
Week 3

June 5, 2026

Contents

1 Linear Combinations of Vectors

When studying vectors, the two main operations we have learned about are vector addition and scalar multiplication. Both are involved in the important concept of a *linear combination* of vectors.

Definition 1. A vector $\mathbf{v}$ is said to be a *linear combination* of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ if

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_n\mathbf{v}_n$$

for some scalars $a_1, a_2, \dots, a_n$.

For example, $\begin{bmatrix} -4 \\ 9 \\ -10 \\ -1 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} -1 \\ 3 \\ -3 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 0 \\ 1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ because

$$\begin{bmatrix} -4 \\ 9 \\ -10 \\ -1 \end{bmatrix} = 2 \begin{bmatrix} -1 \\ 3 \\ -3 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 2 \\ 0 \\ 1 \\ 4 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

In this section we will focus on vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Visualizing Linear Combinations in $\mathbb{R}^2$ and $\mathbb{R}^3$

Let's start by visualizing linear combinations of two vectors in $\mathbb{R}^2$.

Exploration 1 Answer the questions below using the GeoGebra interactive. To use the interactive, you can

1. Change vectors $\mathbf{v}$ and $\mathbf{w}$ by dragging the tips of these vectors.
2. Change the coefficients k_1 and k_2 of the linear combination by using sliders.

Access GeoGebra interactives through the online version of this text at

<https://ximera.osu.edu/linearalgebradv3/LinearAlgebraInteractiveIntro>.

$$1. \text{ Let } \mathbf{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \text{ Find } k_1 \text{ and } k_2 \text{ such that } k_1\mathbf{w} + k_2\mathbf{v} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$k_1 = \dots, \quad k_2 = \dots$$

$$2. \text{ Let } \mathbf{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}. \text{ Find } k_1 \text{ and } k_2 \text{ such that } k_1\mathbf{w} + k_2\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$k_1 = \dots, \quad k_2 = \dots$$

3. Use the same vectors $\mathbf{w}$ and $\mathbf{v}$ as in the previous part. Do you think it is possible to express any vector in $\mathbb{R}^2$ as a linear combination of $\mathbf{w}$ and $\mathbf{v}$? (Yes, / No)

4. Let $\mathbf{w} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$. Do you think it is possible to express any vector in $\mathbb{R}^2$ as a linear combination of $\mathbf{w}$ and $\mathbf{v}$? (Yes, / No)

Visualizing linear combinations of vectors in $\mathbb{R}^3$ is more difficult than doing so in $\mathbb{R}^2$. The following Exploration will help you do this.

Exploration 2 We will start by visualizing linear combinations of two vectors, $\mathbf{u}$ and $\mathbf{v}$, in $\mathbb{R}^3$. To use the interactive below, define vectors $\mathbf{u}$ and $\mathbf{v}$. Use sliders to change the coefficients k_1 and k_2 of the linear combination. You will see the linear combination $k_1\mathbf{u} + k_2\mathbf{v}$ as the pink vector along the diagonal of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$.

RIGHT-CLICK and DRAG the left panel to rotate the image.

Access GeoGebra interactives through the online version of this text at

<https://ximera.osu.edu/linearalgebradv3/LinearAlgebraInteractiveIntro>.

We will now consider three vectors. Define vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. Use sliders to change the coefficients k_1 , k_2 and k_3 of the linear combination. The linear combination $k_1\mathbf{u} + k_2\mathbf{v} + k_3\mathbf{w}$ is shown as the pink vector along the diagonal of the parallelepiped.

RIGHT-CLICK and DRAG the left panel to rotate the image.

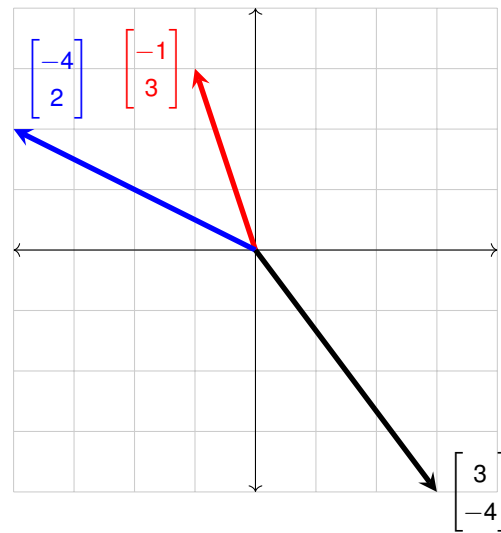
Access GeoGebra interactives through the online version of this text at

<https://ximera.osu.edu/linearalgebradv3/LinearAlgebraInteractiveIntro>.

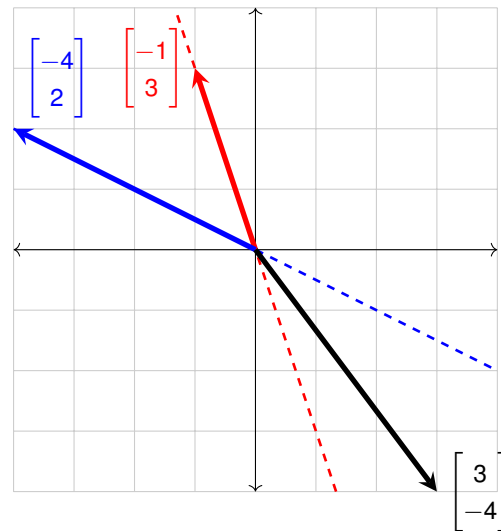
Geometry of Linear Combinations

Example 1. Use geometry to express $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$.

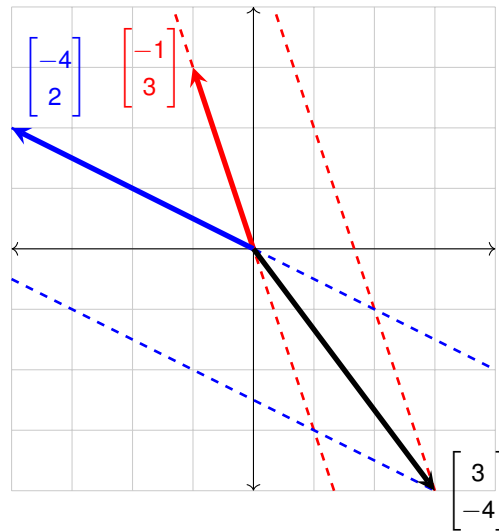
Explanation. We are looking for $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$ to be the diagonal of a parallelogram determined by scalar multiples of $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$.



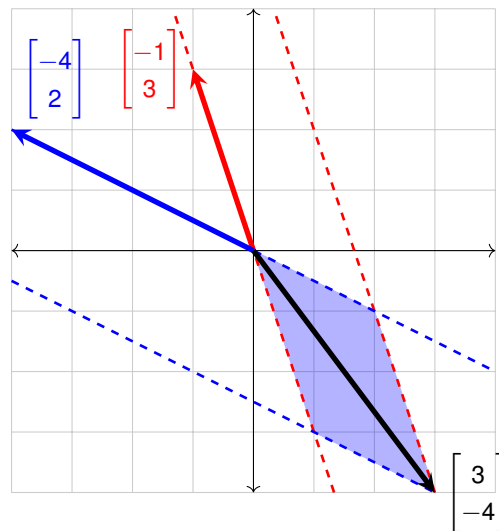
Because a scalar multiple of a vector can point in the same direction as the vector or in the opposite direction, we will start by drawing straight lines determined by the two vectors.



The two lines that we drew will contain the sides of the parallelogram we are looking for. To find the other two sides we will draw lines parallel to $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ through the head of vector $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$.

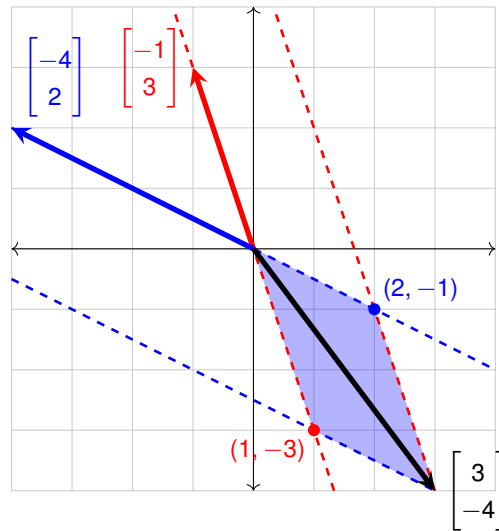


Now the parallelogram is clearly visible.



The last remaining task is to identify the sides of the parallelogram as scalar multiples of $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$. Observe

that vectors $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ determine the parallelogram.



Vector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is half the length of $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ and points in the opposite direction, while the vector $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ is the same length as $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and also points in the opposite direction.

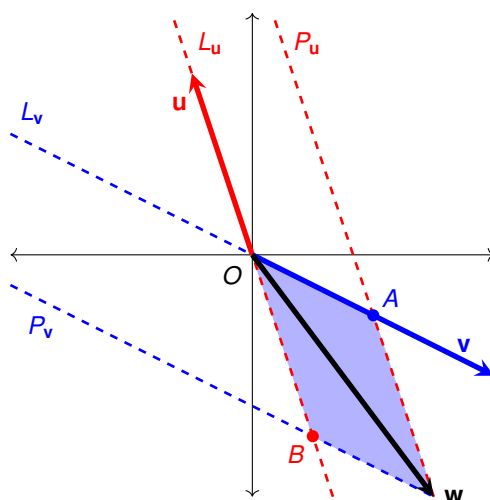
We write $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ as follows

$$\begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} = (-1) \begin{bmatrix} -1 \\ 3 \end{bmatrix} + \left(-\frac{1}{2}\right) \begin{bmatrix} -4 \\ 2 \end{bmatrix}$$

The method we used in Example ?? to express the given vector as a linear combination of two other vectors is sufficiently useful that we summarize the steps.

Procedure 1. Given two non-collinear vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$, and a vector $\mathbf{w}$, we can express $\mathbf{w}$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$ as follows:

1. Draw lines $L_{\mathbf{u}}$ and $L_{\mathbf{v}}$ determined by $\mathbf{u}$ and $\mathbf{v}$, respectively.
2. Through the head of vector $\mathbf{w}$, draw lines $P_{\mathbf{u}}$ and $P_{\mathbf{v}}$, parallel to $L_{\mathbf{u}}$ and $L_{\mathbf{v}}$, respectively.
3. Let A be the point of intersection of $P_{\mathbf{u}}$ and $L_{\mathbf{v}}$.
4. Let B be the point of intersection of $P_{\mathbf{v}}$ and $L_{\mathbf{u}}$.
5. Let O denote the origin. Then $\overrightarrow{OA} = k_1 \mathbf{v}$ and $\overrightarrow{OB} = k_2 \mathbf{u}$ for some scalars k_1 and k_2 .
6. We have $\mathbf{w} = k_2 \mathbf{u} + k_1 \mathbf{v}$.



Exploration 3 This GeoGebra interactive will allow you to go through the steps given in Procedure ?? for a combination of vectors of your choice. To use the interactive

1. Enter components of vectors $\mathbf{u}$ and $\mathbf{v}$.
2. Enter components of vector $\mathbf{w}$ that you want to express as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.
3. Use the navigation bar to go through the steps of Procedure ??

Access GeoGebra interactives through the online version of this text at

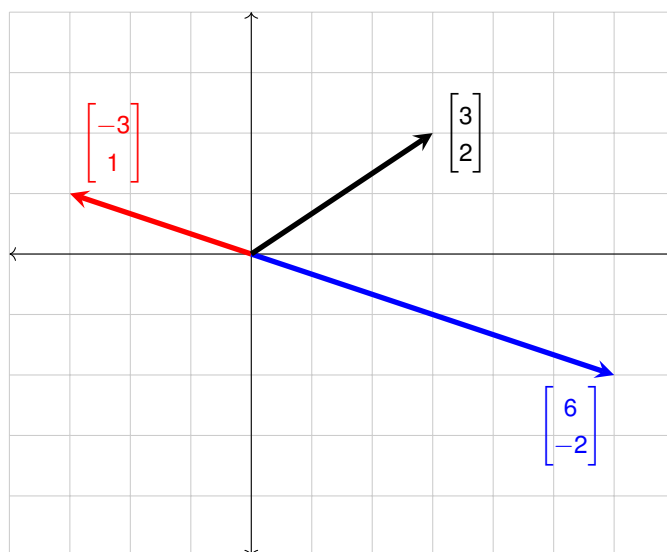
<https://ximera.osu.edu/linearalgebradv3/LinearAlgebraInteractiveIntro>.

From Geometry to Algebra of Linear Combinations

One of the stipulations in Procedure ?? is that vectors $\mathbf{u}$ and $\mathbf{v}$ should be non-collinear. You can use the interactive in Exploration ?? to investigate what happens when $\mathbf{u}$ and $\mathbf{v}$ are collinear. The following example examines what happens from a geometric as well as an algebraic standpoint.

Example 2. Can the vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ be written as a linear combination of vectors $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ -2 \end{bmatrix}$?

Explanation. We will start with a geometric approach.



Observe that $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ -2 \end{bmatrix}$ are scalar multiples of each other and lie on the same line.

A linear combination of $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ -2 \end{bmatrix}$ has the form:

$$a \begin{bmatrix} 6 \\ -2 \end{bmatrix} + b \begin{bmatrix} -3 \\ 1 \end{bmatrix} = a(-2) \begin{bmatrix} -3 \\ 1 \end{bmatrix} + b \begin{bmatrix} -3 \\ 1 \end{bmatrix} = (-2a + b) \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

This shows that all linear combinations of $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ -2 \end{bmatrix}$ will be scalar multiples of $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$, and therefore lie on the same line as $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$. Since $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ does not lie on the line determined by $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ it cannot be expressed as a linear combination of $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ -2 \end{bmatrix}$.

We can also address this question algebraically. To express $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ -2 \end{bmatrix}$, we need to solve the equation.

$$a \begin{bmatrix} -3 \\ 1 \end{bmatrix} + b \begin{bmatrix} 6 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

This gives us a system of equations

$$\begin{aligned} -3a + 6b &= 3 \\ a - 2b &= 2 \end{aligned}$$

When you try to solve this system, you will find that the system is inconsistent. Thus, $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ cannot be written as a linear combination of $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ -2 \end{bmatrix}$.

We know that there is no way to express $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ as a linear combination of vectors $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ -2 \end{bmatrix}$. What would happen if we tried to apply Procedure ?? to these vectors? You can use the GeoGebra interactive in Exploration ?? to find out.

Example 3. Express $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$. Interpret your results geometrically.

Explanation. We need to find scalars a and b such that

$$\begin{bmatrix} 2 \\ 4 \end{bmatrix} = a \begin{bmatrix} 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

This amounts to solving a system of linear equations

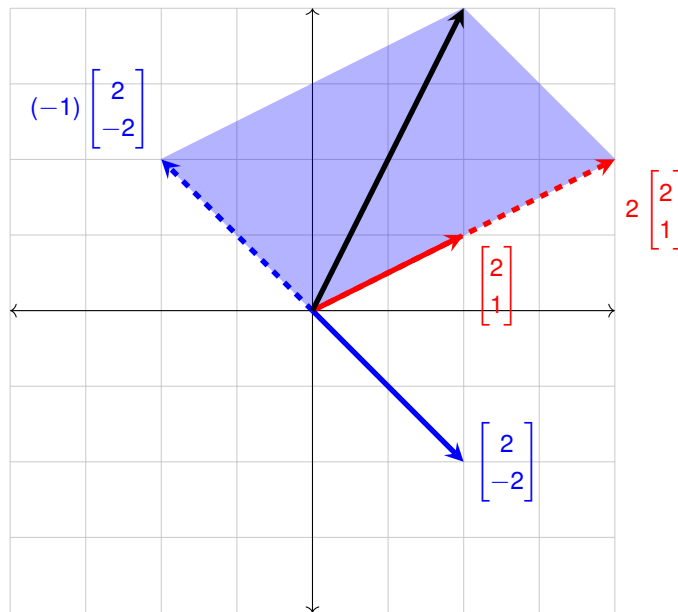
$$2a + 2b = 2$$

$$a - 2b = 4$$

Use your favorite method to solve this system. (Hint: adding the second equation to the first will work well for this system.) You will find that $a = 2$ and $b = -1$. Now we can write $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$ as follows:

$$\begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

Geometrically speaking, this means that the vector $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ is the diagonal of the parallelogram determined by $2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $(-1) \begin{bmatrix} 2 \\ -2 \end{bmatrix}$. The original vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$ are shown below together with the parallelogram and its diagonal.



Example 4. If possible, express $\begin{bmatrix} 7 \\ 4 \\ -5 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$.

Explanation. We are looking for coefficients a and b such that

$$a \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -5 \end{bmatrix}$$

This translates into a system of equations

$$\begin{aligned} a + 3b &= 7 \\ -2a &= 4 \\ a - b &= -5 \end{aligned}$$

Solve this system for a and b , and enter your answers below:

$$a = \dots\dots \text{ and } b = \dots\dots$$

We conclude that $\begin{bmatrix} 7 \\ 4 \\ -5 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$, and write:

$$\begin{bmatrix} 7 \\ 4 \\ -5 \end{bmatrix} = \dots\dots \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \dots\dots \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$$

Example 5. Set up a system of equations that can be used to express $\begin{bmatrix} 2 \\ -1 \\ 3 \\ 0 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 0 \\ 4 \\ -2 \end{bmatrix}$, $\begin{bmatrix} -2 \\ -1 \\ 1 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 4 \\ -3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ -1 \\ 4 \end{bmatrix}$, or to determine that such a combination does not exist. Do not solve the system.

Explanation. We are looking for x_1, x_2, x_3 and x_4 such that

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 4 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -1 \\ 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 4 \\ -3 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 1 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 0 \end{bmatrix}$$

This translates into the following system of equations:

$$\begin{array}{ccccccccc} x_1 & - & 2x_2 & & & + & x_4 & = & 2 \\ & & -x_2 & + & 4x_3 & + & x_4 & = & -1 \\ 4x_1 & + & x_2 & - & 3x_3 & - & x_4 & = & 3 \\ -2x_1 & - & x_2 & + & x_3 & + & 4x_4 & = & 0 \end{array}$$

Practice Problems

Problem 4 Solve a system of linear equations to express $\begin{bmatrix} -1 \\ 7 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

System of linear equations:

$$\begin{array}{ccccccc} \dots\dots a & + & \dots\dots b & = & \dots\dots \\ \dots\dots a & + & \dots\dots b & = & \dots\dots \end{array}$$

Values of a and b :

$$a = \dots\dots \text{ and } b = \dots\dots$$

Linear Combination:

$$\begin{bmatrix} -1 \\ 7 \end{bmatrix} = \dots\dots \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \dots\dots \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Problem 5 Use Procedure ?? to express $\begin{bmatrix} -3 \\ 0 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Linear Combination:

$$\begin{bmatrix} -3 \\ 0 \end{bmatrix} = \dots\dots \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \dots\dots \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Problem 6 Use two different approaches (algebraic and geometric) to explain why the vector $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$ cannot be expressed as a linear combination of vectors $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$.

Problem 7 We have seen Procedure ?? applied to vectors in $\mathbb{R}^2$. The same process can, in certain cases be applied to vectors in $\mathbb{R}^3$. Follow the steps in Procedure ?? to express one vector as a linear combination of two given vectors.

The following GeoGebra interactive shows vectors $\mathbf{u} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 7 \\ 4 \\ -5 \end{bmatrix}$. Use the navigation bar to click through the construction steps. RIGHT-CLICK and DRAG to rotate the image.

Access GeoGebra interactives through the online version of this text at

<https://ximera.osu.edu/linearalgebradv3/LinearAlgebraInteractiveIntro>.

1. Can $\mathbf{w}$ be expressed as a linear combination of $\mathbf{u}$ and $\mathbf{v}$?

Multiple Choice:

- (i) No, because $\mathbf{w}$ is not between $\mathbf{u}$ and $\mathbf{v}$.
- (ii) Yes, because all three vectors are in the same plane.
- (iii) Yes, because all three vectors are in the same plane, AND $\mathbf{u}$ and $\mathbf{v}$ are not collinear.

2. Use the navigation bar at the bottom of the interactive window to view construction steps of Procedure ?? applied to vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. (Right-click and drag to rotate the image.) Use the final image to express $\mathbf{w}$ as a linear combination of $\mathbf{v}$ (blue) and $\mathbf{u}$ (red).

$$\mathbf{w} = \dots \mathbf{v} + \dots \mathbf{u}$$

Problem 8 The following GeoGebra interactive shows vectors $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 7 \\ 4 \\ 0 \end{bmatrix}$. RIGHT-

CLICK and DRAG to rotate the image. Use geometry to explain why $\mathbf{w}$ cannot be expressed as a linear combination of $\mathbf{v}$ and $\mathbf{u}$.

Access GeoGebra interactives through the online version of this text at

<https://ximera.osu.edu/linearalgebradv3/LinearAlgebraInteractiveIntro>.

We can also show that $\mathbf{w}$ is not a linear combination of $\mathbf{v}$ and $\mathbf{u}$ algebraically by attempting to solve a system of equations corresponding to

$$x_1 \mathbf{v} + x_2 \mathbf{u} = \mathbf{w}$$

Set up the system of equations

$$\begin{array}{rcl} \dots x_1 & + & \dots x_2 = \dots \\ \dots x_1 & + & \dots x_2 = \dots \\ \dots x_1 & + & \dots x_2 = \dots \end{array}$$

Find the reduced row echelon form.

$$\left[\begin{array}{cc|c} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{array} \right]$$

2 Span

Linear Combinations Revisited

Recall that a vector $\mathbf{v}$ is said to be a linear combination of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ if

$$\mathbf{v} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + \dots + a_n \mathbf{v}_n$$

for some scalars $a_1, a_2, \dots, a_n$.

Example 6. If possible, express the given vector as a linear combination of

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ -3 \\ 4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$$

Interpret your results geometrically.

1.

$$\mathbf{u} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

2.

$$\mathbf{w} = \begin{bmatrix} 5 \\ 5 \\ 1 \end{bmatrix}$$

Explanation. ?? We need to find coefficients a_1 and a_2 such that $\mathbf{u} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2$. To do this we need to solve the vector equation:

$$a_1 \begin{bmatrix} -2 \\ -3 \\ 4 \end{bmatrix} + a_2 \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

This equation translates into the following system:

$$\begin{aligned} -2a_1 + 2a_2 &= 2 \\ -3a_1 + 3a_2 &= 3 \\ 4a_1 + 2a_2 &= 5 \end{aligned}$$

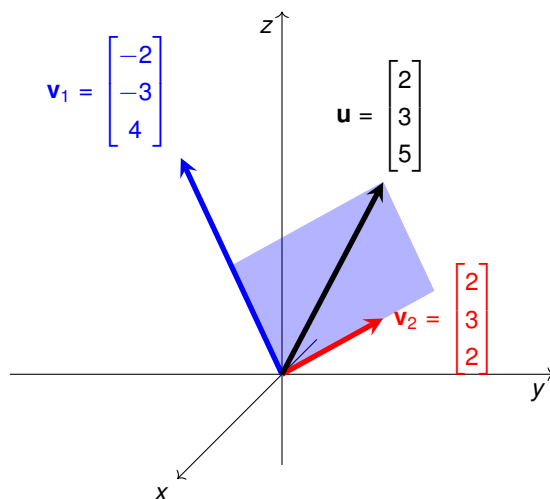
We write the system in augmented matrix form and apply elementary row operations to bring it to reduced row-echelon form.

$$\left[\begin{array}{cc|c} -2 & 2 & 2 \\ -3 & 3 & 3 \\ 4 & 2 & 5 \end{array} \right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 1/2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{array} \right]$$

This shows that $a_1 = \frac{1}{2}$ and $a_2 = \frac{3}{2}$, and we can express $\mathbf{u}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ as follows:

$$\frac{1}{2} \begin{bmatrix} -2 \\ -3 \\ 4 \end{bmatrix} + \frac{3}{2} \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

Observe that because vector $\mathbf{u}$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, $\mathbf{u}$ is the diagonal of a parallelogram whose sides are scalar multiples of $\mathbf{v}_1$ and $\mathbf{v}_2$. As such, $\mathbf{u}$ lies in the same plane as $\mathbf{v}_1$ and $\mathbf{v}_2$, as illustrated below.



?? We need to solve the following vector equation:

$$a_1 \begin{bmatrix} -2 \\ -3 \\ 4 \end{bmatrix} + a_2 \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 1 \end{bmatrix}$$

This equation corresponds to the system:

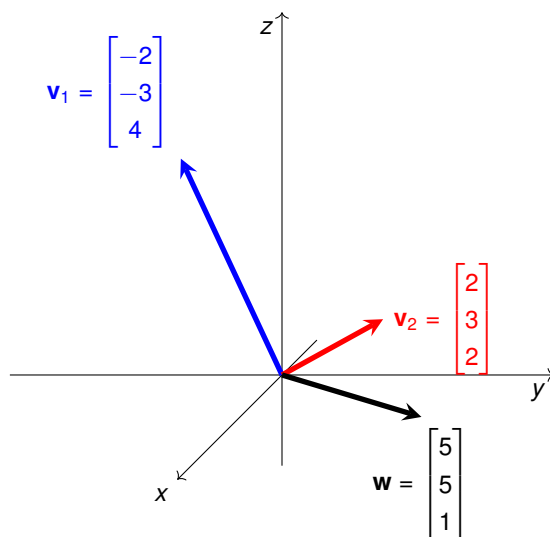
$$\begin{aligned} -2a_1 + 2a_2 &= 5 \\ -3a_1 + 3a_2 &= 5 \\ 4a_1 + 2a_2 &= 1 \end{aligned}$$

Writing the system in augmented matrix form and applying elementary row operations gives us the following reduced row-echelon form:

$$\left[\begin{array}{cc|c} -2 & 2 & 5 \\ -3 & 3 & 5 \\ 4 & 2 & 1 \end{array} \right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]$$

We conclude that there are no solutions, and $\mathbf{w}$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Geometrically, this means that $\mathbf{w}$ is not the diagonal of any parallelogram whose sides are scalar multiples of $\mathbf{v}_1$ and $\mathbf{v}_2$. Thus, $\mathbf{w}$ does not lie in the plane determined by $\mathbf{v}_1$ and $\mathbf{v}_2$.



In part ?? of Example ?? we expressed $\mathbf{u}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, and concluded that $\mathbf{u}$ lies in the plane determined by $\mathbf{v}_1$ and $\mathbf{v}_2$. We say that $\mathbf{u}$ is *in the span* of $\mathbf{v}_1$ and $\mathbf{v}_2$. In fact, every vector in the plane determined by $\mathbf{v}_1$ and $\mathbf{v}_2$ is in the span of $\mathbf{v}_1$ and $\mathbf{v}_2$. We say that $\mathbf{v}_1$ and $\mathbf{v}_2$ *span the plane*.

In contrast, vector $\mathbf{w}$ of part ?? of Example ?? is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. We say that $\mathbf{w}$ is not in the span of $\mathbf{v}_1$ and $\mathbf{v}_2$.

The following video takes another look at Example ?? using our new vocabulary.

YouTube link: <https://www.youtube.com/watch?v=Sum3qo0EXhY>

Definition of Span

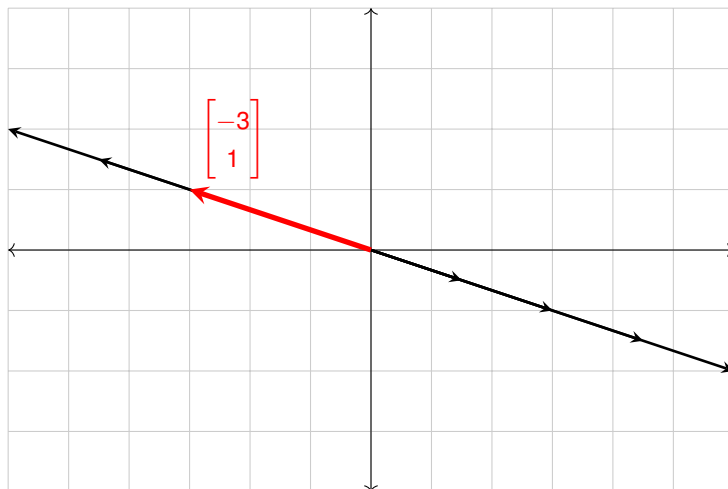
Definition 2. Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ be vectors in $\mathbb{R}^n$. The set S of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ is called the *span* of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$. We write

$$S = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p)$$

and we say that vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ *span* S . Any vector in S is said to be *in the span* of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$. The set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ is called a *spanning set* for S .

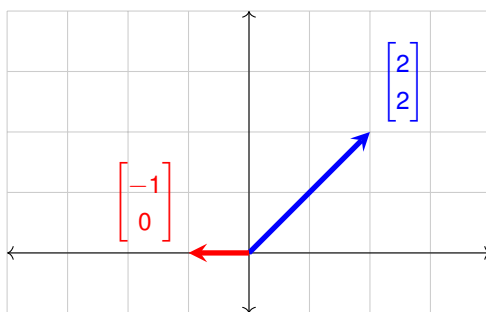
Example 7. Describe $\text{span}\left(\begin{bmatrix} -3 \\ 1 \end{bmatrix}\right)$.

Explanation. The span of $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ is the set of all linear combinations of $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$. Since we are looking for linear combinations of only one vector, we are really looking for all of its scalar multiples. So, the span will be the set of all vectors of the form $\mathbf{v} = a \begin{bmatrix} -3 \\ 1 \end{bmatrix}$. All such vectors lie on the line determined by $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$.



Example 8. Describe $\text{span} \left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right)$.

Explanation. First, observe that $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ are not scalar multiples of each other.



Geometrically, we can use Procedure ?? to express any vector of $\mathbb{R}^2$ as a linear combination of $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$, indicating that the two vectors span all of $\mathbb{R}^2$.

To verify this claim algebraically we will show that an arbitrary vector $\begin{bmatrix} s \\ t \end{bmatrix}$ of $\mathbb{R}^2$ can be written as a linear combination of $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Consider the vector equation:

$$a_1 \begin{bmatrix} 2 \\ 2 \end{bmatrix} + a_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} s \\ t \end{bmatrix}$$

This corresponds to the system:

$$\begin{aligned} 2a_1 - a_2 &= s \\ 2a_1 &= t \end{aligned}$$

Writing the system in augmented matrix form and applying elementary row operations gives us the following reduced row-echelon form:

$$\left[\begin{array}{cc|c} 2 & -1 & s \\ 2 & 0 & t \end{array} \right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & t/2 \\ 0 & 1 & t-s \end{array} \right]$$

This shows that every vector of $\mathbb{R}^2$ can be written as a linear combination of $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$:

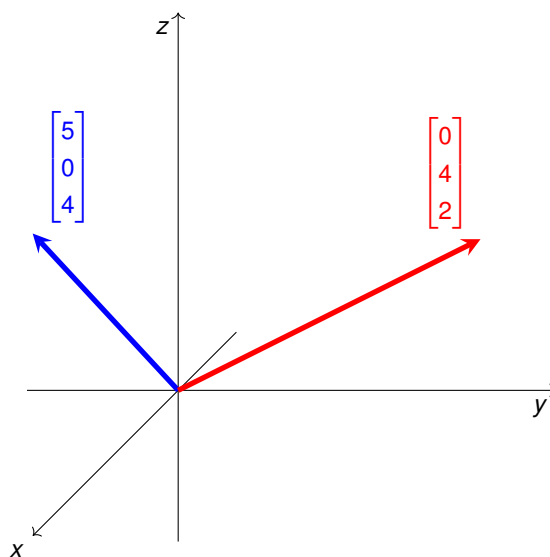
$$(t/2) \begin{bmatrix} 2 \\ 2 \end{bmatrix} + (t-s) \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} s \\ t \end{bmatrix}$$

We conclude that

$$\text{span} \left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) = \mathbb{R}^2$$

Example 9. Describe $\text{span} \left(\begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} \right)$.

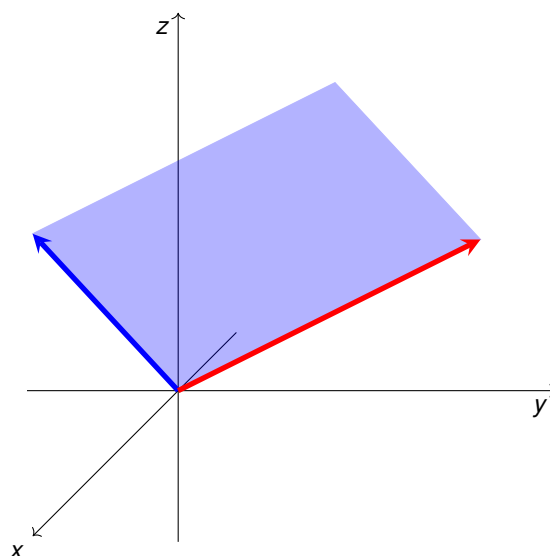
Explanation. First, observe that $\begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$ are not scalar multiples of each other.



The span of $\begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$ consists of elements of the form

$$a_1 \begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$$

Geometrically, we can interpret all such linear combinations as diagonals of parallelograms determined by scalar multiples of $\begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$. All such diagonals will lie in the plane determined by $\begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$. Let this plane be called p . A portion of p is shown below.



Because Procedure ?? can be applied to vectors that lie in p just as easily as it can be applied to vectors of $\mathbb{R}^2$, we

conclude that every vector in p can be expressed as a linear combination of $\begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$. Thus,

$$\text{span} \left(\begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} \right) = p$$

Redundant Vectors

If a friend told you that they have a line spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$, you would probably think that your friend's description is a little excessive. Isn't one of the above vectors sufficient to describe the line? A line can be

described as a span of one vector, but it can also be described as a span of two or more vectors. There are many advantages, however, to using the most efficient description possible. In this section we will begin to explore what makes a description “more efficient.”

Exploration 9 Consider the following collection of vectors:

$$\left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$$

What is the span of these vectors? (A line, / $\mathbb{R}^2$, / A parallelogram, / A parallelepiped)

In this Exploration we will examine what can happen to the span of a collection of vectors when a vector is removed from the collection.

First, let's remove $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ from $\left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$.

Which of the following is true?

Multiple Choice:

1. $\text{span} \left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \end{bmatrix} \right) = \text{span} \left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)$

2. $\text{span} \left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \end{bmatrix} \right)$ is a line

3. $\text{span} \left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \end{bmatrix} \right) = \mathbb{R}^2$

4. $\text{span} \left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \end{bmatrix} \right)$ is a parallelogram.

Removing $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ from $\left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ (changed, / did not change) the span.

Now let's remove $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ from the original collection of vectors.

Which of the following is true?

Multiple Choice:

1. $\text{span} \left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) = \text{span} \left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)$

2. $\text{span} \left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)$ is a line

3. $\text{span} \left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)$ is the right side of the coordinate plane.

4. $\text{span} \left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)$ is a parallelogram.

Removing $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ from $\left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ (changed, / did not change) the span.

As you just discovered, removing a vector from a collection of vectors may or may not affect the span of the collection. We will refer to vectors that can be removed from a collection without changing the span as *redundant*. In Exploration ??, $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ is *redundant*, while $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is not.

Definition 3. Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ be a set of vectors in $\mathbb{R}^n$. If we can remove one vector without changing the span of this set, then that vector is *redundant*. In other words, if

$$\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k) = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \dots, \mathbf{v}_k)$$

we say that $\mathbf{v}_j$ is a redundant element of $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$, or simply redundant.

Our next goal is to see what causes $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ of Exploration ?? to be redundant. The answer lies not in the vector itself, but in its relationship to the other vectors in the collection. Observe that $\begin{bmatrix} -4 \\ 2 \end{bmatrix} = -2 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. In other words, $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ is a scalar multiple of another vector in the set. To see why this matters, let's pick an arbitrary vector $\mathbf{w} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ in $\text{span} \left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)$. Vector $\mathbf{w}$ is in the span because it can be written as a linear combination of the three vectors as follows

$$\mathbf{w} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} -4 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

But $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ is not essential to this linear combination because it can be replaced with $-2 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, as shown below.

$$\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} -4 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} + (-2) \begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = - \begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Regardless of what vector $\mathbf{w}$ we write as a linear combination of $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, we will always be able to replace $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ with $-2 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, placing $\mathbf{w}$ into the span of $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and making $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ redundant. (Note that

we can just as easily write $\begin{bmatrix} 2 \\ -1 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} -4 \\ 2 \end{bmatrix}$, and argue that $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is redundant.) We conclude that only one of $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is needed to maintain the span of the original three vectors. We have

$$\text{span} \left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) = \text{span} \left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) = \text{span} \left(\begin{bmatrix} -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)$$

The left-most collection in this expression contains redundant vectors; the other two collections do not.

In Exploration ?? we found one vector to be redundant because we could replace it with a scalar multiple of another vector in the set. The following Exploration delves into what happens when a vector in a given set is a linear combination of the other vectors.

Exploration 10 Consider the set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ -1 \end{bmatrix} \right\}$$

The three vectors are shown below. RIGHT-CLICK and DRAG to rotate the interactive graph.

Access GeoGebra interactives through the online version of this text at

<https://ximera.osu.edu/linearalgebraadv3/LinearAlgebraInteractiveIntro>.

$\text{span} \left(\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ -1 \end{bmatrix} \right)$ is (A line, / A plane, / $\mathbb{R}^3$, / A parallelepiped)

Can we remove one of the vectors from the set without changing the span? Observe that we can write $\begin{bmatrix} 4 \\ 4 \\ -1 \end{bmatrix}$ as a linear combination of the other two vectors

$$\begin{bmatrix} 4 \\ 4 \\ -1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \quad (1)$$

This means that we can write any vector in $\text{span} \left(\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ -1 \end{bmatrix} \right)$ as a linear combination of only $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and

$\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ by replacing $\begin{bmatrix} 4 \\ 4 \\ -1 \end{bmatrix}$ with the expression in (1). For example,

$$\begin{bmatrix} 7 \\ 6 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \\ -1 \end{bmatrix} = \dots \dots \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \dots \dots \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

We have

$$\text{span} \left(\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ -1 \end{bmatrix} \right) = \text{span} \left(\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right)$$

We conclude that vector $\begin{bmatrix} 4 \\ 4 \\ -1 \end{bmatrix}$ is redundant. Can each of the other two vectors in the set $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ -1 \end{bmatrix} \right\}$ be considered redundant? You will address this question in the problem set.

Collections of vectors that do not contain redundant vectors are very important in linear algebra. In subsequent sections, we will formally introduce such collections as *linearly independent*. Collections of vectors that contain redundant vectors will be called *linearly dependent*.

Practice Problems

Problem 11 Choose the best description for the set below.

$$\text{span} \left(\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -4 \end{bmatrix} \right)$$

Multiple Choice:

1. Plane in $\mathbb{R}^3$
2. Line in $\mathbb{R}^2$
3. Line in $\mathbb{R}^3$
4. $\mathbb{R}^3$
5. $\mathbb{R}^2$

Problem 12 Choose the best description for the set below.

$$\text{span} \left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

Multiple Choice:

1. Plane in $\mathbb{R}^3$
2. Line in $\mathbb{R}^2$
3. Line in $\mathbb{R}^3$

4. $\mathbb{R}^3$

5. $\mathbb{R}^2$

Problem 13 Choose the best description for the set below.

$$\text{span} \left(\begin{bmatrix} -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right)$$

Multiple Choice:

1. Plane in $\mathbb{R}^3$

2. Line in $\mathbb{R}^2$

3. Line in $\mathbb{R}^3$

4. $\mathbb{R}^3$

5. $\mathbb{R}^2$

Problem 14 Which of the following pairs of sets are equal? Select all that apply.

Select All Correct Answers:

$$1. V = \text{span} \left(\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -4 \end{bmatrix} \right) \quad \text{and} \quad W = \text{span} \left(\begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

$$2. V = \text{span} \left(\begin{bmatrix} 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 10 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right) \quad \text{and} \quad W = \text{span} \left(\begin{bmatrix} -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

$$3. V = \text{span} \left(\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} \right) \quad \text{and} \quad W = \text{span} \left(\begin{bmatrix} -1 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

$$4. V = \text{span} \left(\begin{bmatrix} 5 \\ 0 \end{bmatrix} \right) \quad \text{and} \quad W = \text{span} \left(\begin{bmatrix} -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right)$$

Problem 15 Let $\mathbf{v} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$. Give an example of at least one vector $\mathbf{w}$ such that $\mathbf{v}, \mathbf{w}$ do NOT span a plane in $\mathbb{R}^3$.
Describe $\text{span}(\mathbf{v}, \mathbf{w})$.

Problem 16 Prove or disprove. The zero vector of $\mathbb{R}^n$ is contained in the span of any collection of vectors of $\mathbb{R}^n$.

Problem 17 In Exploration ?? we considered the following set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ -1 \end{bmatrix} \right\}$$

and demonstrated that $\begin{bmatrix} 4 \\ 4 \\ -1 \end{bmatrix}$ is redundant by using the fact that it is a linear combination of the other two vectors.

1. Express each of $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ as a linear combination of the remaining vectors.

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \dots\dots \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \dots\dots \begin{bmatrix} 4 \\ 4 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \dots\dots \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \dots\dots \begin{bmatrix} 4 \\ 4 \\ -1 \end{bmatrix}$$

2. Which of the following is NOT true?

Multiple Choice:

- (i) If $\mathbf{w}$ is in $\text{span} \left(\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ -1 \end{bmatrix} \right)$, then $\mathbf{w}$ is in $\text{span} \left(\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ -1 \end{bmatrix} \right)$.
- (ii) Both $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ are redundant in $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ -1 \end{bmatrix} \right\}$.
- (iii) We can remove $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ from $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ -1 \end{bmatrix} \right\}$ at the same time without affecting the span.

Problem 18 Show that if the zero vector is part of a collection of two or more vectors, the zero vector is redundant.

3 Linear Independence

In the last section we considered properties a set of vectors should possess in order to be used to define a “good” coordinate system for a plane. Let S be a set of vectors in the plane. We have already established that in order to use vectors in S to define the axes of a “good” coordinate system, vectors in S must span the plane. This would guarantee that every vector in the plane can be written as a linear combination of elements of S .

In addition, we observed that we want each vector in the plane to be represented as a *unique* linear combination of elements of S . This gives us motivation to consider the question of uniqueness. We will now step away from coordinate systems in the plane and turn to this question in a more general setting.

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ be vectors of $\mathbb{R}^n$. Let $V = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k)$. Then every vector in V can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ in at least one way. Our interest here is in spanning sets where each vector in V has exactly one representation as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$.

Suppose that two linear combinations are equal

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_k\mathbf{v}_k = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \dots + b_k\mathbf{v}_k$$

We are looking for a condition on the set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ that guarantees that this representation is unique. This amounts to showing that $a_i = b_i$ for each i . Taking all terms to the left side gives us

$$(a_1 - b_1)\mathbf{v}_1 + (a_2 - b_2)\mathbf{v}_2 + \dots + (a_k - b_k)\mathbf{v}_k = \mathbf{0}$$

So the condition that guarantees uniqueness is that all coefficients $(a_i - b_i)$ must be zero. This motivates the following definition.

Definition 4 (Linear Independence). Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ be vectors of $\mathbb{R}^n$. We say that the set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is *linearly independent* if the only solution to

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_k\mathbf{v}_k = \mathbf{0} \tag{2}$$

is the *trivial solution* $c_1 = c_2 = \dots = c_k = 0$.

If, in addition to the trivial solution, any *non-trivial solutions* (not all $c_1, c_2, \dots, c_k$ are zero) exist, then we say that the set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is *linearly dependent*.

Observation 1. The trivial solution $c_1 = c_2 = \dots = c_k = 0$ is always a solution to $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_k\mathbf{v}_k = \mathbf{0}$ because $0\mathbf{v}_1 + 0\mathbf{v}_2 + \dots + 0\mathbf{v}_k = \mathbf{0}$. So, when checking for linear independence, the question is not whether the trivial solution is a solution, but whether it is the *ONLY* solution.

Remark 1. Given a set of vectors $X = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ we can now ask the following questions:

1. Are the vectors in X linearly dependent according to Definition ???
2. Can we write one element of X as a linear combination of the others?
3. Does X contain redundant vectors?

It turns out that these questions are equivalent. In other words, if the answer to one of them is “YES”, the answer to the other two is also “YES”. Conversely, if the answer to one of them is “NO”, then the answer to the other two is also “NO”. We will start by illustrating this idea with an example, then conclude this section by formally proving the equivalency in Theorem ??.

Example 10. What can we say about the following sets of vectors in light of Remark ???

1.

$$\left\{ \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$$

2.

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Explanation. ?? We will start by addressing linear independence. To do so, we will solve the vector equation

$$c_1 \begin{bmatrix} 2 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_4 \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \mathbf{0} \quad (3)$$

Clearly $c_1 = c_2 = c_3 = c_4 = 0$ is a solution to the equation. The question is whether another solution exists.

The vector equation translates into the following system:

$$\begin{aligned} 2c_1 & & + & c_3 & + & c_4 & = & 0 \\ -3c_1 & + & 3c_2 & - & c_3 & - & 2c_4 & = & 0 \end{aligned}$$

Writing the system in augmented matrix form and applying elementary row operations gives us the following reduced row-echelon form:

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & 1 & 0 \\ -3 & 3 & -1 & -2 & 0 \end{array} \right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 1/2 & 1/2 & 0 \\ 0 & 1 & 1/6 & -1/6 & 0 \end{array} \right]$$

This shows that (??) has infinitely many solutions:

$$c_1 = -\frac{1}{2}s - \frac{1}{2}t, \quad c_2 = -\frac{1}{6}s + \frac{1}{6}t, \quad c_3 = s, \quad c_4 = t$$

Letting $t = s = 6$, we obtain the following:

$$-6 \begin{bmatrix} 2 \\ -3 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 3 \end{bmatrix} + 6 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 6 \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \mathbf{0} \quad (4)$$

We conclude that the vectors are linearly dependent.

Observe that (??) allows us to solve for one of the vectors and express it as a linear combination of the others. For example,

$$\begin{bmatrix} 2 \\ -3 \end{bmatrix} = 0 \begin{bmatrix} 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad (5)$$

This would not be possible if a nontrivial solution to the equation

$$c_1 \begin{bmatrix} 2 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_4 \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \mathbf{0}$$

did not exist.

Using the linear combination in (??) and the argument of Exploration ??, we conclude that $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ is redundant in

$$\left\{ \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$$

We find that the answer to all questions in Remark ?? is “YES”.

?? To address linear independence, we need to solve the equation

$$c_1 \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} = \mathbf{0}$$

Converting the equation to augmented matrix form and performing row reduction gives us

$$\left[\begin{array}{cc|c} 2 & -3 & 0 \\ 1 & 1 & 0 \\ 4 & 1 & 0 \end{array} \right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

This shows that $c_1 = c_2 = 0$ is the only solution. Therefore the two vectors are linearly independent.

Furthermore, we cannot write one of the vectors as a linear combination of the other. (Do you see that the only way this would be possible with a set of two vectors is if they were scalar multiples of each other?)

Finally, we observe that removing either vector would change the span from a plane in $\mathbb{R}^3$ to a line in $\mathbb{R}^3$, so the answer to all three questions in Remark ?? is “NO”.

We now formalize Remark ?? as a theorem.

Theorem 1. Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ be a set of vectors in $\mathbb{R}^n$ containing two or more vectors. The following conditions are equivalent.

1. $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are linearly dependent.
2. One of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ can be expressed as a linear combination of the others.
3. The set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ contains redundant vectors.

Proof ?? $\Rightarrow$?? If $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are linearly dependent, then

$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_j \mathbf{v}_j + \dots + c_k \mathbf{v}_k = \mathbf{0}$$

has a non-trivial solution. In other words at least one of the constants, say c_j , does not equal zero. This allows us to solve for $\mathbf{v}_j$:

$$-c_j \mathbf{v}_j = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_k \mathbf{v}_k$$

$$\mathbf{v}_j = -\frac{c_1}{c_j} \mathbf{v}_1 - \frac{c_2}{c_j} \mathbf{v}_2 - \dots - \frac{c_k}{c_j} \mathbf{v}_k$$

(Do you see why it was important to have one of the constants nonzero?) This shows that $\mathbf{v}_j$ may be expressed as a linear combination of the other vectors.

?? $\Rightarrow$?? First, suppose $\mathbf{v}_j$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$. We will show that $\mathbf{v}_j$ is redundant by showing that

$$\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k) = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \dots, \mathbf{v}_k).$$

To show equality of the two spans we will pick a vector in the left span and show that it is also an element of the span on the right. Then, we will pick a vector in the right span and show that it is also an element of the span on the left, and we will conclude that the sets are equal.

Observe that if $\mathbf{w}$ is in $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \dots, \mathbf{v}_k)$, then it has to be in $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k)$. (Why?)

Now suppose $\mathbf{w}$ is in $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k)$. We need to show that $\mathbf{w}$ is also in $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \dots, \mathbf{v}_k)$.

By assumption, we can write $\mathbf{v}_j$ as

$$\mathbf{v}_j = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + \dots + a_{j-1} \mathbf{v}_{j-1} + a_{j+1} \mathbf{v}_{j+1} + \dots + a_k \mathbf{v}_k. \quad (6)$$

Since $\mathbf{w}$ is in $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k)$, we have

$$\mathbf{w} = b_1 \mathbf{v}_1 + b_2 \mathbf{v}_2 + \dots + b_j \mathbf{v}_j + \dots + b_k \mathbf{v}_k.$$

Substituting the expression in (6) for $\mathbf{v}_j$ and simplifying, we obtain the following

$$\begin{aligned} \mathbf{w} &= (b_1 + b_j a_1) \mathbf{v}_1 + (b_2 + b_j a_2) \mathbf{v}_2 + \dots \\ &\quad + (b_{j-1} + b_j a_{j-1}) \mathbf{v}_{j-1} \\ &\quad + (b_{j+1} + b_j a_{j+1}) \mathbf{v}_{j+1} \\ &\quad + \dots \\ &\quad + (b_k + b_j a_k) \mathbf{v}_k. \end{aligned}$$

This shows that $\mathbf{w}$ is in $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \dots, \mathbf{v}_k)$. We now have

$$\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k) = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \dots, \mathbf{v}_k),$$

which shows that $\mathbf{v}_j$ is redundant.

?? $\Rightarrow$?? Suppose that $\mathbf{v}_j$ is redundant, so that

$$\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k) = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \dots, \mathbf{v}_k).$$

Consider a vector $\mathbf{w}$ in $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k)$

$$\mathbf{w} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + \dots + a_j \mathbf{v}_j + \dots + a_k \mathbf{v}_k \quad (7)$$

Since the span contains ALL possible linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$, we may choose $\mathbf{w}$ such that $a_j \neq 0$.

By assumption, $\mathbf{w}$ is also in $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \dots, \mathbf{v}_k)$. Therefore, we can express $\mathbf{w}$ as a linear combination

$$\mathbf{w} = b_1 \mathbf{v}_1 + b_2 \mathbf{v}_2 + \dots + b_{j-1} \mathbf{v}_{j-1} + b_{j+1} \mathbf{v}_{j+1} + \dots + b_k \mathbf{v}_k. \quad (8)$$

We complete the proof by showing there exists a non-trivial solution to

$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_j \mathbf{v}_j + \dots + c_k \mathbf{v}_k = \mathbf{0}. \quad (9)$$

Subtracting expression (??) from (??) we obtain

$$\begin{aligned} \mathbf{0} = \mathbf{w} - \mathbf{w} &= (a_1 - b_1)\mathbf{v}_1 + \dots \\ &+ (a_{j-1} - b_{j-1})\mathbf{v}_{j-1} + a_j\mathbf{v}_j + (a_{j+1} - b_{j+1})\mathbf{v}_{j+1} \\ &+ \dots \\ &+ (a_k - b_k)\mathbf{v}_k \end{aligned}$$

Recall that we ensured that $a_j \neq 0$. This implies that we have a non-trivial solution to Equation ??.

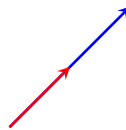
These three parts of the proof show that if one of the conditions is true, all three must be true. It is a logical consequence that if one of the three conditions is false, all three must be false. ■

Geometry of Linearly Dependent and Linearly Independent Vectors

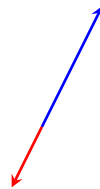
Theorem ?? gives us a convenient ways of looking at linear dependence/independence geometrically. When looking at two or more vectors, we ask, “can one of the vectors be written as a linear combination of the others?” We can also ask, “is one of the vectors redundant?” If the answer to either of these questions is “YES”, then the vectors are linearly dependent.

A Set of Two Vectors

Two vectors are linearly dependent if and only if one is a scalar multiple of the other. Two nonzero linearly dependent vectors may look like this:



or like this:



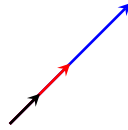
Two linearly independent vectors will look like this:



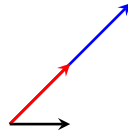
A Set of Three Vectors

Given a set of three nonzero vectors, we have the following possibilities:

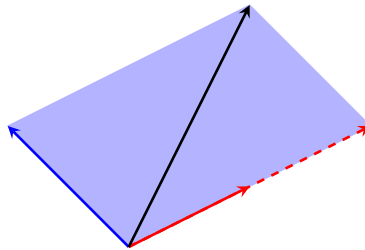
- (Linearly Dependent Vectors) The three vectors are scalar multiples of each other.



- (Linearly Dependent Vectors) Two of the vectors are scalar multiples of each other.



- (Linearly Dependent Vectors) One vector can be viewed as the diagonal of a parallelogram determined by scalar multiples of the other two vectors. All three vectors lie in the same plane.



- (Linearly Independent Vectors) A set of three vectors is linearly independent if the vectors do not lie in the same plane. For example, vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are linearly independent.

Practice Problems

Problem 19 Are the vectors below linearly independent?

$$\begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

Multiple Choice:

1. Yes
2. No

Problem 20 Are the vectors below linearly independent?

$$\begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Multiple Choice:

1. Yes
2. No

Problem 21 Are the vectors below linearly independent?

$$\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -5 \end{bmatrix}$$

Multiple Choice:

1. Yes
2. No

Problem 22 Are the vectors below linearly independent?

$$\begin{bmatrix} 3 \\ 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Multiple Choice:

1. Yes
2. No

Problem 23 (True or False?) Any set containing the zero vector is linearly dependent.

Multiple Choice:

1. TRUE
2. FALSE

Problem 24 (True or False?) A set containing five vectors in $\mathbb{R}^2$ is linearly dependent.

Multiple Choice:

1. TRUE
2. FALSE

Problem 25 Given that

$$0\mathbf{v}_1 + 0\mathbf{v}_2 + 0\mathbf{v}_3 = \mathbf{0}$$

what (if anything) can we conclude about linear independence of vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$?

Multiple Choice:

1. The vectors are linearly independent
2. The vectors are linearly dependent
3. There is not enough information given to make a determination

Problem 26 Given that

$$3\mathbf{v}_1 + 4\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$$

what (if anything) can we conclude about linear independence of vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$?

Multiple Choice:

1. The vectors are linearly independent
2. The vectors are linearly dependent
3. There is not enough information given to make a determination

Problem 27 Given that

$$2\mathbf{v}_1 + 0\mathbf{v}_2 + 0\mathbf{v}_3 = \mathbf{0}$$

what (if anything) can we conclude about linear independence of vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$?

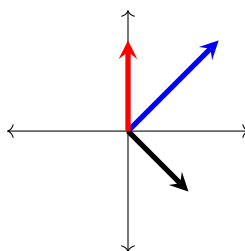
Multiple Choice:

1. The vectors are linearly independent
2. The vectors are linearly dependent
3. There is not enough information given to make a determination

Problem 28 Are the vectors in the diagram linearly dependent or independent?

Multiple Choice:

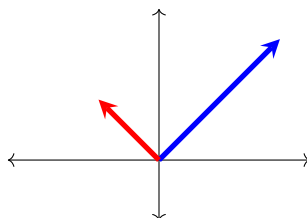
1. The vectors are linearly independent
2. The vectors are linearly dependent
3. There is not enough information given to make a determination



Problem 29 Are the vectors in the diagram linearly dependent or independent?

Multiple Choice:

1. The vectors are linearly independent
2. The vectors are linearly dependent
3. There is not enough information given to make a determination



Problem 30 Suppose $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ is a linearly independent set in $\mathbb{R}^n$, and that $\mathbf{v}_{m+1}$ is not in $\text{span}(\mathbf{v}_1, \dots, \mathbf{v}_m)$. Prove that $\{\mathbf{v}_1, \dots, \mathbf{v}_m, \mathbf{v}_{m+1}\}$ is also linearly independent.

Problem 31 Suppose $\{\mathbf{u}, \mathbf{v}\}$ is a linearly independent set of vectors. Prove that the set $\{\mathbf{u} - \mathbf{v}, \mathbf{u} + 2\mathbf{v}\}$ is also linearly independent.

Problem 32 Suppose $\{\mathbf{u}, \mathbf{v}\}$ is a linearly independent set of vectors in $\mathbb{R}^3$. Is the following set dependent or independent $\{\mathbf{u} - \mathbf{v}, \mathbf{u} + 2\mathbf{v}, \mathbf{u} + \mathbf{v}\}$? Prove your claim.

Text Source

A portion of this section was adapted from Section 5.2 of Keith Nicholson's *Linear Algebra with Applications*. (CC-BY-NC-SA)

W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2018, Open Edition, p. 271.