
Week 6

June 5, 2026

Contents

1 $\mathbb{R}^n$ and Subspaces of $\mathbb{R}^n$

We are familiar with two operations that can be applied to vectors in $\mathbb{R}^n$, namely, addition and scalar multiplication. We learned that addition and scalar multiplication satisfy many nice properties (see Theorem ??). These properties give $\mathbb{R}^n$ an algebraic structure. We begin this section by introducing another property, called *closure*. Adding closure to the properties we studied earlier allows us to show that $\mathbb{R}^n$ satisfies all of the properties of a *vector space*.

Closure

Definition 1. A set V is said to be *closed under addition* if for each element $\mathbf{u} \in V$ and $\mathbf{v} \in V$ the sum $\mathbf{u} + \mathbf{v}$ is also in V .

Definition 2. A set V is said to be *closed under scalar multiplication* if for each element $\mathbf{v} \in V$ and for each scalar $k \in \mathbb{R}$ the product $k\mathbf{v}$ is also in V .

Example 1. Let E be the set of positive even integers. Then E is closed under addition, because the sum of two even integers is again an even integer.

Example 2. Let D be the set of positive odd integers. Then D is not closed under addition, for the sum of two odd integers need not be an odd integer (in fact, it will always be even).

Example 3. Let X_1 be defined as follows.

$$X_1 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0 \right\}$$

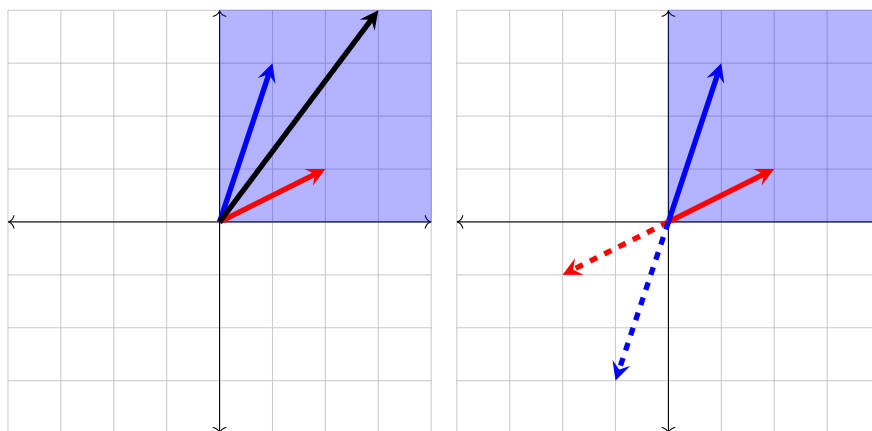
In other words, X_1 is the set of vectors in $\mathbb{R}^2$ in the first quadrant (or on an axis on the boundary of the first quadrant). Show that X_1 is closed under addition, but X_1 is not closed under scalar multiplication.

Explanation. Suppose $\mathbf{v}_1 = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} c \\ d \end{bmatrix}$ are in X_1 . This means that $a, b, c, d \geq 0$. But then we have

$a + c, b + d \geq 0$. Therefore $\mathbf{v}_1 + \mathbf{v}_2 = \begin{bmatrix} a + c \\ b + d \end{bmatrix}$ is also in X_1 . We conclude that X_1 is closed under addition.

X_1 is not closed under scalar multiplication because $(-1)\mathbf{v}_1 = \begin{bmatrix} -a \\ -b \end{bmatrix}$ is not in X_1 .

The figure below helps us see that the sum of any two vectors in X_1 also lies in X_1 , but any negative scalar multiple of a vector in X_1 does not lie in X_1 .



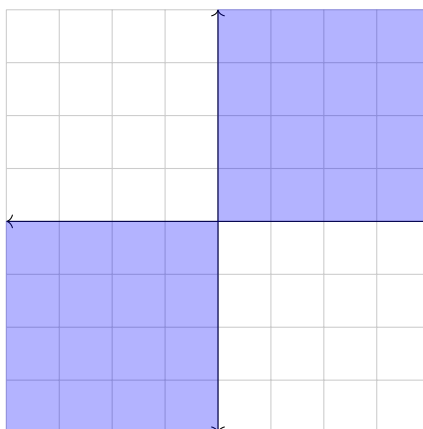
Example 4. Let X_1 be as in Example ?? . Let

$$X_3 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 : x_1 \leq 0, x_2 \leq 0 \right\}$$

In other words, X_3 is the set of vectors in $\mathbb{R}^2$ in the third quadrant (or on an axis on the boundary of the third quadrant). Now let

$$Y = X_1 \cup X_3$$

In other words, Y is the set of vectors in $\mathbb{R}^2$ that are either in the first quadrant, the third quadrant, or lie along one of the axes, as shown below.



Then Y is closed under scalar multiplication, but Y is not closed under addition.

$\mathbb{R}^n$ as a Vector Space

In Theorem ?? we learned that vector addition and scalar multiplication in $\mathbb{R}^n$ satisfy the following eight properties:

For all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, and scalars $k, p \in \mathbb{R}$,

1. Commutative Property of Addition: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. Associative Property of Addition: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. Existence of Additive Identity: $\mathbf{u} + \mathbf{0} = \mathbf{u}$
4. Existence of Additive Inverse: $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5. Distributive Property over Vector Addition: $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
6. Distributive Property over Scalar Addition: $(k + p)\mathbf{u} = k\mathbf{u} + p\mathbf{u}$
7. Associative Property for Scalar Multiplication: $k(p\mathbf{u}) = (kp)\mathbf{u}$
8. Multiplication by 1: $1\mathbf{u} = \mathbf{u}$

In addition, observe that

$\mathbb{R}^n$ is closed under addition (Why?)

$\mathbb{R}^n$ is closed under scalar multiplication (Why?)

The eight properties of vector operations, together with closure, constitute the criteria for a set with two operations to be considered a *vector space*. So, $\mathbb{R}^n$ is a vector space.

We will encounter other vector spaces later. Any vector space must be closed under both of its operations, and must satisfy the other eight properties in the list above. We will see (in [Abstract Vector Spaces](#), for instance) that a wide variety of sets, with a wide variety of operations, are vector spaces (one important reason to study linear algebra). As we shall see, these sets and their operations may look very different, but the behavior of the elements under the two operations makes them vector spaces. For now, we simply focus on $\mathbb{R}^n$.

Subspaces of $\mathbb{R}^n$

Now that we understand what it means for a set to be closed under addition and scalar multiplication, we are ready for the main definition.

Definition 3. Suppose that V is a nonempty subset of $\mathbb{R}^n$ that is closed under addition and closed under scalar multiplication. Then V is a *subspace* of $\mathbb{R}^n$.

We use the term *subspace* because it turns out that any subset of $\mathbb{R}^n$ closed under both addition and scalar multiplication is also a vector space. In other words, by inheriting vector addition and scalar multiplication from $\mathbb{R}^n$, and satisfying the properties of closure, a subset of $\mathbb{R}^n$ will automatically satisfy all vector space properties. We will prove this in Theorem ?? of [Abstract Vector Spaces](#).

Example 5. Let V be the set of vectors in $\mathbb{R}^3$ on the y -axis. Then V is a subspace of $\mathbb{R}^3$.

Explanation. To verify this, note that any vector in V is of the form $\begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix}$. If we multiply such a vector by a scalar c ,

we get a vector of the form $\begin{bmatrix} 0 \\ cy \\ 0 \end{bmatrix}$, which is clearly still in V . This proves V is closed under scalar multiplication. Next,

note that $\begin{bmatrix} 0 \\ y_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ y_1 + y_2 \\ 0 \end{bmatrix}$, so V is closed under addition.

Recall that the *span* of a set of vectors is the set of all linear combinations of those vectors (see Definition ??). It is easy to see from this definition, that the span of any set of vectors in $\mathbb{R}^n$ must be closed under both addition and scalar multiplication, and therefore the span of those vectors is a subspace of $\mathbb{R}^n$. This argument proves the following result, giving us an abundance of examples of subspaces:

Theorem 1. Let S be any set of vectors in $\mathbb{R}^n$. Then $\text{span}(S)$ is a subspace of $\mathbb{R}^n$.

In particular, if we take S to be the single vector $\mathbf{v}$, we have that $\text{span}(\mathbf{v})$ is a subspace of $\mathbb{R}^n$. Geometrically, this subspace is a line with a direction vector $\mathbf{v}$. Similarly, the span of two vectors is a subspace of $\mathbb{R}^n$. If the two vectors are linearly independent, then the subspace is a plane in $\mathbb{R}^n$.

Not every line or plane in $\mathbb{R}^n$ is a subspace, however. The following important result provides us with a quick way to determine that some subsets are not subspaces.

Theorem 2. If V is a subspace of $\mathbb{R}^n$, then the zero vector $\mathbf{0}$ is in V .

Proof Take any vector $\mathbf{v}$ in V , and note that $0\mathbf{v} = \mathbf{0}$ is in V because V is closed under scalar multiplication. ■

Theorem ?? shows that the only lines in $\mathbb{R}^n$ that are subspaces are those that pass through the origin. The same holds true for planes and hyperplanes. For example, the plane $z = 3$ in $\mathbb{R}^3$ is not a subspace of $\mathbb{R}^3$, while any plane containing the origin is a subspace.

Theorem 3. If V is a subspace of $\mathbb{R}^n$, then for any vector $\mathbf{v} \in V$, the opposite vector, $-\mathbf{v}$, is also in V .

The proof is similar to what was done for the previous theorem and is left as an exercise.

Practice Problems

Problem 1 Let Y^+ be the set of all vectors in $\mathbb{R}^2$ whose y components are non-negative. Is Y^+ closed under vector addition?

Multiple Choice:

1. Yes
2. No

Problem 2 Let Y^+ be the set of all vectors in $\mathbb{R}^2$ whose y components are non-negative. Is Y^+ closed under scalar multiplication?

Multiple Choice:

1. Yes

2. No

Problem 3 Let X be the set of all vectors in $\mathbb{R}^3$ that lie on either the x -axis, the y -axis, or the z -axis. Is X closed under vector addition?

Multiple Choice:

1. Yes

2. No

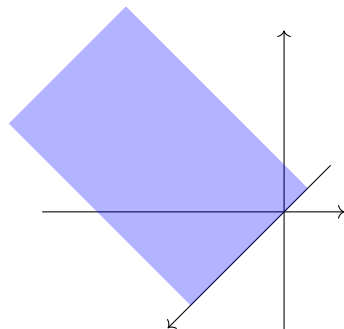
Problem 4 Let X be the set of all vectors in $\mathbb{R}^3$ that lie on either the x -axis, the y -axis, or the z -axis. Is X closed under scalar multiplication?

Multiple Choice:

1. Yes

2. No

Problem 5 Determine whether the set V of vectors shown in the figure is closed under vector addition and scalar multiplication. Justify your responses. V consists of all vectors in $\mathbb{R}^3$ in a slanted half-plane which has the x -axis as a boundary.



Is V closed under scalar multiplication?

Multiple Choice:

1. Yes

2. No

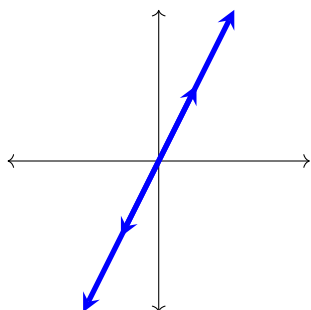
Is V closed under addition?

Multiple Choice:

1. Yes

2. No

Problem 6 Determine whether the set V of vectors shown in the figure is closed under vector addition and scalar multiplication. Justify your responses. V consists of all vectors along the line, as shown.



Is V closed under scalar multiplication?

Multiple Choice:

1. Yes
2. No

Is V closed under addition?

Multiple Choice:

1. Yes
2. No

Problem 7 Prove that if V is a subspace of $\mathbb{R}^n$, then for any vector $\mathbf{v} \in V$, the opposite vector, $-\mathbf{v}$, is also in V . (Theorem ??)

Problem 8 Let A be an $m \times n$ matrix. Let V be the subset of $\mathbb{R}^n$ consisting of all vectors $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$. Prove that V is a subspace of $\mathbb{R}^n$. (This subspace is called the null space of the matrix A . We will denote it $\text{null}(A)$.)

2 Introduction to Bases

Coordinate Vectors

When we first introduced vectors we learned to represent them using component notation. If $\mathbf{u} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ we know that the head of $\mathbf{u}$ is located at the point (2, 5). But there is another way to look at the component form. Observe that $\mathbf{u}$ can be expressed as a linear combination of the standard unit vectors $\mathbf{i}$ and $\mathbf{j}$:

$$\mathbf{u} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2\mathbf{i} + 5\mathbf{j}$$

In fact, any vector $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ of $\mathbb{R}^2$ can be written as a linear combination of $\mathbf{i}$ and $\mathbf{j}$:

$$\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix} = a\mathbf{i} + b\mathbf{j}$$

This gives us an alternative way of interpreting the component notation:

$$\begin{bmatrix} a \\ b \end{bmatrix} \begin{array}{l} \longleftarrow \text{coefficient in front of } \mathbf{i} \\ \longleftarrow \text{coefficient in front of } \mathbf{j} \end{array}$$

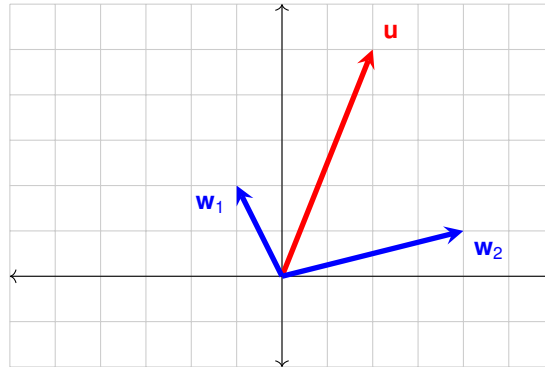
We say that a and b are *coordinates* of $\mathbf{v}$ with respect to $\{\mathbf{i}, \mathbf{j}\}$, and $\begin{bmatrix} a \\ b \end{bmatrix}$ is said to be the *coordinate vector* for $\mathbf{v}$ with respect to $\{\mathbf{i}, \mathbf{j}\}$. Every vector $\mathbf{v}$ of $\mathbb{R}^2$ can be thus represented using $\mathbf{i}$ and $\mathbf{j}$. Moreover, such representation in terms of $\mathbf{i}$ and $\mathbf{j}$ is unique for each vector, meaning that we will never have two different coordinate vectors representing the same vector. We will refer to $\{\mathbf{i}, \mathbf{j}\}$ as a *basis* of $\mathbb{R}^2$.

The order in which the basis elements are written matters. For example, $\mathbf{u}$ is represented by the coordinate vector $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ with respect to $\{\mathbf{i}, \mathbf{j}\}$, but changing the basis to $\{\mathbf{j}, \mathbf{i}\}$ would change the coordinate vector to $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$.

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix} \begin{array}{l} \longleftarrow \text{coefficient in front of the first basis element} \\ \longleftarrow \text{coefficient in front of the second basis element} \end{array}$$

Clearly, standard unit vectors $\mathbf{i}$ and $\mathbf{j}$ are very convenient, but other vectors can also be used in place of $\mathbf{i}$ and $\mathbf{j}$ to represent $\mathbf{u}$.

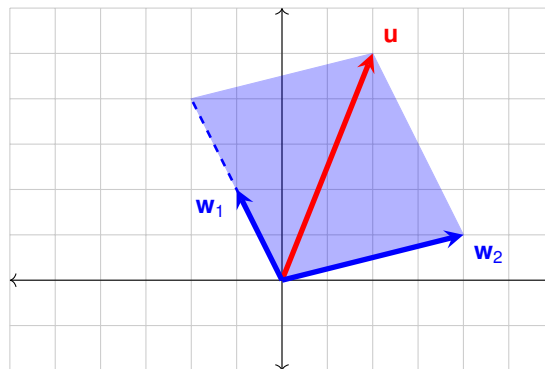
Exploration 9 The diagram below shows $\mathbf{u}$ together with vectors $\mathbf{w}_1$ and $\mathbf{w}_2$.



It is easy to see that

$$\mathbf{u} = 2\mathbf{w}_1 + \mathbf{w}_2$$

as shown below.



If we declare $\{\mathbf{w}_1, \mathbf{w}_2\}$ to be a basis of $\mathbb{R}^2$, then we can say that the coordinate vector for $\mathbf{u}$ with respect to $\{\mathbf{w}_1, \mathbf{w}_2\}$ is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

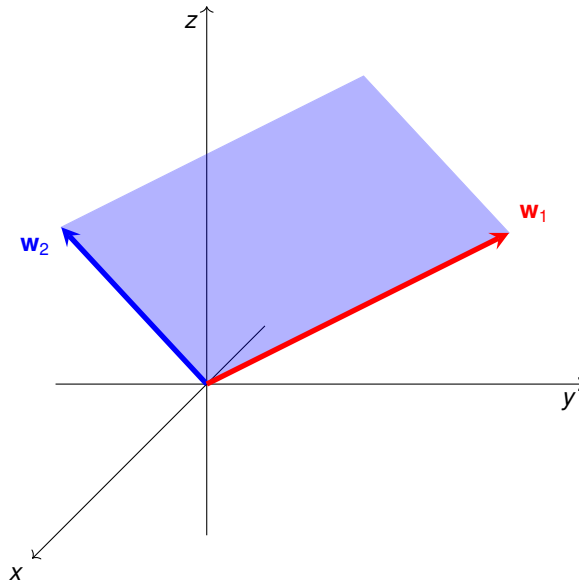
$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{array}{l} \leftarrow \text{coefficient in front of the first basis element} \\ \leftarrow \text{coefficient in front of the second basis element} \end{array}$$

What Constitutes a Basis?

In the previous section we had used the term *basis* without defining it. Now is the time to pause and think about what we *want* a basis to do. Let's focus on $\mathbb{R}^n$ and subspaces of $\mathbb{R}^n$. What we establish here will easily generalize to other vector spaces.

Based on our previous discussion, given any vector $\mathbf{v}$ of $\mathbb{R}^n$ (or a subspace V of $\mathbb{R}^n$), we want to be able to write a coordinate vector for $\mathbf{v}$ with respect to the given basis of $\mathbb{R}^n$ (or V). Based on this condition, we will require that basis vectors span $\mathbb{R}^n$ (or V).

For example, consider $\mathbf{w}_1$ and $\mathbf{w}_2$ shown below.



The set $\{\mathbf{w}_1, \mathbf{w}_2\}$ cannot be a basis for $\mathbb{R}^3$ because $\mathbf{w}_1$ and $\mathbf{w}_2$ span a plane in $\mathbb{R}^3$, and any vector not in that plane cannot be written as a linear combination of $\mathbf{w}_1$ and $\mathbf{w}_2$.

On the other hand, the plane spanned by $\mathbf{w}_1$ and $\mathbf{w}_2$ is a subspace of $\mathbb{R}^3$. Because every vector in that plane can be written as a linear combination of $\mathbf{w}_1$ and $\mathbf{w}_2$, the set $\{\mathbf{w}_1, \mathbf{w}_2\}$ can potentially be a basis for the plane, provided that the set satisfies our second requirement.

Our second requirement is that for a fixed basis of $\mathbb{R}^n$ (or V), the coordinate vector for each $\mathbf{v}$ in $\mathbb{R}^n$ (or V) should be unique. Uniqueness of representation in terms of the basis elements will play an important role in our future study of functions that map vector spaces to vector spaces.

The following theorem shows that the uniqueness requirement is equivalent to the requirement that the basis vectors be linearly independent.

Theorem 4. Suppose $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_p\}$ is a spanning set for a subspace V of $\mathbb{R}^n$. Then every element $\mathbf{v}$ of V has a unique representation as a linear combination of $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_p$ if and only if the vectors $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_p$ are linearly independent.

Proof Suppose that every $\mathbf{v}$ in V can be expressed as a unique linear combination of $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_p$. This means that $\mathbf{0}$ has a unique representation as a linear combination of $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_p$. But

$$\mathbf{0} = 0\mathbf{w}_1 + 0\mathbf{w}_2 + \dots + 0\mathbf{w}_p$$

is a representation of $\mathbf{0}$ in terms of $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_p$. Since we are assuming that such a representation is unique, we conclude that there is no other. This means that the vectors $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_p$ are linearly independent.

Conversely, suppose that vectors $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_p$ are linearly independent. An arbitrary element $\mathbf{v}$ of V can be expressed as a linear combination of $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_p$:

$$\mathbf{v} = a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + \dots + a_p\mathbf{w}_p$$

Suppose this representation is not unique. Then there may be another linear combination that is also equal to $\mathbf{v}$:

$$\mathbf{v} = b_1\mathbf{w}_1 + b_2\mathbf{w}_2 + \dots + b_p\mathbf{w}_p$$

But then

$$a_1 \mathbf{w}_1 + a_2 \mathbf{w}_2 + \dots + a_p \mathbf{w}_p = b_1 \mathbf{w}_1 + b_2 \mathbf{w}_2 + \dots + b_p \mathbf{w}_p$$

This gives us

$$(a_1 - b_1) \mathbf{w}_1 + (a_2 - b_2) \mathbf{w}_2 + \dots + (a_p - b_p) \mathbf{w}_p = \mathbf{0}$$

Because we assumed that $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_p$ are linearly independent, we must have

$$a_1 - b_1 = 0, a_2 - b_2 = 0, \dots, a_p - b_p = 0$$

so that

$$a_1 = b_1, a_2 = b_2, \dots, a_p = b_p$$

This proves the representation of $\mathbf{v}$ in terms of $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_p$ is unique. ■

Example 6. Use $V = \text{span}(S)$, where $S = \left\{ \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} \right\}$ to illustrate why a set of linearly dependent vectors cannot be used as a basis for a subspace by showing that linearly dependent vectors fail to ensure uniqueness of coordinate vectors for vectors in V .

Explanation. We will first show that the elements of S are linearly dependent. Let A be a matrix whose columns are the vectors in S .

$$A = \begin{bmatrix} 5 & 4 & -3 \\ 2 & 1 & 0 \\ 4 & 1 & 2 \end{bmatrix}$$

We find that

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore the matrix equation $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions:

$$\mathbf{x} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} t$$

This tells us that there are infinitely many nontrivial linear relations among the elements of S . Letting $t = 1$ gives us one such nontrivial relation.

$$-\begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} = \mathbf{0}$$

Now let's pick an arbitrary vector $\mathbf{v}$ in V . Any vector will do, so let

$$\mathbf{v} = \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix} + (-1)\begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$$

Based on this representation of $\mathbf{v}$, the coordinate vector for $\mathbf{v}$ with respect to $\mathcal{S}$ is

$$\begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix}$$

But

$$\begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$$

So, by substitution, we have:

$$\mathbf{v} = \left(2 \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} \right) + (-1) \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} = 0 \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix} + 1 \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$$

Based on this representation, the coordinate vector for $\mathbf{v}$ with respect to $\mathcal{S}$ is

$$\begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix}$$

The set $\mathcal{S}$ is linearly dependent. As a result, coordinate vectors for elements of V are not unique and we do not want to use $\mathcal{S}$ as a basis for V .

Definition of a Basis

Definition 4. A set $\mathcal{S}$ of vectors is called a *basis* of $\mathbb{R}^n$ (or a basis of a subspace V of $\mathbb{R}^n$) provided that

1. $\text{span}(\mathcal{S}) = \mathbb{R}^n$ (or V)
2. $\mathcal{S}$ is linearly independent.

Example 7. The standard unit vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$ are linearly independent and span $\mathbb{R}^n$. Thus $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is a basis of $\mathbb{R}^n$.

Definition 5. The set $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is called the *standard basis* of $\mathbb{R}^n$.

Bases are not unique. For example, we know that vectors $\mathbf{i}$ and $\mathbf{j}$ form the standard basis of $\mathbb{R}^2$. But, as we discussed in Example ??, vectors

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

are linearly independent vectors that span $\mathbb{R}^2$. Therefore

$$\left\{ \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\}$$

is also a basis for $\mathbb{R}^2$.

Any linearly independent spanning set in $\mathbb{R}^n$ (or a subspace of $\mathbb{R}^n$) is a basis of $\mathbb{R}^n$ (or the subspace). The plural form of the word *basis* is *bases*. It is easy to see that $\mathbb{R}^n$ and its subspaces each has infinitely many bases.

Example 8. Let $V = \text{span} \left(\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix} \right)$. The set

$$\mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix} \right\}$$

is a basis for V because the two vectors in $\mathcal{B}$ are linearly independent and span V . Find the coordinate vector for

$\mathbf{v} = \begin{bmatrix} 2 \\ -10 \\ 9 \end{bmatrix}$ with respect to $\mathcal{B}$.

Explanation. We need to express $\begin{bmatrix} 2 \\ -10 \\ 9 \end{bmatrix}$ as a linear combination of the elements of $\mathcal{B}$. To this end, we need to solve the vector equation:

$$a_1 \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} + a_2 \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -10 \\ 9 \end{bmatrix}$$

The augmented matrix and the reduced row-echelon form are:

$$\left[\begin{array}{cc|c} -2 & 2 & 2 \\ -1 & -4 & -10 \\ 3 & 1 & 9 \end{array} \right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array} \right]$$

We conclude that $a_1 = 2$, $a_2 = 3$. This gives us

$$2 \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -10 \\ 9 \end{bmatrix}$$

The coefficient in front of the first basis vector is 2, the coefficient in front of the second basis vector is 3. This means

that the coordinate vector for $\begin{bmatrix} 2 \\ -10 \\ 9 \end{bmatrix}$ with respect to $\mathcal{B}$ is $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Remark 1. It may seem strange to you that the coordinate vector for a vector in $\mathbb{R}^3$ only has two components. But remember that subspace V is a plane. When viewed as a vector in the plane, it makes sense that the coordinate

vector for $\begin{bmatrix} 2 \\ -10 \\ 9 \end{bmatrix}$ only requires two components. This issue is related to the question of dimension, which will be addressed in [Bases and Dimension](#).

Remark 2. To construct the coordinate vector for $\begin{bmatrix} 2 \\ -10 \\ 9 \end{bmatrix}$ with respect to $\mathcal{B}$, we had to be mindful of the order of the elements in $\mathcal{B}$. Ordinarily, the order of elements in a set is irrelevant, and the basis

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix} \right\}$$

is considered to be the same as

$$\left\{ \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \right\}$$

When dealing with coordinate vectors, however, the order of the elements dictates the order of the components of the coordinate vector coefficients. If we switch the order of the elements in $\mathcal{B}$, the coordinate vector becomes $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$. For this reason, when we come back to studying coordinate vectors in more detail, we will use the term *ordered basis* to avoid confusion.

Practice Problems

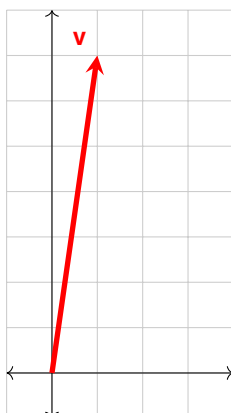
Problem 10 Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$ be a basis for $\mathbb{R}^2$. Find the coordinate vector for $\mathbf{v}$ with respect to $\mathcal{B}$.



Answer:

$$\begin{bmatrix} \dots \dots \\ \dots \dots \end{bmatrix}$$

Problem 11 Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$ be a basis for $\mathbb{R}^2$. Find the coordinate vector for $\mathbf{v}$ with respect to $\mathcal{B}$.



Answer:

$$\begin{bmatrix} \dots \dots \\ \dots \dots \end{bmatrix}$$

Problem 12 Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right\}$ be a basis for $\text{span} \left(\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right)$. Find the coordinate vector for

$$\begin{bmatrix} -4 \\ -2 \\ 2 \end{bmatrix} \text{ with respect to } \mathcal{B}.$$

Answer:

$$\begin{bmatrix} \dots \dots \\ \dots \dots \end{bmatrix}$$

Problem 13 Suppose $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{w} \right\}$ is a basis for $\mathbb{R}^3$. Find $\mathbf{w}$ if the coordinate vector for $\begin{bmatrix} -2 \\ -7 \\ 4 \end{bmatrix}$ is $\begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}$.

Answer:

$$\begin{bmatrix} \dots \dots \\ \dots \dots \\ \dots \dots \end{bmatrix}$$

Problem 14 Which of the following is a basis for $\mathbb{R}^2$?

Select All Correct Answers:

1. $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

2. $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

$$3. \left\{ \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \end{bmatrix} \right\}$$

$$4. \left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ 6 \end{bmatrix} \right\}$$

Problem 15 Which of the following is a basis for $V = \text{span} \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right)$?

Select All Correct Answers:

$$1. \left\{ \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$$

$$2. \left\{ \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix} \right\}$$

$$3. \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$4. \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$$

3 Bases and Dimension

Recall that a *basis* of a subspace V of $\mathbb{R}^n$ is a subset of V that is linearly independent and spans V . A basis allows us to uniquely express every element of V as a linear combination of the elements of the basis. Several questions may come to mind at this time. Does every subspace of $\mathbb{R}^n$ have a basis? We know that bases are not unique. If there is more than one basis, what, if anything, do they have in common?

Exploring Dimension

Exploration 16 *How would you describe*

$$V = \text{span} \left(\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ -6 \end{bmatrix} \right) ?$$

If you answered that V is a line in $\mathbb{R}^3$, you are correct. While the two vectors span the line, it is not necessary to have both of them in the spanning set to describe the line.

What is the minimum number of vectors needed to span a line?

Answer:

Observe also that the vectors in the given spanning set are not linearly independent, so they do not form a basis for V . How many vectors would a basis for V have?

Answer:

Now consider another subspace of $\mathbb{R}^3$:

$$W = \text{span} \left(\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix} \right)$$

Geometrically, W is a plane in $\mathbb{R}^3$. Note that the vectors in the spanning set are linearly independent. Can we remove one of the vectors and have the remaining vector span the plane?

What is the minimum number of vectors needed to span a plane?

Answer:

How many vectors would a basis for a plane have?

Answer:

Our observations in Exploration ?? hint at the idea of dimension. We know that a line is a one-dimensional object, a plane is a two-dimensional object, and the space we reside in is three-dimensional.

Based on our observations in Exploration ??, it makes sense for us to define dimension of a vector space (or a subspace) as the minimum number of vectors required to span the space (subspace). We can accomplish this by defining dimension as the number of elements in a basis.

We have to proceed carefully because we don't want the dimension to depend on our choice of a basis. So, before we state our definition, we need to make sure that every basis for a given vector space (or subspace) has the same number of elements.

Theorem 5. Suppose $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_t\}$ and $\mathcal{C} = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_s\}$ be two bases of $\mathbb{R}^n$ (or a subspace V of $\mathbb{R}^n$). Then $s = t$.

Proof Suppose $s \neq t$. Without loss of generality, assume that $s > t$. Because $\mathcal{B}$ spans V , every $\mathbf{w}_i$ of $\mathcal{C}$ can be written as a linear combination of elements of $\mathcal{B}$:

$$\mathbf{w}_i = a_{i1}\mathbf{v}_1 + a_{i2}\mathbf{v}_2 + \dots + a_{it}\mathbf{v}_t$$

Consider the vector equation

$$b_1\mathbf{w}_1 + b_2\mathbf{w}_2 + \dots + b_s\mathbf{w}_s = \mathbf{0} \quad (1)$$

By substitution, we have:

$$\begin{aligned} & b_1\mathbf{w}_1 + b_2\mathbf{w}_2 + \dots + b_s\mathbf{w}_s = \\ & = b_1(a_{11}\mathbf{v}_1 + a_{21}\mathbf{v}_2 + \dots + a_{t1}\mathbf{v}_t) + b_2(a_{12}\mathbf{v}_1 + a_{22}\mathbf{v}_2 + \dots + a_{t2}\mathbf{v}_t) + \dots \\ & \quad + b_s(a_{1s}\mathbf{v}_1 + a_{2s}\mathbf{v}_2 + \dots + a_{ts}\mathbf{v}_t) \\ & = (b_1a_{11} + b_2a_{12} + \dots + b_sa_{1s})\mathbf{v}_1 + (b_1a_{21} + b_2a_{22} + \dots + b_sa_{2s})\mathbf{v}_2 + \dots \\ & \quad + (b_1a_{t1} + b_2a_{t2} + \dots + b_sa_{ts})\mathbf{v}_t \\ & = \mathbf{0} \end{aligned}$$

Because $\mathbf{v}_j$'s are linearly independent, we must have

$$b_1a_{j1} + b_2a_{j2} + \dots + b_sa_{js} = 0$$

For all $1 \leq j \leq t$. This gives us a system of t equations and s unknowns. We can write the system as a matrix equation.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1s} \\ a_{21} & a_{22} & \dots & a_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{t1} & a_{t2} & \dots & a_{ts} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_s \end{bmatrix} = \mathbf{0}$$

Recall our assumption that $s > t$. By Theorem ??, we know that the system has infinitely many solutions. This shows that equation (??) has a nontrivial solution (in fact, infinitely many of them). But this shows that $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_s\}$ is linearly dependent and contradicts our assumption that $\mathcal{C}$ is a basis of V . We conclude that $s = t$. ■

Definition 6. Let V be a subspace of $\mathbb{R}^n$. The *dimension* of V is the number, m , of elements in any basis of V . We write

$$\dim(V) = m$$

Example 9. We know that vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$ form a basis of $\mathbb{R}^n$. Therefore $\dim(\mathbb{R}^n) = n$.

Remark 3. It is easy to verify that $\{\mathbf{0}\}$ is a subspace of $\mathbb{R}^n$. Because $\{\mathbf{0}\}$ is linearly dependent, it does not have a basis. We define $\dim\{\mathbf{0}\} = 0$.

The following section will guarantee that dimension is defined for every subspace of $\mathbb{R}^n$.

Every Subspace of $\mathbb{R}^n$ has a Basis

Lemma 1. If a linearly independent subset of $\mathbb{R}^n$ contains m vectors, then $m \leq n$.

Proof See Practice Problem ??.

Lemma 2. Let $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ be a linearly independent subset of $\mathbb{R}^n$. If $\mathbf{u}$ is not in $\text{span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$, then $\{\mathbf{u}, \mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly independent.

Proof Consider the equation

$$a\mathbf{u} + a_1\mathbf{v}_1 + \dots + a_k\mathbf{v}_k = \mathbf{0} \quad (2)$$

We need to show that $a = a_1 = \dots = a_k = 0$. Suppose $a \neq 0$, then $\mathbf{u} = \frac{-a_1}{a}\mathbf{v}_1 + \dots + \frac{-a_k}{a}\mathbf{v}_k$. But this contradicts the assumption that $\mathbf{u}$ is not in the span of $\mathbf{v}_1, \dots, \mathbf{v}_k$. So, $a = 0$. But $a_1 = \dots = a_k = 0$ because $\mathbf{v}_1, \dots, \mathbf{v}_k$ are linearly independent.

This means that (??) has only the trivial solution, and $\{\mathbf{u}, \mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly independent.

Theorem 6. Let V be a subspace of $\mathbb{R}^n$. Any linearly independent subset of V can be expanded to a basis of V .

Proof Suppose that $X = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a linearly independent subset of V . If $\text{span}(X) = V$ then X is already a basis of V . If $\text{span}(X) \neq V$, choose $\mathbf{u}_1$ in V such that $\mathbf{u}_1$ is not in $\text{span}(X)$. The set $\{\mathbf{u}_1, \mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly independent by Lemma ??.

If $\text{span}(\mathbf{u}_1, \mathbf{v}_1, \dots, \mathbf{v}_k) = V$ we are done; otherwise choose $\mathbf{u}_2 \in V$ such that $\mathbf{u}_2$ is not in $\text{span}(\mathbf{u}_1, \mathbf{v}_1, \dots, \mathbf{v}_k)$. Then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly independent, and the process continues. We claim that a basis of V will be reached eventually. If no basis of V is ever reached, the process creates arbitrarily large independent sets in $\mathbb{R}^n$. But this is impossible by Lemma ??.

Practice Problems

Problem 17 Find $\dim(\text{span}(S))$.

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Answer: $\dim(\text{span}(S)) = \dots$

Problem 18 Find $\dim(\text{span}(S))$.

$$S = \left\{ \begin{bmatrix} 3 \\ -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \\ -2 \\ 3 \end{bmatrix} \right\}$$

Answer: $\dim(\text{span}(S)) = \dots$

Problem 19 Find $\dim(\text{span}(S))$.

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix} \right\}$$

Answer: $\dim(\text{span}(S)) = \dots$

Problem 20 Prove that if a linearly independent subset of $\mathbb{R}^n$ contains m vectors, then $m \leq n$. (Lemma ??)

Exercise Source

Practice Problem ?? is adopted from Keith Nicholson's Example 5.2.12 *Linear Algebra with Applications*. (CC-BY-NC-SA)

W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2018, Open Edition, p 277.