (Theorem ??). Thus, we should be able to find the standard matrix for T . To do this, find the images of the standard unit vectors and use them to create the standard matrix A for T .

$$A = \begin{bmatrix} \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

We say that A is the matrix of τ with respect to ordered bases $\mathcal{B}$ and $\mathcal{C}$.

Exploration 15 Let

$$\begin{aligned} \mathbf{v}_1 &= \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} & \text{and} & \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \\ \mathbf{w}_1 &= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} & \text{and} & \mathbf{w}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

Matrices of Linear Transformations with Respect to Arbitrary Bases

Define V and W as follows:

$$V = \text{span}(\mathbf{v}_1, \mathbf{v}_2) \quad \text{and} \quad W = \text{span}(\mathbf{w}_1, \mathbf{w}_2)$$

Geometrically, V and W are planes in $\mathbb{R}^3$. Let's choose

$$\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\} \quad \text{and} \quad \mathcal{C} = \{\mathbf{w}_1, \mathbf{w}_2\}$$

as ordered bases of V and W , respectively.

Define a transformation $T : V \rightarrow W$ by

$$T(\mathbf{v}_1) = 2\mathbf{w}_1 - 3\mathbf{w}_2 \quad \text{and} \quad T(\mathbf{v}_2) = -\mathbf{w}_1 + 4\mathbf{w}_2$$

(You should mentally verify that T is linear.)

Our goal now is to find a matrix for T with respect to $\mathcal{B}$ and $\mathcal{C}$.

The information given in this problem is slightly different from the information in Exploration ???. Instead of being given an expression for the image of a generic vector of V , we are only given the images of the two basis vectors of V . But this information is sufficient to determine the linear transformation.

As before, we will map vectors of V and W to their coordinate vectors. Where are the coordinate vectors located?

1. $[\mathbf{v}_1]_{\mathcal{B}}$ and $[\mathbf{v}_2]_{\mathcal{B}}$ are elements of $(\mathbb{R}, / \mathbb{R}^2, / \mathbb{R}^3)$

2. $[\mathbf{w}_1]_{\mathcal{C}}$ and $[\mathbf{w}_2]_{\mathcal{C}}$ are elements of $(\mathbb{R}, / \mathbb{R}^2, / \mathbb{R}^3)$

Now we find the coordinate vectors.

1.

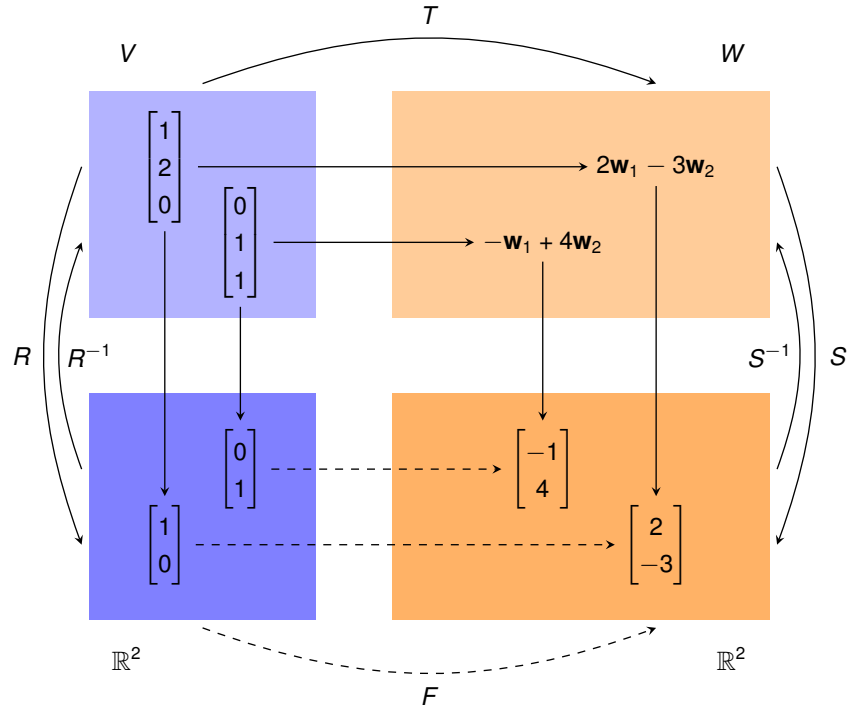
$$[\mathbf{v}_1]_{\mathcal{B}} = \begin{bmatrix} \cdots \cdots \\ \cdots \cdots \end{bmatrix} \quad \text{and} \quad [\mathbf{v}_2]_{\mathcal{B}} = \begin{bmatrix} \cdots \cdots \\ \cdots \cdots \end{bmatrix}$$

2.

$$[T(\mathbf{v}_1)]_{\mathcal{C}} = \begin{bmatrix} \cdots \cdots \\ \cdots \cdots \end{bmatrix} \quad \text{and} \quad [T(\mathbf{v}_2)]_{\mathcal{C}} = \begin{bmatrix} \cdots \cdots \\ \cdots \cdots \end{bmatrix}$$

The diagram below summarizes this information.

Matrices of Linear Transformations with Respect to Arbitrary Bases



Define $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $F = S \circ T \circ R^{-1}$. F is a linear transformation that maps $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$, respectively. Thus, the standard matrix for F is:

$$A = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$$

We say that A is a matrix for T with respect to $\mathcal{B}$ and $\mathcal{C}$.

Let's take a look at what this matrix can do for us. Suppose we want to find the image of $\mathbf{v} = 2\mathbf{v}_1 + \mathbf{v}_2$ under T . We can compute this directly, as follows:

$$T(\mathbf{v}) = T(2\mathbf{v}_1 + \mathbf{v}_2) = 2T(\mathbf{v}_1) + T(\mathbf{v}_2) = 3\mathbf{w}_1 - 2\mathbf{w}_2$$

This result can also be obtained by finding the product of A with the coordinate vector of $\mathbf{v}$.

$$\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Given any vector $\mathbf{u} = a\mathbf{v}_1 + b\mathbf{v}_2$ of V , we can find $T(\mathbf{u})$ as follows:

$$\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a - b \\ -3a + 4b \end{bmatrix}$$

This gives us

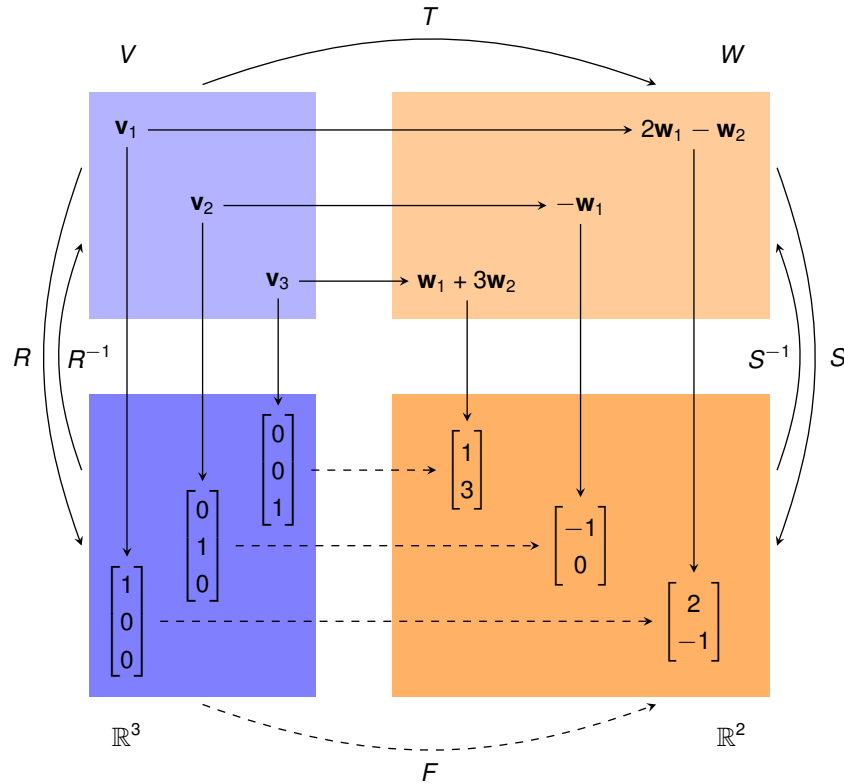
$$T(\mathbf{u}) = T(a\mathbf{v}_1 + b\mathbf{v}_2) = (2a - b)\mathbf{w}_1 + (-3a + 4b)\mathbf{w}_2$$

Example 11. Let V and W be vector spaces with ordered bases $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\mathcal{C} = \{\mathbf{w}_1, \mathbf{w}_2\}$, respectively. Define a linear transformation $T : V \rightarrow W$ by

$$T(\mathbf{v}_1) = 2\mathbf{w}_1 - \mathbf{w}_2, \quad T(\mathbf{v}_2) = -\mathbf{w}_1, \quad T(\mathbf{v}_3) = \mathbf{w}_1 + 3\mathbf{w}_2$$

Find the matrix of T with respect to $\mathcal{B}$ and $\mathcal{C}$, and use it to find $T(2\mathbf{v}_1 - \mathbf{v}_2 + 3\mathbf{v}_3)$. Verify your answer by computing $T(2\mathbf{v}_1 - \mathbf{v}_2 + 3\mathbf{v}_3)$ directly.

Explanation. We start with a diagram:



Looking at the images of the standard unit vectors in $\mathbb{R}^3$, we can construct the standard matrix A of T with respect to $\mathcal{B}$ and $\mathcal{C}$.

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 0 & 3 \end{bmatrix}$$

Applying this matrix to the coordinate vector of $2\mathbf{v}_1 - \mathbf{v}_2 + 3\mathbf{v}_3$ we get

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \end{bmatrix}$$

This means that $T(2\mathbf{v}_1 - \mathbf{v}_2 + 3\mathbf{v}_3) = 8\mathbf{w}_1 + 7\mathbf{w}_2$. We can verify this by direct computation as follows:

$$\begin{aligned} T(2\mathbf{v}_1 - \mathbf{v}_2 + 3\mathbf{v}_3) &= 2T(\mathbf{v}_1) - T(\mathbf{v}_2) + 3T(\mathbf{v}_3) \\ &= 2(2\mathbf{w}_1 - \mathbf{w}_2) - (-\mathbf{w}_1) + 3(\mathbf{w}_1 + 3\mathbf{w}_2) \\ &= 8\mathbf{w}_1 + 7\mathbf{w}_2 \end{aligned}$$

The Matrix of a Linear Transformation

In this section we will formalize the process for finding the matrix of a linear transformation with respect to arbitrary bases that we established through earlier examples.

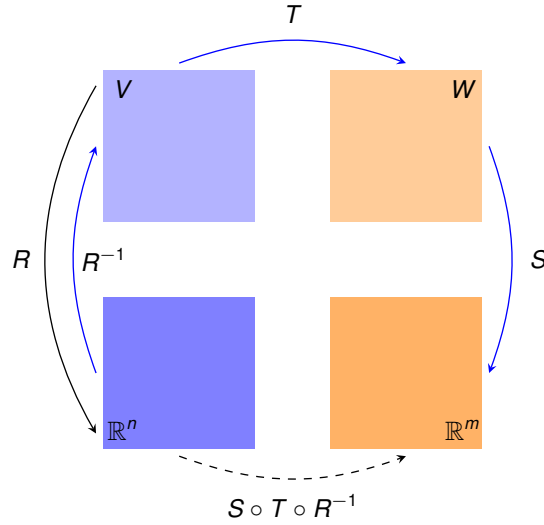
Let V and W be vector spaces with ordered bases $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ and $\mathcal{C} = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m\}$, respectively. Suppose $T : V \rightarrow W$ is a linear transformation. Our goal is to find a matrix for T with respect to $\mathcal{B}$ and $\mathcal{C}$.

Observe that $\dim(V) = n = \dim(\mathbb{R}^n)$ and $\dim(W) = m = \dim(\mathbb{R}^m)$. Let $R : V \rightarrow \mathbb{R}^n$ and $S : W \rightarrow \mathbb{R}^m$ be coordinate isomorphisms defined by

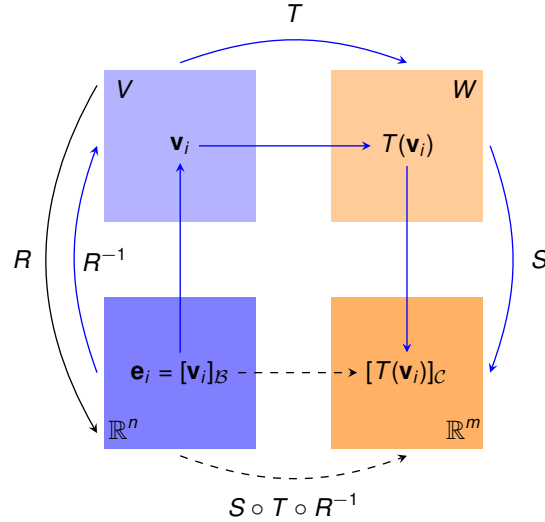
$$R(\mathbf{v}) = [\mathbf{v}]_{\mathcal{B}} \quad \text{and} \quad S(\mathbf{w}) = [\mathbf{w}]_{\mathcal{C}}$$

We know that R is an isomorphism and R^{-1} exists. Consider the transformation

$$S \circ T \circ R^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^m$$



As a composition of linear transformation, $S \circ T \circ R^{-1}$ is linear and thus has a standard matrix. To find it, we need to determine the images of standard unit vectors $\mathbf{e}_i$ under $S \circ T \circ R^{-1}$. We have the following:



Vectors $[T(\mathbf{v}_i)]_{\mathcal{C}}$ will become the columns of the standard matrix. We summarize this discussion as a theorem.

Theorem 9. Let V and W be finite-dimensional vector spaces with ordered bases $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ and $\mathcal{C}$, respectively. Suppose $T : V \rightarrow W$ is a linear transformation.

$$\text{Let } A = \begin{bmatrix} | & | & & | \\ [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \dots & [T(\mathbf{v}_n)]_{\mathcal{C}} \\ | & | & & | \end{bmatrix}$$

Then $A[\mathbf{v}]_{\mathcal{B}} = [T(\mathbf{v})]_{\mathcal{C}}$ for all vectors $\mathbf{v}$ in V .

Definition 10. Matrix A of Theorem ?? is called the matrix of T with respect to ordered bases $\mathcal{B}$ and $\mathcal{C}$.

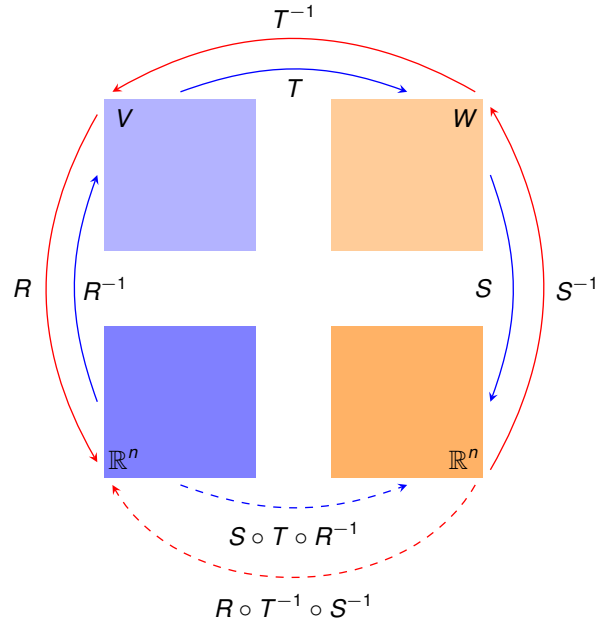
In conclusion, observe how isomorphisms helped us solve the matrix of a linear transformation problem. The coordinate mappings R and S are isomorphisms. This means that V and $\mathbb{R}^n$ are isomorphic and have the same structural properties. The same is true for W and $\mathbb{R}^m$. In this abstract discussion, we do not know anything about the elements of V and W , but isomorphisms allow us to take a problem that we do not know much about and transform it to a familiar problem involving familiar spaces.

The Inverse of a Linear Transformation and its Matrix

Let V and W be vector spaces. Suppose $T : V \rightarrow W$ is an invertible linear transformation. This, of course, means that T is an isomorphism, which means that

$$\dim(V) = \dim(W)$$

Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ and $\mathcal{C} = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$ be ordered bases of V and W , respectively. We can find the matrix of T^{-1} with respect to $\mathcal{C}$ and $\mathcal{B}$ by finding the standard matrix of the linear transformation $R \circ T^{-1} \circ S^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$.



Observe that $R \circ T^{-1} \circ S^{-1}$ is the inverse of $S \circ T \circ R^{-1}$. So, if A is the standard matrix of $S \circ T \circ R^{-1}$, then A^{-1} is the standard matrix of $R \circ T^{-1} \circ S^{-1}$. Thus, A^{-1} is the matrix of T^{-1} with respect to $\mathcal{C}$ and $\mathcal{B}$.

Example 12. Let

$$V = \text{span} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

Define a linear transformation

$$T : V \rightarrow \mathbb{R}^2$$

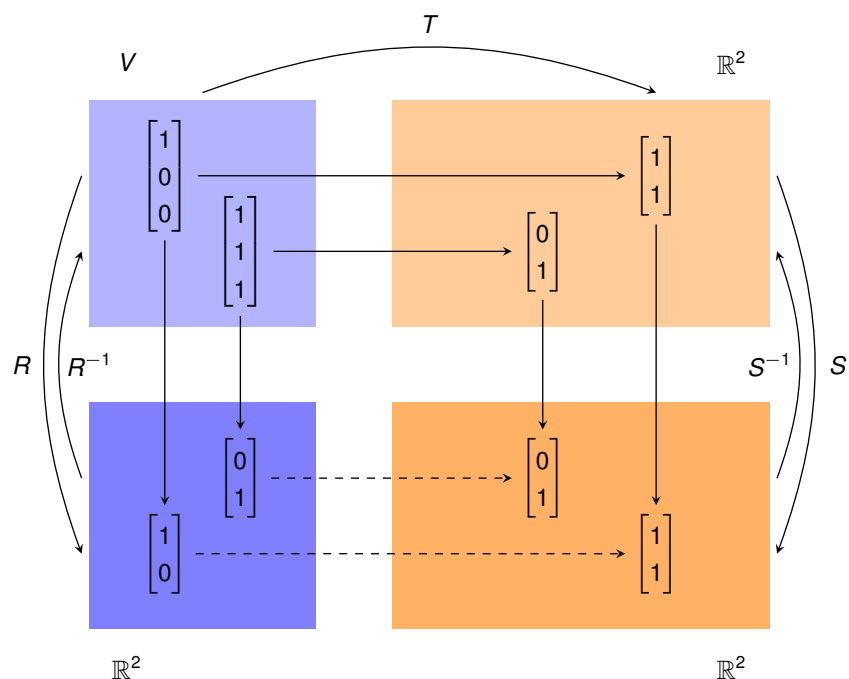
by

$$T \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad T \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Conclude that T is invertible by finding the matrix of T^{-1} with respect to ordered bases $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ of $\mathbb{R}^2$, and

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ of } V.$$

Explanation. Consider the diagram:



This gives us the matrix of T with respect to $\mathcal{B}$ and $\mathcal{C}$:

$$A = \begin{bmatrix} \cdots & \cdots \\ \cdots & \cdots \end{bmatrix}$$

We now find A^{-1} .

$$A^{-1} = \begin{bmatrix} \cdots & \cdots \\ \cdots & \cdots \end{bmatrix}$$

A^{-1} is the matrix of T^{-1} with respect to $\mathcal{C}$ and $\mathcal{B}$.

Practice Problems

Problem 16 Verify that τ of Exploration ?? is linear.

Problem 17 Define $T : \mathbb{M}_{2,2} \rightarrow \mathbb{L}$ by

$$T \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = (a + b) + (c + d)x$$

1. Show that T is linear.

2. Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$\mathcal{C} = \{1, x\}$$

be ordered bases for $\mathbb{M}_{2,2}$ and $\mathbb{L}$, respectively.

Find the matrix for T with respect to $\mathcal{B}$ and $\mathcal{C}$.

Answer:

$$\begin{bmatrix} \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

Problem 18 Let V and W be vector spaces with ordered bases $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $\mathcal{C} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$, respectively. Define a linear transformation $T : V \rightarrow W$ by

$$T(\mathbf{v}_1) = \mathbf{w}_1 + 2\mathbf{w}_2 - \mathbf{w}_3, \quad T(\mathbf{v}_2) = 3\mathbf{w}_2 + \mathbf{w}_3$$

Find the matrix A of T with respect to $\mathcal{B}$ and $\mathcal{C}$, and use it to find $T(-\mathbf{v}_1 - 3\mathbf{v}_2)$. Verify your answer by computing $T(-\mathbf{v}_1 - 3\mathbf{v}_2)$ directly.

$$A = \begin{bmatrix} \dots & \dots \\ \dots & \dots \\ \dots & \dots \end{bmatrix}$$

$$T(-\mathbf{v}_1 - 3\mathbf{v}_2) = \dots \mathbf{w}_1 + \dots \mathbf{w}_2 + \dots \mathbf{w}_3$$

Problem 19 Let

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{w}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{w}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Let V and W be subspaces of $\mathbb{R}^3$ with ordered bases

$$\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\} \quad \text{and} \quad \mathcal{C} = \{\mathbf{w}_1, \mathbf{w}_2\}$$

respectively.

Let $T : V \rightarrow W$ be a linear transformation such that

$$T(\mathbf{v}_1) = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} \quad \text{and} \quad T(\mathbf{v}_2) = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$$

1. Show that $\begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$ lie in W by expressing them as linear combinations of $\mathbf{w}_1$ and $\mathbf{w}_2$.

$$\begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \dots \mathbf{w}_1 + \dots \mathbf{w}_2$$

$$\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} = \dots \mathbf{w}_1 + \dots \mathbf{w}_2$$

2. Find the matrix of T with respect to $\mathcal{B}$ and $\mathcal{C}$.

$$\begin{bmatrix} \dots & \dots \\ \dots & \dots \end{bmatrix}$$

3. Find the matrix of T^{-1} with respect to $\mathcal{C}$ and $\mathcal{B}$.

$$\begin{bmatrix} \dots & \dots \\ \dots & \dots \end{bmatrix}$$

4. Find $T^{-1} \left(\begin{bmatrix} 10 \\ 10 \\ 0 \end{bmatrix} \right)$.

$$T^{-1} \left(\begin{bmatrix} 10 \\ 10 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix}$$