
Week 4

June 5, 2026

Contents

1 Addition and Scalar Multiplication of Matrices

Introduction to Matrices

A *matrix* is a rectangular array of numbers. The plural form of *matrix* is *matrices*. You have encountered matrices before in the context of augmented matrices and coefficient matrices associate with linear systems.

Consider the matrix

$$M = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 2 & 8 & 7 \\ 6 & -9 & 1 & 2 \end{bmatrix}$$

The *dimension of a matrix* is defined as $m \times n$ where m is the number of rows and n is the number of columns. The above matrix is a 3×4 matrix because there are three rows and four columns.

A *column vector* in $\mathbb{R}^n$ is an $n \times 1$ matrix. A *row vector* in $\mathbb{R}^n$ is a $1 \times n$ matrix.

The individual *entries* in the matrix are identified according to their position. The (i, j) -entry of a matrix is the entry in the i^{th} row and j^{th} column. For example, in matrix M above, 8 is called the $(2, 3)$ -entry because it is in the second row and the third column.

We denote the entry in the i^{th} row and the j^{th} column of matrix A by a_{ij} , and write A in terms of its entries as

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

Occasionally it will be convenient to talk about columns and rows of a matrix A as vectors. We will use the following notation:

$$A = \begin{bmatrix} | & | & & | \\ \mathbf{c}_1 & \mathbf{c}_2 & \dots & \mathbf{c}_n \\ | & | & & | \end{bmatrix} \quad \text{or} \quad A = [\mathbf{c}_1 \quad \mathbf{c}_2 \quad \dots \quad \mathbf{c}_n]$$

$$A = \begin{bmatrix} - & \mathbf{r}_1 & - \\ - & \mathbf{r}_2 & - \\ & \vdots & \\ - & \mathbf{r}_m & - \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_m \end{bmatrix}$$

A matrix is called a *square matrix* if it has the same number of rows and columns. If $B = [b_{ij}]$ is an $n \times n$ square

matrix, the entries of the form b_{ii} are said to lie on the *main diagonal*. For example, if

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 4 & 9 \end{bmatrix}$$

then the main diagonal consists of entries $b_{11} = 1$, $b_{22} = 5$ and $b_{33} = 9$.

There are various operations which are done on matrices of appropriate sizes. Matrices can be added to and subtracted from other matrices, multiplied by a scalar, and multiplied by other matrices. We will never divide a matrix by another matrix, but we will see later how multiplication by a matrix inverse (if an inverse exists) plays a similar role to division.

In doing arithmetic with matrices, we often define the action by what happens in terms of the entries (or components) of the matrices. Before looking at these operations in depth, consider a few general definitions.

Definition 1 (The Zero Matrix). The $m \times n$ zero matrix is the $m \times n$ matrix having every entry equal to zero. The zero matrix is denoted by O .

Definition 2 (Equality of Matrices). Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two $m \times n$ matrices. Then $A = B$ means that $a_{ij} = b_{ij}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

Addition of Matrices

Given two matrices of the same dimensions, we can add them together by adding their corresponding entries.

Definition 3 (Addition of Matrices). Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two $m \times n$ matrices. Then the *sum of matrices* A and B , denoted by $A + B$, is an $m \times n$ matrix given by

$$A + B = [a_{ij} + b_{ij}]$$

Example 1. Find the sum of A and B , if possible.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 4 \end{bmatrix}, B = \begin{bmatrix} 5 & 2 & 3 \\ -6 & 2 & 1 \end{bmatrix}$$

Explanation. Notice that both A and B are of size 2×3 . Since A and B are of the same size, addition is possible.

$$\begin{aligned} A + B &= \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 2 & 3 \\ -6 & 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1+5 & 2+2 & 3+3 \\ 1+(-6) & 0+2 & 4+1 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 4 & 6 \\ -5 & 2 & 5 \end{bmatrix} \end{aligned}$$

Going forward, whenever we write $A + B$ it will be assumed that the two matrices are of equal size and addition is possible.

Theorem 1 (Properties of Matrix Addition). Let A, B and C be matrices. Then, the following properties hold.

1. Commutative Law of Addition

$$A + B = B + A$$

2. Associative Law of Addition

$$(A + B) + C = A + (B + C)$$

3. Additive Identity

There exists a zero matrix such that

$$A + O = A$$

4. Additive Inverse

There exists a matrix, $-A$, such that

$$A + (-A) = O$$

We will prove Properties ?? and ??. The remaining properties are left as exercises.

Proof of Property ??: The (i, j) -entry of $A + B$ is given by

$$a_{ij} + b_{ij}$$

The (i, j) -entry of $B + A$ is given by

$$b_{ij} + a_{ij}$$

Since $a_{ij} + b_{ij} = b_{ij} + a_{ij}$, for all i, j , we conclude that $A + B = B + A$. ■

Proof of Property ??: Let $-A$ be defined by

$$-A = \begin{bmatrix} -a_{ij} \end{bmatrix}$$

Then $A + (-A) = O$. ■

You will recognize the zero matrix of Theorem ???? as the zero matrix of Definition ??.

Scalar Multiplication of Matrices

When a matrix is multiplied by a scalar, the new matrix is obtained by multiplying every entry of the original matrix by the given scalar.

Definition 4 (Scalar Multiplication of Matrices). If $A = [a_{ij}]$ and k is a scalar, then $kA = [ka_{ij}]$.

Example 2. Find $7A$ if

$$A = \begin{bmatrix} 2 & 0 \\ 1 & -4 \end{bmatrix}$$

Explanation. By Definition ??, we multiply each entry of A by 7. Therefore,

$$7A = 7 \begin{bmatrix} 2 & 0 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} 7(2) & 7(0) \\ 7(1) & 7(-4) \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 7 & -28 \end{bmatrix}$$

Theorem 2 (Properties of Scalar Multiplication). Let A, B be matrices, and k, p be scalars. Then, the following properties of scalar multiplication hold.

1. Distributive Law over Matrix Addition

$$k(A + B) = kA + kB$$

2. Distributive Law over Scalar Addition

$$(k + p)A = kA + pA$$

3. Associative Law for Scalar Multiplication

$$k(pA) = (kp)A$$

4. Multiplication by 1

$$1A = A$$

The proof of this theorem is similar to the proof of Theorem ?? and is left as an exercise.

Practice Problems

Problem 1 If

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix}$$

then

$$2(A + B) = \begin{bmatrix} \dots & \dots \\ \dots & \dots \end{bmatrix}$$

$$3A - 2B = \begin{bmatrix} \dots & \dots \\ \dots & \dots \end{bmatrix}$$

Text Source

The text in this section is an adaptation of Section 2.1 of Ken Kuttler's *A First Course in Linear Algebra*. (CC-BY)
Ken Kuttler, *A First Course in Linear Algebra*, Lyryx 2017, Open Edition, p. 53-58.

2 Matrix Multiplication

We will introduce matrix multiplication by first considering the special case of a matrix-vector product. In other words, the first matrix is $m \times n$ and the second matrix is $n \times 1$ for some positive integers m, n .

Matrix-Vector Multiplication

Example 3. Let

$$A = \begin{bmatrix} 2 & -1 & 3 & 2 \\ 0 & 3 & -2 & 1 \\ -2 & 4 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} 3 \\ -1 \\ 4 \\ 1 \end{bmatrix}$$

One way to understand the matrix-vector product $A\mathbf{x}$ is by thinking of it as a linear combination of the columns of A , using the entries in $\mathbf{x}$ as our coefficients.

$$A\mathbf{x} = \begin{bmatrix} 2 & -1 & 3 & 2 \\ 0 & 3 & -2 & 1 \\ -2 & 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 4 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} + (-1) \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix} + 4 \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 21 \\ -10 \\ -6 \end{bmatrix}$$

We can also compute the product one entry at a time. First, let's focus on the first row of A .

$$\begin{bmatrix} 2 & -1 & 3 & 2 \\ 0 & 3 & -2 & 1 \\ -2 & 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} (2)(3) + (-1)(-1) + (3)(4) + (2)(1) \\ \\ \\ \end{bmatrix} = \begin{bmatrix} 21 \\ \\ \\ \end{bmatrix}$$

Next, let's look at the second row of A .

$$\begin{bmatrix} 2 & -1 & 3 & 2 \\ 0 & 3 & -2 & 1 \\ -2 & 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 21 \\ (0)(3) + (3)(-1) + (-2)(4) + (1)(1) \\ \\ \end{bmatrix} = \begin{bmatrix} 21 \\ -10 \\ \\ \end{bmatrix}$$

Finally, let's do the third row of A .

$$\begin{bmatrix} 2 & -1 & 3 & 2 \\ 0 & 3 & -2 & 1 \\ -2 & 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 21 \\ -10 \\ (-2)(3) + (4)(-1) + (1)(4) + (0)(1) \end{bmatrix} = \begin{bmatrix} 21 \\ -10 \\ -6 \end{bmatrix}$$

Definition 5. Let A be an $m \times n$ matrix, and let $\mathbf{x}$ be an $n \times 1$ vector. The product $A\mathbf{x}$ is the $m \times 1$ vector given by:

$$A\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & \dots & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

or, equivalently,

$$A\mathbf{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{bmatrix}$$

We can now make a couple of observations about the matrix-vector product. The first observation is part of the definition, but it is still worth pointing out.

Observation 1. In order for the product $A\mathbf{x}$ to exist, A and $\mathbf{x}$ must have compatible dimensions. In particular, vector $\mathbf{x}$ must have as many components as the number of columns of A . (Otherwise, we would not be have a well-defined linear combination of the columns.) So, if A is an $m \times n$ matrix, $\mathbf{x}$ must be an $n \times 1$ vector. If we write these dimensions next to each other, we will notice that the inner dimensions (n) must match, while the outer dimensions, m and 1 , give us the dimensions of the product.

$$\begin{array}{ccc} A & \mathbf{x} & = & A\mathbf{x} \\ (m \times n) & (n \times 1) & & (m \times 1) \end{array}$$

Observation 2. If you are familiar with the *dot product* (see [Dot Product and its Properties](#)), you may have noticed that each individual entry in the product matrix $A\mathbf{x}$ is the dot product of a row of A with $\mathbf{x}$. Thus, if the rows of A are vectors $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_n$ we can restate Definition ?? as follows:

$$A\mathbf{x} = \begin{bmatrix} - & \mathbf{r}_1 & - \\ - & \mathbf{r}_2 & - \\ & \vdots & \\ - & \mathbf{r}_n & - \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{x} \\ \mathbf{r}_2 \cdot \mathbf{x} \\ \vdots \\ \mathbf{r}_n \cdot \mathbf{x} \end{bmatrix}$$

Let's find another matrix-vector product.

Example 4. Let

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ -2 & 1 \\ 4 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

Find $A\mathbf{x}$.

Explanation. Fill in the blanks below.

$$A\mathbf{x} = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ -2 & 1 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} \dots \\ \dots \\ \dots \\ \dots \end{bmatrix}$$

Matrix-Matrix Multiplication

Matrix-matrix multiplication is simply an extension of the idea of matrix-vector multiplication. In order for the product definition to work, matrix dimensions must be compatible. Let A be an $m \times n$ matrix, and let B be an $n \times p$ matrix, then the product AB will be an $m \times p$ matrix.

$$\begin{array}{ccc} A & B & = \\ (m \times n) & (n \times p) & (m \times p) \end{array} \quad AB$$

Just like with vector products, the inner dimensions must be the same, while the outer dimensions, m and p , give us the dimensions of the product.

Definition 6. Let A be an $m \times n$ matrix whose rows are vectors $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_m$. Let B be an $n \times p$ matrix with columns $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_p$. Then the entries of the matrix product AB are given by the dot products

$$\begin{aligned} AB &= \begin{bmatrix} - & \mathbf{r}_1 & - \\ - & \mathbf{r}_2 & - \\ & \vdots & \\ - & \mathbf{r}_i & - \\ & \vdots & \\ - & \mathbf{r}_m & - \end{bmatrix} \begin{bmatrix} | & | & & | & | \\ \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_j & \dots & \mathbf{b}_p \\ | & | & & | & | \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{b}_1 & \mathbf{r}_1 \cdot \mathbf{b}_2 & \dots & \mathbf{r}_1 \cdot \mathbf{b}_j & \dots & \mathbf{r}_1 \cdot \mathbf{b}_p \\ \mathbf{r}_2 \cdot \mathbf{b}_1 & \mathbf{r}_2 \cdot \mathbf{b}_2 & \dots & \mathbf{r}_2 \cdot \mathbf{b}_j & \dots & \mathbf{r}_2 \cdot \mathbf{b}_p \\ \vdots & \vdots & & \vdots & & \vdots \\ \mathbf{r}_i \cdot \mathbf{b}_1 & \mathbf{r}_i \cdot \mathbf{b}_2 & \dots & \mathbf{r}_i \cdot \mathbf{b}_j & \dots & \mathbf{r}_i \cdot \mathbf{b}_p \\ \vdots & \vdots & & \vdots & & \vdots \\ \mathbf{r}_m \cdot \mathbf{b}_1 & \mathbf{r}_m \cdot \mathbf{b}_2 & \dots & \mathbf{r}_m \cdot \mathbf{b}_j & \dots & \mathbf{r}_m \cdot \mathbf{b}_p \end{bmatrix} \end{aligned}$$

So the (i, j) -entry of AB is the dot product of the i^{th} row of A and the j^{th} column of B .

In terms of components, if the i^{th} row of A is

$$\begin{bmatrix} a_{i1} & a_{i2} & \dots & a_{in} \end{bmatrix}$$

and the j^{th} column of B is

$$\begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$$

then the (i, j) -entry of AB is given by

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj} = \sum_{k=1}^n a_{ik}b_{kj} \quad (1)$$

Example 5. Let

$$A = \begin{bmatrix} 3 & 2 & -1 & 1 \\ 0 & 3 & 1 & 1 \\ 1 & 4 & -1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -2 \\ -1 & 4 \\ 2 & 3 \\ 2 & 0 \end{bmatrix}$$

Find AB .

Explanation. Observe that A is a 3×4 matrix and B is a 4×2 matrix. So, we expect the product AB to be a 3×2 matrix.

$$\begin{aligned} AB &= \begin{bmatrix} 3 & 2 & -1 & 1 \\ 0 & 3 & 1 & 1 \\ 1 & 4 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 4 \\ 2 & 3 \\ 2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} (3)(1) + (2)(-1) + (-1)(2) + (1)(2) & (3)(-2) + (2)(4) + (-1)(3) + (1)(0) \\ (0)(1) + (3)(-1) + (1)(2) + (1)(2) & (0)(-2) + (3)(4) + (1)(3) + (1)(0) \\ (1)(1) + (4)(-1) + (-1)(2) + (0)(2) & (1)(-2) + (4)(4) + (-1)(3) + (0)(0) \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \\ 1 & 15 \\ -5 & 11 \end{bmatrix} \end{aligned}$$

Remark 1. It is possible to use linear combinations rather than dot products to compute a matrix-matrix product. If the columns of matrix B are given by $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_p$, then the matrix product consists of p columns, each of which is a matrix-vector product:

$$AB = \begin{bmatrix} | & | & \dots & | \\ A\mathbf{b}_1 & A\mathbf{b}_2 & \dots & A\mathbf{b}_p \\ | & | & \dots & | \end{bmatrix}.$$

If we return to Example ?? above, we may use this approach to compute

$$A\mathbf{b}_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -5 \end{bmatrix}$$

and

$$A\mathbf{b}_2 = -2 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + 4 \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} + 0 = \begin{bmatrix} -1 \\ 15 \\ 11 \end{bmatrix}.$$

Example 6. Let

$$A = \begin{bmatrix} -2 & 1 \\ 3 & 0 \\ 1 & -3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & -2 & 3 & -2 & 2 \\ 1 & -3 & 1 & 1 & 0 \end{bmatrix}$$

Find AB .

Explanation. Fill in the blanks below.

$$AB = \begin{bmatrix} -2 & 1 \\ 3 & 0 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 4 & -2 & 3 & -2 & 2 \\ 1 & -3 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

Example 7. Suppose that we know that $\mathbf{x} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$ satisfies

$$\begin{bmatrix} -2 & 3 \\ 4 & -1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 19 \\ -13 \end{bmatrix}$$

Use this information to express $\begin{bmatrix} 19 \\ -13 \end{bmatrix}$ as a linear combination of the columns of $\begin{bmatrix} -2 & 3 \\ 4 & -1 \end{bmatrix}$.

Explanation.

$$-2 \begin{bmatrix} -2 \\ 4 \end{bmatrix} + 5 \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 19 \\ -13 \end{bmatrix}$$

Properties of Matrix Multiplication

Exploration 2 Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

Observe that both AB and BA are defined, and both products are 2×2 matrices. Let's compute the two products

$$AB = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$$

Clearly $AB \neq BA$. We say that A and B do not commute.

While it is possible to find specific matrices that commute, matrix multiplication is not commutative in general.

Warning 1. Matrix multiplication is not commutative.

One example of an $n \times n$ square matrix that commutes with all $n \times n$ matrices is the matrix I_n defined by

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

I_n has 1's along the main diagonal and 0's everywhere else. It is often useful to think of I_n as a matrix whose j^{th} column (and j^{th} row) is $\mathbf{e}_j$, the j^{th} standard unit vector of $\mathbb{R}^n$. When the dimensions of I_n are clear from the context, or irrelevant, we will omit the subscript n and simply refer to this matrix as I .

You can easily convince yourself that I commutes with all square matrices of appropriate dimensions. Let

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \quad \text{and} \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Verify that $AI = A$ and $IA = A$.

Because I acts like the multiplicative identity 1 in regular multiplication, I (or I_n) is called the *identity matrix*.

Next we list several important properties of matrix multiplication. These properties hold only when matrix sizes are such that the products are defined.

Theorem 3 (Properties of Matrix Multiplication). The following hold for matrices A, B , and C and for scalar c ,

1. Left Distributive Property

$$A(B + C) = AB + AC$$

2. Right Distributive Property

$$(B + C)A = BA + CA$$

3. Associativity

$$A(BC) = (AB)C$$

- 4.

$$c(AB) = (cA)B = A(cB)$$

5. Multiplicative Identity

$$AI = IA = A$$

Proof We prove ?? using the expression in (??) for the (i, j) -entry of a matrix product. (The proof of ?? is similar.) The (i, j) -entry of $A(B + C)$ is given by

$$\sum_k a_{ik}(b_{kj} + c_{kj}) = \sum_k a_{ik}b_{kj} + \sum_k a_{ik}c_{kj}$$

We recognize the right hand side as the (i, j) -entry of $AB + AC$. Thus $A(B + C) = AB + AC$.

For ??,

$$c \left(\sum_k a_{ik} b_{kj} \right) = \sum_k (ca_{ik}) b_{kj} = \sum_k a_{ik} (b_{kj}c).$$

For ??, the (i, j) -entry of the product AI is given by the dot product of the i^{th} row of A with the standard unit vector $\mathbf{e}_j$. Clearly, this dot product is a_{ij} . Because the (i, j) -entry of the product AI is equal to the (i, j) -entry A , we conclude that $AI = A$. The proof that $IA = A$ is similar.

Note that we skipped the proof of ??, which is quite cumbersome using sigma notation. We will easily tackle this proof later in the course when we cover [Composition of Linear Transformations](#). ■

Practice Problems

Problem 3 Explain why the following product is not defined.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Problem 4 Express the given product as a linear combination of the columns of the matrix.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ -5 \\ 7 \end{bmatrix} = \dots \begin{bmatrix} 1 \\ 5 \end{bmatrix} + \dots \begin{bmatrix} 2 \\ 6 \end{bmatrix} + \dots \begin{bmatrix} 3 \\ 7 \end{bmatrix} + \dots \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

Problem 5 Find the dimensions of the product.

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

Dimensions of product: $\dots \times \dots$

Problem 6 Find the dimensions of the product.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

Dimensions of product: $\dots \times \dots$

Problem 7 Find the product.

$$\begin{bmatrix} 1 & 3 & -2 & 1 \\ -2 & 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \dots \\ \dots \end{bmatrix}$$

Problem 8 Find the product.

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 & -3 & 1 \\ 1 & 1 & -1 & -2 \\ 0 & 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} \dots & \dots & \dots & \dots \end{bmatrix}$$

Problem 9 Find the product.

$$\begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 2 & 3 \\ -4 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} \dots & \dots \\ \dots & \dots \\ \dots & \dots \\ \dots & \dots \end{bmatrix}$$

Text Source

The section on Properties of Matrix Multiplication is an adaptation of Section 2.1 of Ken Kuttler's *A First Course in Linear Algebra*. (CC-BY)

Ken Kuttler, *A First Course in Linear Algebra*, Lyryx 2017, Open Edition, p. 67-68.

3 Transpose of a Matrix

Another important operation on matrices is that of taking the *transpose*. For a matrix A , we denote the *transpose* of A by A^T . Before formally defining the transpose, we explore this operation on the following matrix.

$$\begin{bmatrix} 1 & 4 \\ 3 & 1 \\ 2 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 1 & 6 \end{bmatrix}$$

What happened? The first column became the first row and the second column became the second row. Thus the 3×2 matrix became a 2×3 matrix. The number 4 was in the first row and the second column and it ended up in the second row and first column.

The definition of the transpose is as follows.

Definition 7 (The Transpose of a Matrix). Let $A = [a_{ij}]$ be an $m \times n$ matrix. Then the *transpose* of A , denoted by A^T , is the $n \times m$ matrix given by

$$A^T = [a_{ij}]^T = [a_{ji}]$$

The (i, j) -entry of A becomes the (j, i) -entry of A^T .

Example 8. Calculate A^T for the following matrix

$$A = \begin{bmatrix} 1 & 2 & -6 \\ 3 & 5 & 4 \end{bmatrix}$$

Explanation.

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 5 \\ -6 & 4 \end{bmatrix}$$

Note that A is a 2×3 matrix, while A^T is a 3×2 matrix. The columns of A are the rows of A^T , and the rows of A are the columns of A^T .

Theorem 4 (Properties of the Transpose of a Matrix). Let A be an $m \times n$ matrix, B an $n \times p$ matrix, and k a scalar. Then

1. $(A^T)^T = A$
2. $(AB)^T = B^T A^T$ (Shoes and Socks rule)
3. $(A + B)^T = A^T + B^T$
4. $(kA)^T = kA^T$

We will prove Property ???. The remaining properties are left as exercises.

Proof of Property ??: Note that A and B have compatible dimensions, so that AB is defined and has dimensions $m \times p$. Thus, $(AB)^T$ has dimensions $p \times m$. On the right side of the equality, A^T has dimensions $n \times m$, and B^T has dimensions $p \times n$. Therefore $B^T A^T$ is defined and has dimensions $p \times m$. Now we know that $(AB)^T$ and $B^T A^T$ have the same dimensions.

To show that $(AB)^T = B^T A^T$ we need to show that their corresponding entries are equal. Recall that the (i, j) -entry of AB is given by the dot product of the i^{th} row of A and the j^{th} column of B . The same dot product is also the (j, i) -entry of $(AB)^T$.

The (j, i) -entry of $B^T A^T$ is given by the dot product of the j^{th} row of B^T and the i^{th} column of A^T . But the j^{th} row of B^T has the same entries as the j^{th} column of B , and the i^{th} column of A^T has the same entries as the i^{th} row of A . Therefore the (j, i) -entry of $B^T A^T$ is also equal to the (i, j) -entry of AB .

Thus, the corresponding components of $(AB)^T$ are equal and we conclude that $(AB)^T = B^T A^T$. ■

Symmetric and Skew Symmetric Matrices

Definition 8 (Symmetric and Skew Symmetric Matrices). An $n \times n$ matrix A is said to be *symmetric* if $A = A^T$. It is said to be *skew symmetric* if $A = -A^T$.

We will explore these definitions in the following examples.

Example 9. Let

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 5 & -3 \\ 3 & -3 & 7 \end{bmatrix}$$

Show that A is symmetric.

Explanation.

$$A^T = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 5 & -3 \\ 3 & -3 & 7 \end{bmatrix}$$

Hence, $A = A^T$, so A is symmetric.

Example 10. Let

$$A = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 2 \\ -3 & -2 & 0 \end{bmatrix}$$

Show that A is skew symmetric.

Explanation.

$$A^T = \begin{bmatrix} 0 & -1 & -3 \\ 1 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix}$$

Each entry of A^T is equal to -1 times the same entry of A . Hence, $A^T = -A$ and so by Definition ??, A is skew symmetric.

A special case of a symmetric matrix is a *diagonal matrix*. A diagonal matrix is a square matrix whose entries outside of the main diagonal are all zero. The identity matrix I is a diagonal matrix. Here is another example.

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Practice Problems

Problem 10 Prove Properties ??, ?? and ?? of Theorem ??.

Problem 11 Let A be an arbitrary matrix. What can you say about the dimensions of the product $A^T A$?

Problems ??-??

Classify each matrix as symmetric, skew symmetric, or neither.

Problem 12

$$\begin{bmatrix} 0 & -1 & -3 \\ 1 & -1 & -2 \\ 3 & 2 & 0 \end{bmatrix}$$

Multiple Choice:

1. Symmetric
2. Skew symmetric
3. Neither

Problem 13

$$\begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix}$$

Multiple Choice:

1. Symmetric
2. Skew symmetric
3. Neither

Problem 14

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & -1 & 2 \\ 3 & 2 & 4 \end{bmatrix}$$

Multiple Choice:

1. Symmetric
2. Skew symmetric
3. Neither

Problem 15 Give your own example of a 4×4 skew symmetric matrix.

Problem 16 Make a conjecture about the main diagonal entries of a skew symmetric matrix. Prove your conjecture.

Text Source

The text in this section is an adaptation of Section 2.1 of Ken Kuttler's [*A First Course in Linear Algebra*](#). (CC-BY)

Ken Kuttler, *A First Course in Linear Algebra*, Lyryx 2017, Open Edition, p. 68-70.

4 Linear Systems as Matrix and Linear Combination Equations

A Linear System as a Matrix Equation

Exploration 17 Consider the linear system

$$\begin{array}{ccccccccc} 3x_1 & - & 2x_2 & + & 4x_3 & + & x_4 & = & 5 \\ -x_1 & & & & + & 5x_3 & - & 2x_4 & = & 1 \\ 2x_1 & + & x_2 & - & x_3 & + & 3x_4 & = & -4 \end{array}$$

Let's construct the coefficient matrix A and multiply it by $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ on the right.

$$A\mathbf{x} = \begin{bmatrix} 3 & -2 & 4 & 1 \\ -1 & 0 & 5 & -2 \\ 2 & 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_1 - 2x_2 + 4x_3 + x_4 \\ -x_1 + 5x_3 - 2x_4 \\ 2x_1 + x_2 - x_3 + 3x_4 \end{bmatrix}$$

Observe that each component of the product vector corresponds to one of the equations in the system. Let $\mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ -4 \end{bmatrix}$. Then

$$A\mathbf{x} = \mathbf{b}$$

$$\begin{bmatrix} 3 & -2 & 4 & 1 \\ -1 & 0 & 5 & -2 \\ 2 & 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -4 \end{bmatrix}$$

is a matrix equation that corresponds to our system of equations.

In general, a system of linear equations

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ & & & & \vdots & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \end{array}$$

can be written as a matrix equation as follows:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & \dots & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Solving this matrix equation (or showing that a solution does not exist) amounts to finding the reduced row-echelon form of the augmented matrix

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \dots & \dots & a_{mn} & b_m \end{array} \right]$$

Example 11. Given a linear system

$$\begin{array}{rcrcrcrcl} x & + & 2y & = & 0 \\ -x & + & y & = & -3 \\ & & y & = & -1 \\ x & & & = & 2 \end{array}$$

1. Write the system as a matrix equation
2. Solve the system and the matrix equation

Explanation. ?? The matrix equation that corresponds to the system is

$$\begin{bmatrix} \dots & \dots \\ \dots & \dots \\ \dots & \dots \\ \dots & \dots \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \dots \\ \dots \\ \dots \\ \dots \end{bmatrix}$$

?? The augmented matrix that corresponds to the original system and its reduced row-echelon form are

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ -1 & 1 & -3 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{array} \right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

This shows that the ordered pair $(2, -1)$ is a solution to the system. We conclude that $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is a solution to the matrix equation in ?? . A quick verification confirms this

$$\begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ -1 \\ 2 \end{bmatrix}$$

Example 12. Let

$$A = \begin{bmatrix} 2 & 1 & -1 & 2 \\ 1 & 1 & 0 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

Solve $A\mathbf{x} = \mathbf{b}$.

Explanation. We write the equation $A\mathbf{x} = \mathbf{b}$ in augmented matrix form and apply elementary row operations to find its reduced row-echelon form.

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 2 & 0 \\ 1 & 1 & 0 & 3 & -2 \end{array} \right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 2 \\ 0 & 1 & 1 & 4 & -4 \end{array} \right]$$

One way to obtain a solution is to convert this to a system of equations. It is not necessary to write the system down, but it helps to think about it as you write out your solution vector.

$$\begin{array}{ccccccccc} x_1 & & & - & x_3 & - & x_4 & = & 2 \\ & x_2 & + & x_3 & + & 4x_4 & = & -4 \end{array}$$

We see that x_1 and x_2 are leading variables because they correspond to leading 1s in the reduced row-echelon form, while x_3 and x_4 are free variables. We start by assigning parameters s and t to x_3 and x_4 , respectively, then solve for x_1 and x_2 .

$$\begin{aligned} x_1 &= 2 + s + t \\ x_2 &= -4 - s - 4t \\ x_3 &= s \\ x_4 &= t \end{aligned}$$

We can now write the solution vector as follows

$$\mathbf{x} = \begin{bmatrix} 2 + s + t \\ -4 - s - 4t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix} t \quad (2)$$

The solution given in (2) is an example of a *general solution* because it accounts for all of the solutions to the system. Letting s and t take on specific values produces *particular solutions*. For example, $\begin{bmatrix} 2 \\ -1 \\ 1 \\ -1 \end{bmatrix}$ is a particular solution that corresponds to $s = 1, t = -1$.

Singular and Nonsingular Matrices

Our examples so far involved non-square matrices. Square matrices, however, play a very important role in linear algebra. This section will focus on square matrices.

Example 13. Let

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Solve $A\mathbf{x} = \mathbf{b}$.

Explanation. We apply elementary row operations to bring the augmented matrix to its reduced row-echelon form.

$$\left[\begin{array}{ccc|c} 3 & -1 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & 2 & 0 \end{array} \right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

We can immediately see that the solution vector is

$$\mathbf{x} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Observe that the left-hand side of the augmented matrix in Example ?? is the identity matrix I . This means that $\text{rref}(A) = I$.

The elementary row operations that carried A to I were not dependent on the vector $\mathbf{b}$. In fact, the same row reduction process can be applied to the matrix equation $A\mathbf{x} = \mathbf{b}$ for *any* vector $\mathbf{b}$ to obtain a unique solution.

$$\left[\begin{array}{ccc|c} 3 & -1 & 1 & a \\ 0 & 1 & 2 & b \\ 1 & 2 & 2 & c \end{array} \right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & a^* \\ 0 & 1 & 0 & b^* \\ 0 & 0 & 1 & c^* \end{array} \right]$$

$$\mathbf{x} = \begin{bmatrix} a^* \\ b^* \\ c^* \end{bmatrix}$$

Given a matrix A such that $\text{rref}(A) = I$, the system $A\mathbf{x} = \mathbf{b}$ will never be inconsistent because we will never have a row like this: $\left[\begin{array}{ccc|c} 0 & 0 & \dots & 0 & 1 \end{array} \right]$. Neither will we have infinitely many solutions because there will never be free variables. Matrices such as A deserve special attention.

Definition 9. A square matrix A is said to be *nonsingular* provided that $\text{rref}(A) = I$. Otherwise we say that A is *singular*.

Non-singular matrices have many useful properties.

Theorem 5. The following statements are equivalent for an $n \times n$ matrix A .

1. A is nonsingular
2. $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $\mathbf{b}$ in $\mathbb{R}^n$
3. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$

We will prove equivalence of the three statements by showing that

$$?? \Rightarrow ?? \Rightarrow ?? \Rightarrow ??$$

Proof of $?? \Rightarrow ??$ Suppose $\text{rref}(A) = I$. Given any vector $\mathbf{b}$ in $\mathbb{R}^n$, the augmented matrix $[A|\mathbf{b}]$ can be carried to its reduced row-echelon form $[I|\mathbf{b}^*]$. Uniqueness of the reduced row-echelon form guarantees that $\mathbf{b}^*$ is the unique solution of $A\mathbf{x} = \mathbf{b}$. ■

Proof of $?? \Rightarrow ??$ Suppose $A\mathbf{x} = \mathbf{b}$ has a unique solution for all vectors $\mathbf{b}$. Then $A\mathbf{x} = \mathbf{0}$ has a unique solution. But $\mathbf{x} = \mathbf{0}$ is always a solution to $A\mathbf{x} = \mathbf{0}$. Therefore $\mathbf{x} = \mathbf{0}$ is the only solution. ■

Proof of $?? \Rightarrow ??$ Suppose $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. This means that $x_1 = 0, x_2 = 0, \dots, x_n = 0$ is the only solution of $A\mathbf{x} = \mathbf{0}$. But then, we know that the augmented matrix $[A|\mathbf{0}]$ can be reduced to $[I|\mathbf{0}]$. The same row operations will carry A to I . ■

Not all square matrices are nonsingular. For example,

$$\text{rref} \left(\begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & 1 \\ 3 & 0 & 2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 2/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{bmatrix} \neq I$$

By Theorem ??, a matrix equation $A\mathbf{x} = \mathbf{b}$ involving a singular matrix A cannot have a unique solution. The following example illustrates the two scenarios that arise when solving equations that involve singular matrices.

Example 14. Let

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & 1 \\ 3 & 0 & 2 \end{bmatrix}$$

Solve the equation $A\mathbf{x} = \mathbf{b}$ or show that the equation is inconsistent.

1.

$$\mathbf{b} = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$$

2.

$$\mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Explanation. ?? Row reduction gives us

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 4 \\ 1 & 1 & 1 & -1 \\ 3 & 0 & 2 & 3 \end{array} \right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 2/3 & 1 \\ 0 & 1 & 1/3 & -2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

There are infinitely many solutions.

$$\mathbf{x} = \begin{bmatrix} 1 - (2/3)t \\ -2 - (1/3)t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} -2/3 \\ -1/3 \\ 1 \end{bmatrix} t$$

?? When the vector $\mathbf{b}$ is changed, the row operations that take A to its reduced row-echelon form produce a 1 in the last row of the vector on the right, which shows that the system is inconsistent.

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 3 & 0 & 2 & 1 \end{array} \right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 2/3 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

A Linear System as a Linear Combination Equation

Recall that the product of a matrix and a vector can be interpreted as a linear combination of the columns of the matrix. For example,

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 7 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

Example 15. For each given matrix A and vector $\mathbf{b}$, determine whether $\mathbf{b}$ is a linear combination of the columns of A . If possible, express $\mathbf{b}$ as a linear combination of the columns of A .

1.

$$A = \begin{bmatrix} 3 & 1 & -2 \\ 1 & 0 & 3 \\ -2 & 1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -2 \\ 7 \\ -1 \end{bmatrix}$$

2.

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & -1 & 1 \\ 0 & 1 & -5 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$$

Explanation. ?? We are looking for x_1, x_2, x_3 such that

$$x_1 \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ -1 \end{bmatrix}$$

Solving this equation amounts to finding $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ such that $A\mathbf{x} = \mathbf{b}$. The augmented matrix corresponding to this equation, together with its reduced row-echelon form are

$$\left[\begin{array}{ccc|c} 3 & 1 & -2 & -2 \\ 1 & 0 & 3 & 7 \\ -2 & 1 & 1 & -1 \end{array} \right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

So, $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ is a solution to the matrix equation. We conclude that $\mathbf{b}$ is a linear combination of the columns of A , and write

$$\mathbf{b} = \begin{bmatrix} -2 \\ 7 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$

?? We begin by attempting to solve the matrix equation $A\mathbf{x} = \mathbf{b}$. The augmented matrix corresponding to this equation, together with its reduced row-echelon form are

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 4 \\ 2 & -1 & 1 & -1 \\ 0 & 1 & -5 & 2 \end{array} \right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

This shows that this matrix equation has no solutions. We conclude that $\mathbf{b}$ is not a linear combination of the columns of A .

Practice Problems

Problem 18 Given a system of linear equations, write (a) the corresponding matrix equation, and (b) the corresponding linear combination equation. DO NOT SOLVE.

$$\begin{array}{rrcr} 3x & - & y & - & 2z & = & 4 \\ -x & & & & + & z & = & -1 \\ & & -y & + & 5z & = & 0 \end{array}$$

Problem 19 Use an augmented matrix and elementary row operations to find coefficients x_1 and x_2 that make the expression true, or demonstrate that such coefficients do not exist.

$$x_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

If coefficients x_1 and x_2 do not exist, enter NA in each answer box.

$$X_1 = \dots, X_2 = \dots$$

Problem 20 Use an augmented matrix and elementary row operations to find coefficients x_1 and x_2 that make the expression true, or demonstrate that such coefficients do not exist.

$$x_1 \begin{bmatrix} 4 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -8 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

If coefficients x_1 and x_2 do not exist, enter NA in each answer box.

$$X_1 = \dots, X_2 = \dots$$

Problem 21 Determine whether vector **b** is in the span of the given set of vectors.

$$\mathbf{b} = \begin{bmatrix} 2 \\ 14 \\ 7 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} \right\}$$

Multiple Choice:

1. **b** is in the span of the vectors.
2. **b** is NOT in the span of the vectors.

Problem 22 Determine whether vector **b** is in the span of the given set of vectors.

$$\mathbf{b} = \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$$

Multiple Choice:

1. **b** is in the span of the vectors.
2. **b** is NOT in the span of the vectors.

Problem 23 Determine whether vector **b** is in the span of the given set of vectors.

$$\mathbf{b} = \begin{bmatrix} 2 \\ 4 \\ -7 \\ -5 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ -3 \\ 1 \end{bmatrix} \right\}$$

Multiple Choice:

1. **b** is in the span of the vectors.
2. **b** is NOT in the span of the vectors.

5 The Inverse of a Matrix

Definition and Properties of Matrix Inverses

Consider the equation $2x = 6$. It takes little time to recognize that the solution to this equation is $x = 3$. In fact, the solution is so obvious that we do not think about the algebraic steps necessary to find it. Let's take a look at these steps in detail.

$$\begin{aligned} 2x &= 6 \\ \frac{1}{2} \times (2x) &= \frac{1}{2} \times 6 \\ \left(\frac{1}{2} \times 2\right)x &= 3 \\ 1x &= 3 \\ x &= 3 \end{aligned}$$

This process utilizes many properties of real-number multiplication. In particular, we make use of existence of *multiplicative inverses*. Every non-zero real number a has a multiplicative inverse $a^{-1} = \frac{1}{a}$ with the property that $\frac{1}{a} \times a = a \times \frac{1}{a} = 1$. We say that 1 is the *multiplicative identity* because $a \times 1 = 1 \times a = a$.

Given a matrix equation $A\mathbf{x} = \mathbf{b}$, we would like to follow a process similar to the one above to solve this matrix equation for $\mathbf{x}$.

Observe that the role of the multiplicative identity for $n \times n$ square matrices is filled by I_n because $AI_n = I_nA = A$. Given an $n \times n$ matrix A , a multiplicative inverse of A would have to be some $n \times n$ matrix B such that

$$BA = AB = I_n$$

Assuming that such an inverse B exists, this is what the process of solving the equation $A\mathbf{x} = \mathbf{b}$ would look like:

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ B(A\mathbf{x}) &= B\mathbf{b} \\ (BA)\mathbf{x} &= B\mathbf{b} \\ I\mathbf{x} &= B\mathbf{b} \\ \mathbf{x} &= B\mathbf{b} \end{aligned}$$

Definition 10. Let A be an $n \times n$ matrix. An $n \times n$ matrix B is called an *inverse* of A if

$$AB = BA = I$$

where I is an $n \times n$ identity matrix. If such an inverse matrix exists, we say that A is *invertible*. If an inverse does not exist, we say that A is not invertible.

It follows directly from the way the definition is stated that if B is an inverse of A , then A is an inverse of B . We say that A and B are inverses of each other.

The following theorem shows that matrix inverses are unique.

Theorem 6. Suppose A is an invertible matrix, and B is an inverse of A . Then B is unique.

Proof Because B is an inverse of A , we have:

$$AB = BA = I$$

Suppose there exists another $n \times n$ matrix C such that

$$AC = CA = I$$

Then

$$C(AB) = CI$$

$$(CA)B = C$$

$$IB = C$$

$$B = C$$

■

Now that we know that a matrix A cannot have more than one inverse, we can safely refer to the inverse of A as A^{-1} .

Example 16. Let

$$A = \begin{bmatrix} 1 & -1 \\ -5 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2/3 & -1/3 \\ -5/3 & -1/3 \end{bmatrix}$$

Verify that A and B are inverses of each other.

Explanation. We will show that $AB = BA = I$.

$$AB = \begin{bmatrix} 1 & -1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} -2/3 & -1/3 \\ -5/3 & -1/3 \end{bmatrix} = \begin{bmatrix} -2/3 + 5/3 & -1/3 + 1/3 \\ 10/3 - 10/3 & 5/3 - 2/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} -2/3 & -1/3 \\ -5/3 & -1/3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} -2/3 + 5/3 & 2/3 - 2/3 \\ -5/3 + 5/3 & 5/3 - 2/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example 17. Use what we found in Example ?? to solve the matrix equation:

$$\begin{bmatrix} 1 & -1 \\ -5 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

Explanation. We multiply both sides of the equation by the inverse of $\begin{bmatrix} 1 & -1 \\ -5 & 2 \end{bmatrix}$.

$$\begin{bmatrix} -2/3 & -1/3 \\ -5/3 & -1/3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -5 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -2/3 & -1/3 \\ -5/3 & -1/3 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} 5/3 \\ 14/3 \end{bmatrix}$$

We now prove several useful properties of matrix inverses.

Theorem 7. The following properties are stated for square matrices of appropriate sizes.

1. $I^{-1} = I$
2. For an invertible matrix A , $(A^{-1})^{-1} = A$.
3. If A and B are invertible matrices, then (AB) is invertible and $(AB)^{-1} = B^{-1}A^{-1}$ (Shoes and socks rule).
4. If A is an invertible matrix and k is a non-zero number, then $(kA)^{-1} = \frac{1}{k}A^{-1}$.
5. If A is an invertible matrix, then A^T is also invertible, and $(A^T)^{-1} = (A^{-1})^T$.

We will prove ???. The remaining properties are left as exercises.

Proof of Property ???: We will check to see if $B^{-1}A^{-1}$ is the inverse of AB .

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$$

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

Thus (AB) is invertible and $(AB)^{-1} = B^{-1}A^{-1}$. ■

Computing the Inverse

We now turn to the question of how to find the inverse of a square matrix, or determine that the inverse does not exist.

Given a square matrix A , we are looking for a square matrix B such that

$$AB = I \quad \text{and} \quad BA = I$$

We will start by attempting to satisfy $AB = I$. Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be the columns of B , then

$$A \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \mathbf{e}_1 & \mathbf{e}_2 & \dots & \mathbf{e}_n \\ | & | & & | \end{bmatrix}$$

where each $\mathbf{e}_i$ is a standard unit vector of $\mathbb{R}^n$. This gives us a system of equations $A\mathbf{v}_i = \mathbf{e}_i$ for each $1 \leq i \leq n$. If each $A\mathbf{v}_i = \mathbf{e}_i$ has a unique solution, then finding these solutions will give us the columns of the desired matrix B .

First, suppose that $\text{rref}(A) = I$, then we can use elementary row operations to carry each $[A|\mathbf{e}_i]$ to its reduced row-echelon form.

$$[A|\mathbf{e}_i] \rightsquigarrow [I|\mathbf{v}_i]$$

Observe that the row operations that carry A to I will be the same for each $A\mathbf{v}_i = \mathbf{e}_i$. We can, therefore, combine the process of solving n systems of equations into a single process

$$[A|I] \rightsquigarrow [I|B]$$

Each $\mathbf{v}_i$ is a unique solution of $A\mathbf{v}_i = \mathbf{e}_i$, and we conclude that

$$B = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \\ | & | & & | \end{bmatrix}$$

is a solution to $AB = I$.

By Problem ??, we can reverse the elementary row operations to obtain

$$[I|B] \rightsquigarrow [A|I]$$

But the same row operations would also give us

$$[B|I] \rightsquigarrow [I|A]$$

We conclude that $BA = I$, and $B = A^{-1}$.

Next, suppose that $\text{rref}(A) \neq I$. Then $\text{rref}(A)$ must contain a row of zeros. Because one of the rows of A was completely wiped out by elementary row operations, one of the rows of A must be a linear combination of the other rows. Suppose row p is a linear combination of the other rows. Then row p can be carried to a row of zeros. But then the system $A\mathbf{v}_p = \mathbf{e}_p$ is inconsistent. This is because $\mathbf{e}_p$ has a 1 as the p^{th} entry and zeros everywhere else. The 1 in the p^{th} spot will not be affected by elementary row operations, and the p^{th} row will eventually look like this

$$[0 \dots 0 | 1]$$

This shows that a matrix B such that $AB = I$ does not exist, and A does not have an inverse.

We have just proved the following theorem.

Theorem 8 (Row-reduction Method for Computing the Inverse of a Matrix). Let A be a square matrix. If it is possible to use elementary row operations to carry the augmented matrix $[A|I]$ to $[I|B]$, then $B = A^{-1}$. If such a reduction is not possible, then A does not have an inverse.

Corollary 1. A square matrix A has an inverse if and only if $\text{rref}(A) = I$.

Example 18. Find A^{-1} or demonstrate that A^{-1} does not exist.

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

Explanation. We start with the augmented matrix

$$\begin{bmatrix} 1 & -1 & 2 & | & 1 & 0 & 0 \\ 1 & 1 & 1 & | & 0 & 1 & 0 \\ 1 & 3 & -1 & | & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\substack{R_2 - R_1 \\ R_3 - R_1}} \begin{bmatrix} 1 & -1 & 2 & | & 1 & 0 & 0 \\ 0 & 2 & -1 & | & -1 & 1 & 0 \\ 0 & 4 & -3 & | & -1 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 - 2R_2}$$

$$\begin{aligned}
 & \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -2 & 1 \end{array} \right] \xrightarrow{(-1)R_3} \\
 & \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array} \right] \xrightarrow{\begin{array}{l} R_1 - 2R_3 \\ R_2 + R_3 \end{array}} \\
 & \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 3 & -4 & 2 \\ 0 & 2 & 0 & -2 & 3 & -1 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array} \right] \xrightarrow{\frac{1}{2}R_2} \\
 & \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 3 & -4 & 2 \\ 0 & 1 & 0 & -1 & 3/2 & -1/2 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array} \right] \xrightarrow{R_1 + R_2} \\
 & \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -5/2 & 3/2 \\ 0 & 1 & 0 & -1 & 3/2 & -1/2 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array} \right]
 \end{aligned}$$

We conclude that

$$A^{-1} = \begin{bmatrix} 2 & -5/2 & 3/2 \\ -1 & 3/2 & -1/2 \\ -1 & 2 & -1 \end{bmatrix}$$

Example 19. Find A^{-1} or demonstrate that A is not invertible.

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 2 & 1 \\ 4 & 4 & -3 \end{bmatrix}$$

Explanation. We start with the augmented matrix

$$\begin{aligned}
 & \left[\begin{array}{ccc|ccc} 2 & 3 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 4 & 4 & -3 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\begin{array}{l} \frac{1}{2}R_1 \\ \frac{1}{2}R_2 \end{array}} \\
 & \left[\begin{array}{ccc|ccc} 1 & 3/2 & -1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 1/2 & 0 & 1/2 & 0 \\ 4 & 4 & -3 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R_3 - 4R_1}
 \end{aligned}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3/2 & -1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 1/2 & 0 & 1/2 & 0 \\ 0 & -2 & -1 & -2 & 0 & 1 \end{array} \right] \xrightarrow{R_3+2R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3/2 & -1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{array} \right]$$

At this point we see that the left-hand side cannot be turned into I through elementary row operations. We conclude that A^{-1} does not exist.

Remark 2. Recall that a square matrix whose reduced row-echelon form is the identity matrix is called *nonsingular*. (Definition ??) According to Corollary ??, a matrix is invertible if and only if it is nonsingular. For this reason many linear algebra texts use the terms *invertible* and *nonsingular* as synonyms.

Inverse of a 2×2 Matrix

We will conclude this section by discussing the inverse of a nonsingular 2×2 matrix. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a nonsingular matrix. We can find A^{-1} by using the row reduction method described above, that is, by computing the reduced row-echelon form of $[A|I]$. Row reduction yields the following:

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array} \right] \rightsquigarrow \left[\begin{array}{cc|cc} 1 & 0 & d/(ad-bc) & -b/(ad-bc) \\ 0 & 1 & -c/(ad-bc) & a/(ad-bc) \end{array} \right]$$

Note that the denominator of each term in the inverse matrix is the same. Factoring it out, gives us the following formula for A^{-1} .

Formula 1.

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Clearly, the expression for A^{-1} is defined, if and only if $ad - bc \neq 0$. So, what happens when $ad - bc = 0$? In Practice Problem ?? you will be asked to fill in the steps of the row reduction procedure that produces this formula, and show that if $ad - bc = 0$ then A does not have an inverse.

Practice Problems

Problem 24 Verify that the matrix $\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$ is its own inverse.

Problem 25 Use the row-reduction method for computing matrix inverses to explain why the given matrix does not have an inverse.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$$

Problem 26 Let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ -3 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1/12 & 1/6 & -1/4 \\ -1/4 & 1/2 & 1/4 \\ 5/12 & 1/6 & 1/4 \end{bmatrix}$$

Are A and B inverses of each other?

Multiple Choice:

1. Yes, A and B are inverses.
2. No, A and B are not inverses.

Problem 27 Use the following invertible matrices to illustrate Properties ??, ??, ?? and ?? of Theorem ??.

$$A = \begin{bmatrix} 2 & 0 \\ 4 & -3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 \\ 1 & 5 \end{bmatrix}$$

Problem 28 In Example ?? we found the inverse of

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

to be

$$A^{-1} = \begin{bmatrix} 2 & -5/2 & 3/2 \\ -1 & 3/2 & -1/2 \\ -1 & 2 & -1 \end{bmatrix}$$

Use A^{-1} to solve the equation

$$A\mathbf{x} = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$$

Answer:

$$\mathbf{x} = \begin{bmatrix} \dots \dots \\ \dots \dots \\ \dots \dots \end{bmatrix}$$

Problem 29 Find the inverse of each matrix by using the row-reduction procedure.

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 0 & 0 \\ 1 & -2 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{bmatrix}$$

Problem 30 Use Formula ?? to find the inverse of

$$A = \begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix}$$

Use A^{-1} to solve the equation

$$A\mathbf{x} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

Answer:

$$\mathbf{x} = \begin{bmatrix} \dots \\ \dots \end{bmatrix}$$

Problem 31 Find the value of x for which the matrix is not invertible.

$$\begin{bmatrix} 1 & 2 \\ 4 & x \end{bmatrix}$$

Answer: $x = \dots$

Problem 32 Suppose $AB = AC$ and A is an invertible $n \times n$ matrix. Does it follow that $B = C$? Explain why or why not.

Problem 33 Suppose $AB = AC$ and A is a non-invertible $n \times n$ matrix. Does it follow that $B = C$? Explain why or why not.

Problem 34 Suppose A and B are invertible $n \times n$ matrices. Does it follow that $(A + B)$ is invertible?