
Week 7

June 5, 2026

Contents

1 Subspaces of $\mathbb{R}^n$ Associated with Matrices

Row Space of a Matrix

Recall that in [Gaussian Elimination and Rank](#), we claimed that every row-echelon form of a given matrix has the same number of nonzero rows. This result suggests that there are certain characteristics associated with the rows of a matrix that are not affected by elementary row operations. We are now in the position to examine this question and to supply the proof we omitted earlier.

Definition 1. Let A be an $m \times n$ matrix. The *row space* of A , denoted by $\text{row}(A)$, is the subspace of $\mathbb{R}^n$ spanned by the rows of A .

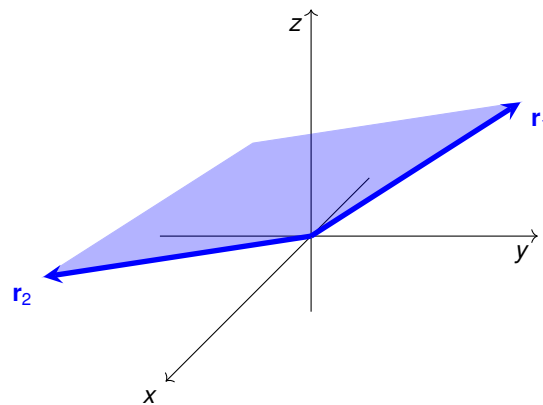
Exploration 1 Consider the matrix

$$A = \begin{bmatrix} -2 & 2 & 1 \\ 4 & -2 & 1 \end{bmatrix}$$

Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the rows of A :

$$\mathbf{r}_1 = \begin{bmatrix} -2 & 2 & 1 \end{bmatrix}, \quad \mathbf{r}_2 = \begin{bmatrix} 4 & -2 & 1 \end{bmatrix}$$

Then $\text{row}(A) = \text{span}(\mathbf{r}_1, \mathbf{r}_2)$ is a plane through the origin containing $\mathbf{r}_1$ and $\mathbf{r}_2$.



We will use elementary row operations to reduce A to $\text{rref}(A)$.

$$\begin{bmatrix} -2 & 2 & 1 \\ 4 & -2 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3/2 \end{bmatrix}$$

Let ρ_1 and ρ_2 be the rows of $\text{rref}(A)$:

$$\rho_1 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, \quad \rho_2 = \begin{bmatrix} 0 & 1 & 3/2 \end{bmatrix}$$

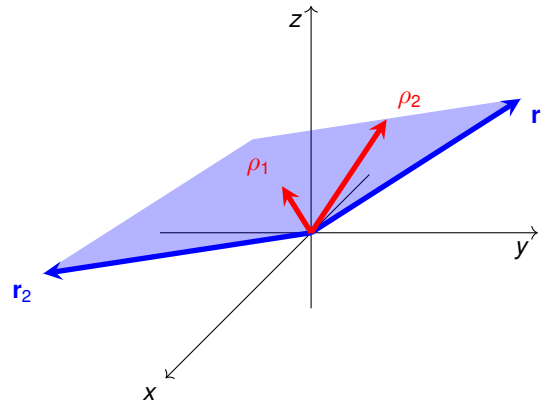
What do you think $\text{span}(\rho_1, \rho_2)$ looks like?

The following video will help us visualize $\text{span}(\rho_1, \rho_2)$ and compare it to $\text{span}(\mathbf{r}_1, \mathbf{r}_2)$.

YouTube link: <https://www.youtube.com/watch?v=6wz-5G14jo8>

Based on what we observed in the video, we may conjecture that

$$\text{span}(\rho_1, \rho_2) = \text{span}(\mathbf{r}_1, \mathbf{r}_2)$$



But why does this make sense? Vectors ρ_1 and ρ_2 were obtained from $\mathbf{r}_1$ and $\mathbf{r}_2$ by repeated applications of elementary row operations. At every stage of the row reduction process, the rows of the matrix are linear combinations of $\mathbf{r}_1$ and $\mathbf{r}_2$. Thus, at every stage of the row reduction process, the rows of the matrix lie in the span of $\mathbf{r}_1$ and $\mathbf{r}_2$. Our next video shows a step-by-step row reduction process accompanied by sketches of vectors.

YouTube link: <https://www.youtube.com/watch?v=KpoewUQ3wkY>

Exploration ?? makes a convincing case for the following theorem.

Theorem 1. If matrix B was obtained from matrix A by applying an elementary row operation to A then

$$\text{row}(B) = \text{row}(A)$$

Proof Let $\mathbf{r}_1, \dots, \mathbf{r}_m$ be the rows of A .

There are three elementary row operations. Clearly, switching the order of vectors in $\text{span}(\mathbf{r}_1, \dots, \mathbf{r}_m)$ will not affect the span.

Suppose that B was obtained from A by multiplying the i^{th} row of A by a non-zero constant k . We need to show that

$$\text{span}(\mathbf{r}_1, \dots, k\mathbf{r}_i, \dots, \mathbf{r}_m) = \text{span}(\mathbf{r}_1, \dots, \mathbf{r}_i, \dots, \mathbf{r}_m)$$

To do this we will assume that some vector $\mathbf{v}$ is in $\text{span}(\mathbf{r}_1, \dots, k\mathbf{r}_i, \dots, \mathbf{r}_m)$, and show that $\mathbf{v}$ is in $\text{span}(\mathbf{r}_1, \dots, \mathbf{r}_i, \dots, \mathbf{r}_m)$. We will then assume that some vector $\mathbf{w}$ is in $\text{span}(\mathbf{r}_1, \dots, \mathbf{r}_i, \dots, \mathbf{r}_m)$ and show that $\mathbf{w}$ must be in $\text{span}(\mathbf{r}_1, \dots, k\mathbf{r}_i, \dots, \mathbf{r}_m)$.

Suppose that $\mathbf{v}$ is in $\text{span}(\mathbf{r}_1, \dots, k\mathbf{r}_i, \dots, \mathbf{r}_m)$. Then

$$\mathbf{v} = a_1\mathbf{r}_1 + \dots + a_i(k\mathbf{r}_i) + \dots + a_m\mathbf{r}_m$$

But then

$$\mathbf{v} = a_1\mathbf{r}_1 + \dots + (a_ik)\mathbf{r}_i + \dots + a_m\mathbf{r}_m$$

So $\mathbf{v}$ is in $\text{span}(\mathbf{r}_1, \dots, \mathbf{r}_i, \dots, \mathbf{r}_m)$.

Now suppose $\mathbf{w}$ is in $\text{span}(\mathbf{r}_1, \dots, \mathbf{r}_i, \dots, \mathbf{r}_m)$, then

$$\mathbf{w} = b_1 \mathbf{r}_1 + \dots + b_i \mathbf{r}_i + \dots + b_m \mathbf{r}_m$$

But because $k \neq 0$, we can do the following:

$$\mathbf{w} = b_1 \mathbf{r}_1 + \dots + \frac{b_i}{k}(k\mathbf{r}_i) + \dots + b_m \mathbf{r}_m$$

So $\mathbf{w}$ is in $\text{span}(\mathbf{r}_1, \dots, k\mathbf{r}_i, \dots, \mathbf{r}_m)$.

We leave it to the reader to verify that adding a multiple of one row of A to another does not change the row space. ■

Corollary 1. If matrix B was obtained from matrix A by applying a sequence of elementary row operations to A then

$$\text{row}(B) = \text{row}(A)$$

Proof This follows from repeated applications of Theorem ??.

Corollary 2.

$$\text{row}(A) = \text{row}(\text{rref}(A))$$

Example 1. Let

$$A = \begin{bmatrix} 2 & -1 & 1 & -4 & 1 \\ 1 & 0 & 3 & 3 & 0 \\ -2 & 1 & -1 & 5 & 2 \\ 4 & -1 & 7 & 2 & 1 \end{bmatrix}$$

Find two distinct bases for $\text{row}(A)$.

Explanation. By Corollary ?? a basis for $\text{row}(\text{rref}(A))$ will also be a basis for $\text{row}(A)$. Row reduction gives us:

$$\begin{bmatrix} 2 & -1 & 1 & -4 & 1 \\ 1 & 0 & 3 & 3 & 0 \\ -2 & 1 & -1 & 5 & 2 \\ 4 & -1 & 7 & 2 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 3 & 0 & -9 \\ 0 & 1 & 5 & 0 & -31 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \text{rref}(A)$$

Since the zero row contributes nothing to the span, we conclude that the nonzero rows of $\text{rref}(A)$ span $\text{row}(\text{rref}(A))$. Therefore

$$\text{row}(A) = \text{span}\left(\begin{bmatrix} 1 & 0 & 3 & 0 & -9 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 5 & 0 & -31 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 3 \end{bmatrix}\right)$$

By Problem ??, the nonzero rows of $\text{rref}(A)$ are linearly independent. It follows that the nonzero rows of $\text{rref}(A)$ form a basis for $\text{row}(A)$.

To find a second basis for $\text{row}(A)$, observe that by Corollary ?? the row space of *any* row-echelon form of A will be equal to $\text{row}(A)$. Matrix A has many row-echelon forms. Here is one of them:

$$B = \begin{bmatrix} 1 & 0 & 3 & 3 & 0 \\ 0 & -1 & -5 & -10 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The nonzero rows of B span $\text{row}(A)$. By Theorem ??, the nonzero rows of B are linearly independent. Thus the nonzero rows of B form a basis for $\text{row}(A)$.

Our observations in Example ?? can be generalized to all matrices. Given any matrix A ,

1. The nonzero rows of $\text{rref}(A)$ are linearly independent (Why?) and span $\text{row}(A)$ (Corollary ??).
2. The nonzero rows of any row-echelon form of A are linearly independent (Why?) and span $\text{row}(A)$ (Corollary ??).

Therefore nonzero rows of $\text{rref}(A)$ or the nonzero rows of any row-echelon form of A constitute a basis of $\text{row}(A)$. Since all bases for $\text{row}(A)$ must have the same number of elements (Theorem ??), we have just proved the following theorem.

Theorem 2. All row-echelon forms of a given matrix have the same number of nonzero rows.

This result was first introduced without proof in [Gaussian Elimination and Rank](#) where we used it to define the *rank* of a matrix as the number of nonzero rows in its row-echelon forms. We can now update the definition of rank as follows.

Definition 2. For any matrix A ,

$$\text{rank}(A) = \dim(\text{row}(A))$$

Column Space of a Matrix

Definition 3. Let A be an $m \times n$ matrix. The *column space* of A , denoted by $\text{col}(A)$, is the subspace of $\mathbb{R}^m$ spanned by the columns of A .

Exploration 2 Let

$$B = \begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & -1 & 2 & 2 \\ 1 & 3 & -2 & -3 \end{bmatrix}$$

Our goal is to find a basis for $\text{col}(B)$. To do this we need to find a linearly independent subset of the columns of B that spans $\text{col}(B)$.

Consider the linear relation:

$$a_1 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix} + a_3 \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix} + a_4 \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = \mathbf{0} \quad (1)$$

Solving this equation amounts to finding $\text{rref}(B)$.

$$\begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & -1 & 2 & 2 \\ 1 & 3 & -2 & -3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \text{rref}(B)$$

We see that (??) has infinitely many solutions. This means that the columns of B are linearly dependent, and contain a redundant vector. Which vector can we remove from the set without changing the span?

Choosing a non-zero value for the free variable a_3 , we can express the third column as a linear combination of the other columns. We conclude that

$$\text{col}(B) = \text{span} \left(\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \right) = \text{span} \left(\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \right)$$

Removing the third column also gives us

$$\begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & -1 & 2 & 2 \\ 1 & 3 & -2 & -3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This implies that the remaining columns are linearly independent, and the set

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \right\}$$

is a basis for $\text{col}(B)$.

The approach we took to find a basis for $\text{col}(B)$ in Exploration ?? uses the reduced row-echelon form of B . It is true, however, that any row-echelon form of B could have been used in place of $\text{rref}(B)$. (Why?). We generalize the steps as follows:

Procedure 1. Given a matrix B , a basis for $\text{col}(B)$ can be found as follows:

1. Find $\text{rref}(B)$ (or any row-echelon form B' of B .)
2. Identify the pivot columns of $\text{rref}(B)$ (or B').
3. The columns of B corresponding to the pivot columns of $\text{rref}(B)$ (or B') form a basis for $\text{col}(B)$.

Proof Let $\mathbf{b}_1, \dots, \mathbf{b}_n$ be the columns of B , and let $\mathbf{b}'_1, \dots, \mathbf{b}'_n$ be the columns of $\text{rref}(B)$ (or B'). Observe that the equations

$$a_1 \mathbf{b}_1 + \dots + a_n \mathbf{b}_n = \mathbf{0} \tag{2}$$

$$a_1 \mathbf{b}'_1 + \dots + a_n \mathbf{b}'_n = \mathbf{0} \tag{3}$$

have the same solution set. This means that any non-trivial relation among the columns of $\text{rref}(B)$ (or B') translates into a non-trivial relation among the columns of B . Likewise, any collection of linearly independent columns of $\text{rref}(B)$ (or B') corresponds to linearly independent columns of B .

By Problem ??, the pivot columns of $\text{rref}(B)$ (or B') are linearly independent. Therefore the corresponding columns of B are linearly independent. Non-pivot columns can be expressed as linear combinations of the pivot columns, therefore they contribute nothing to the span and can be removed from the spanning set. (See Problem ??) ■

The proof of Procedure ?? shows that the number of basis elements for the column space of a matrix is equal to the number of pivot columns. But the number of pivot columns is the same as the number of pivots in a row-echelon form, which is equal to the number of nonzero rows and the rank of the matrix. This gives us the following important result.

Theorem 3. Let A be a matrix.

$$\dim(\text{row}(A)) = \text{rank}(A) = \dim(\text{col}(A))$$

Example 2. We will return to matrix A of Example ?? and find a basis for $\text{col}(A)$.

Explanation. We begin by finding $\text{rref}(A)$.

$$\begin{bmatrix} 2 & -1 & 1 & -4 & 1 \\ 1 & 0 & 3 & 3 & 0 \\ -2 & 1 & -1 & 5 & 2 \\ 4 & -1 & 7 & 2 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 3 & 0 & -9 \\ 0 & 1 & 5 & 0 & -31 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \text{rref}(A)$$

Columns 1, 2 and 4 of $\text{rref}(A)$ contain leading 1's. Therefore columns 1, 2 and 4 of A form a basis for $\text{col}(A)$.

The Null Space

Definition 4. Let A be an $m \times n$ matrix. The *null space* of A , denoted by $\text{null}(A)$, is the set of all vectors $\mathbf{x}$ in $\mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{0}$.

Example 3. Find $\text{null}(A)$ if

$$A = \begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}$$

Explanation. We need to solve the equation $A\mathbf{x} = \mathbf{0}$. Row reduction gives us

$$\begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1/3 \\ 0 & 0 \end{bmatrix} = \text{rref}(A)$$

We conclude that $\mathbf{x} = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix} t$. Thus $\text{null}(A)$ consists of all vectors of the form $\begin{bmatrix} 1/3 \\ 1 \end{bmatrix} t$. We might write

$$\text{null}(A) = \left\{ \begin{bmatrix} 1/3 \\ 1 \end{bmatrix} t \right\}$$

or

$$\text{null}(A) = \text{span} \left(\begin{bmatrix} 1/3 \\ 1 \end{bmatrix} \right)$$

Example ?? allows us to make an important observation. Note that every scalar multiple of $\begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$ is contained in $\text{null}(A)$. This means that $\text{null}(A)$ is closed under vector addition and scalar multiplication. Recall that this property makes $\text{null}(A)$ a *subspace* of $\mathbb{R}^n$. This result was first presented as Practice Problem ?. We now formalize it as a theorem.

Theorem 4. Let A be an $m \times n$ matrix. Then $\text{null}(A)$ is a subspace of $\mathbb{R}^n$.

Proof To show that $\text{null}(A)$ is closed under vector addition and scalar multiplication we will show that a linear combination of any two elements of $\text{null}(A)$ is contained in $\text{null}(A)$.

Suppose $\mathbf{x}_1$ and $\mathbf{x}_2$ are in $\text{null}(A)$. Then $A\mathbf{x}_1 = \mathbf{0}$ and $A\mathbf{x}_2 = \mathbf{0}$. But then

$$A(a_1\mathbf{x}_1 + a_2\mathbf{x}_2) = a_1A\mathbf{x}_1 + a_2A\mathbf{x}_2 = \mathbf{0}$$

We conclude that $a_1\mathbf{x}_1 + a_2\mathbf{x}_2$ is also in $\text{null}(A)$. ■

Example 4. Find a basis for $\text{null}(A)$, where A is the matrix in Example ??.

Explanation. Elements in the null space of A are solutions to the equation

$$\begin{bmatrix} 2 & -1 & 1 & -4 & 1 \\ 1 & 0 & 3 & 3 & 0 \\ -2 & 1 & -1 & 5 & 2 \\ 4 & -1 & 7 & 2 & 1 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

Row reduction yields $\text{rref}(A)$

$$\begin{bmatrix} 2 & -1 & 1 & -4 & 1 \\ 1 & 0 & 3 & 3 & 0 \\ -2 & 1 & -1 & 5 & 2 \\ 4 & -1 & 7 & 2 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 3 & 0 & -9 \\ 0 & 1 & 5 & 0 & -31 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, elements of $\text{null}(A)$ are of the form

$$\mathbf{x} = \begin{bmatrix} 9t - 3s \\ 31t - 5s \\ s \\ -3t \\ t \end{bmatrix} = \begin{bmatrix} -3 \\ -5 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} 9 \\ 31 \\ 0 \\ -3 \\ 1 \end{bmatrix} t$$

Thus

$$\text{null}(A) = \text{span} \left(\begin{bmatrix} -3 \\ -5 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 9 \\ 31 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right)$$

To find a basis for $\text{null}(A)$ we need to find linearly independent vectors that span $\text{null}(A)$. Take a closer look at the vectors

$$\begin{bmatrix} -3 \\ -5 \\ \boxed{1} \\ 0 \\ \boxed{0} \end{bmatrix}, \begin{bmatrix} 9 \\ 31 \\ \boxed{0} \\ -3 \\ \boxed{1} \end{bmatrix}$$

Because of the locations of 1's and 0's, it is clear that one vector is not a scalar multiple of the other. Therefore the two vectors are linearly independent. We conclude that

$$\left\{ \begin{bmatrix} -3 \\ -5 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 9 \\ 31 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}$$

is a basis of $\text{null}(A)$, and $\dim(\text{null}(A)) = 2$.

It is not a coincidence that the steps we used in Example ?? produced linearly independent vectors, and it is worth while to try to understand why this procedure will always produce linearly independent vectors.

Take a closer look at the elements of the null space:

$$\mathbf{x} = \begin{bmatrix} 9t - 3s \\ 31t - 5s \\ \boxed{s} \\ -3t \\ \boxed{t} \end{bmatrix} = \begin{bmatrix} -3 \\ -5 \\ \boxed{1} \\ 0 \\ \boxed{0} \end{bmatrix} s + \begin{bmatrix} 9 \\ 31 \\ \boxed{0} \\ -3 \\ \boxed{1} \end{bmatrix} t$$

The parameter s in the third component of $\mathbf{x}$ produces a 1 in the third component of the first vector and a 0 in the third component of the second vector, while parameter t in the fifth component of $\mathbf{x}$ produces a 1 in the fifth component of the second vector and a 0 in the fifth component of the first vector. This makes it clear that the two vectors are linearly independent.

This pattern will hold for any number of parameters, each parameter producing a 1 in exactly one vector and 0's in the corresponding components of the other vectors.

$$\begin{bmatrix} \vdots \\ t_1 \\ \vdots \\ t_2 \\ \vdots \\ t_3 \\ \vdots \\ t_n \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \end{bmatrix} t_1 + \dots + \begin{bmatrix} \vdots \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \end{bmatrix} t_2 + \dots + \begin{bmatrix} \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ \vdots \end{bmatrix} t_3 + \dots + \begin{bmatrix} \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 1 \\ \vdots \end{bmatrix} t_n$$

Therefore, vectors obtained in this way will always be linearly independent.

Rank and Nullity Theorem

Definition 5. Let A be a matrix. The dimension of the null space of A is called the *nullity* of A .

$$\dim(\text{null}(A)) = \text{nullity}(A)$$

We know that the dimension of the row space and the dimension of the column space of a matrix are the same and are equal to the rank of the matrix (or the number of nonzero rows in any row-echelon form of the matrix).

As we observed in Example ??, the dimension of the null space of a matrix is equal to the number of free variables in the solution vector of the homogeneous system associated with the matrix. Since the number of pivots and the number of free variables add up to the number of columns in a matrix (Theorem ??) we have the following significant result.

Theorem 5. Let A be an $m \times n$ matrix. Then

$$\text{rank}(A) + \text{nullity}(A) = n$$

We will see the geometric implications of this theorem when we study linear transformations.

Practice Problems

Problem 3 Let A be a matrix. Prove that non-zero rows of $\text{rref}(A)$ are linearly independent.

Problem 4 Let A be a matrix. Prove that the pivot columns of $\text{rref}(A)$ are linearly independent.

Problem 5 Let A be a matrix. Show that the non-pivot columns of $\text{rref}(A)$ can be expressed as linear combinations of the pivot columns.

Problem 6 Let

$$A = \begin{bmatrix} 2 & 0 & 2 & 4 \\ 1 & 3 & -2 & -1 \\ -1 & -2 & 1 & 0 \end{bmatrix}$$

Question 6.1 Find $\text{rref}(A)$. (You may use technology.)

$$\text{rref}(A) = \begin{bmatrix} \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

Question 6.2 Complete the statement.

$$\text{rank}(A) = \dim(\text{row}(A)) = \dim(\text{col}(A)) = \dots$$

Question 6.3 Use $\text{rref}(A)$ and the procedure outlined in Example ?? to find a basis for $\text{row}(A)$.

Basis for $\text{row}(A)$: $\left\{ \left[\dots \quad \dots \quad \dots \quad \dots \right], \left[\dots \quad \dots \quad \dots \quad \dots \right] \right\}$

Question 6.4 Use Procedure ?? to find a basis for $\text{col}(A)$.

$$\text{Basis for } \text{col}(A) : \left\{ \begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix}, \begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix} \right\}$$

Question 6.5 Find a basis for $\text{null}(A)$.

$$\text{Basis for } \text{null}(A) : \left\{ \begin{bmatrix} \dots \\ \dots \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \dots \\ \dots \\ 0 \\ 1 \end{bmatrix} \right\}$$

Demonstrate that the Rank-Nullity Theorem (Theorem ??) holds for A .

Explain how you can quickly tell that the two vectors you selected for your basis are linearly independent.

Problem 7 Let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 3 \\ 2 & 0 & -2 \\ 1 & -2 & -5 \\ 0 & 1 & 2 \end{bmatrix}$$

Question 7.1 Find $\text{rref}(B)$. (You may use technology.)

$$\text{rref}(B) = \begin{bmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{bmatrix}$$

Question 7.2 Complete the statement.

$$\text{rank}(B) = \dim(\text{row}(B)) = \dim(\text{col}(B)) = \dots$$

Question 7.3 Use $\text{rref}(B)$ and the procedure outlined in Example ?? to find a basis for $\text{row}(B)$.

$$\text{Basis for } \text{row}(B) : \left\{ \begin{bmatrix} \dots & \dots & \dots \end{bmatrix}, \begin{bmatrix} \dots & \dots & \dots \end{bmatrix} \right\}$$

Question 7.4 Use Procedure ?? to find a basis for $\text{col}(B)$.

$$\text{Basis for } \text{col}(B) : \left\{ \begin{bmatrix} \dots \\ \dots \\ \dots \\ \dots \end{bmatrix}, \begin{bmatrix} \dots \\ \dots \\ \dots \\ \dots \end{bmatrix} \right\}$$

Question 7.5 Find a basis for $\text{null}(B)$.

$$\text{Basis for } \text{null}(B) : \left\{ \begin{bmatrix} \dots \\ \dots \\ 1 \end{bmatrix} \right\}$$

Demonstrate that the Rank-Nullity Theorem (Theorem ??) holds for B .

Problem 8 Suppose matrix M is such that

$$\text{rref}(M) = \begin{bmatrix} 1 & 0 & 2 & 0 & 3 & 1 \\ 0 & 1 & -1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Question 8.1 Follow the process used in Example ?? to find a basis for $\text{null}(M)$. Explain why the basis elements obtained in this way are linearly independent.

$$\text{Basis of } \text{null}(M) : \left\{ \begin{bmatrix} \dots \\ \dots \\ 1 \\ \dots \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \dots \\ \dots \\ 0 \\ \dots \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \dots \\ \dots \\ 0 \\ \dots \\ 0 \\ 1 \end{bmatrix} \right\}$$

Question 8.2 Let $\mathbf{v}_1, \dots, \mathbf{v}_6$ denote the columns of M . Express $\mathbf{v}_3$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Answer:

$$\mathbf{v}_3 = \dots \mathbf{v}_1 + \dots \mathbf{v}_2$$

Problem 9 Suppose A is a 3×5 matrix. Which of the following statements could be true?

Select All Correct Answers:

1. $\dim(\text{col}(A)) = 5$
2. $\dim(\text{row}(A)) = 3$
3. $\dim(\text{null}(A)) = 1$
4. $\dim(\text{null}(A)) = 2$
5. $\dim(\text{null}(A)) = 3$

Problem 10 Suppose A is a 7×3 matrix. Which of the following statements could be true?

Select All Correct Answers:

1. $\dim(\text{col}(A)) = 3$

2. $\dim(\text{row}(A)) = 3$

3. $\dim(\text{row}(A)) = 7$

4. $\dim(\text{null}(A)) = 0$

5. $\dim(\text{null}(A)) = 4$

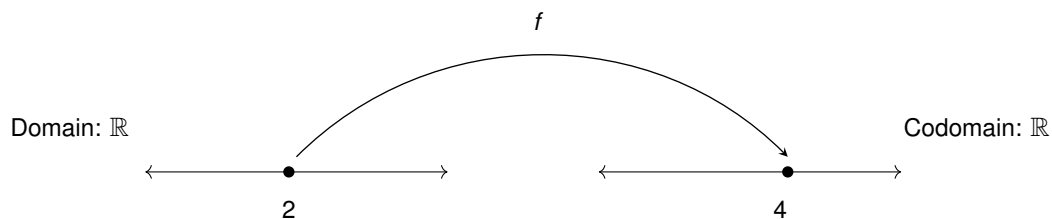
Problem 11 Complete the proof of Theorem ?? by showing that adding a scalar multiple of one row of a matrix to another row does not change the row space.

Problem 12 Prove that $\text{rank}(A) = \text{rank}(A^T)$

2 Matrix Transformations

Functions from $\mathbb{R}^n$ into $\mathbb{R}^m$

In the past you have worked with functions $f : \mathbb{R} \rightarrow \mathbb{R}$. Most of the time such functions were defined algebraically. For example, we can define f by $f(x) = x^2$. This function takes a number in the *domain* ($\mathbb{R}$) and *maps* it to the square of the number in the *codomain* (also $\mathbb{R}$). Previously, you might have visualized function f by looking at its graph, the set of all points of the form $(x, f(x))$ in $\mathbb{R}^2$. In this course, we will find it more useful to look at functions diagrammatically. For instance, the diagram below shows that f maps 2 to 4. We say that 4 is the *image of 2 under f* .



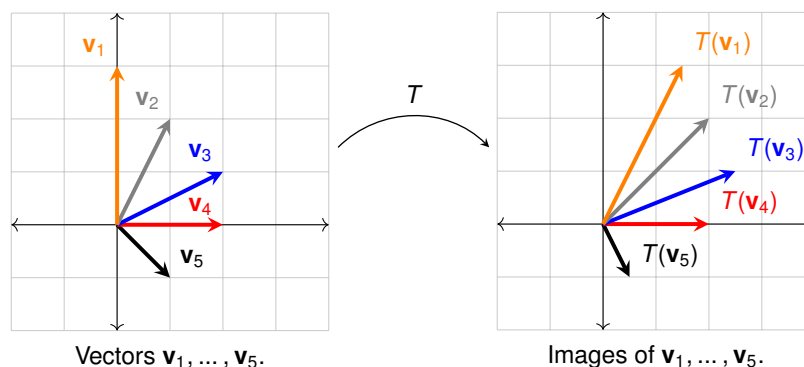
We will now consider functions that map $\mathbb{R}^n$ into $\mathbb{R}^m$. We will refer to such functions as *transformations*. There are two ways of thinking of transformations. A transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ can take a vector in $\mathbb{R}^n$ and map it to a vector in $\mathbb{R}^m$, or it can map a point in $\mathbb{R}^n$ to a point in $\mathbb{R}^m$. We will think of transformations as acting on vectors or

points interchangeably because every point $(x_1, x_2, \dots, x_n)$ of $\mathbb{R}^n$ can be interpreted as the tip of a vector $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ in $\mathbb{R}^n$.

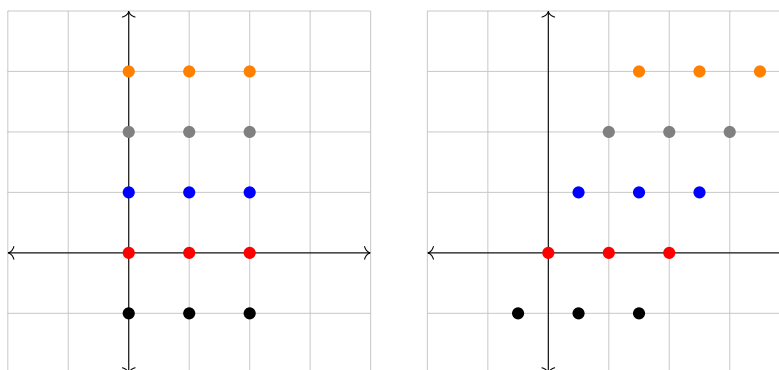
Matrix multiplication will provide us with initial tools for defining some transformations.

Examples of Matrix Transformations

Consider the matrix $A = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$. The product of A with a 2×1 vector is a 2×1 vector. We can define a transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$. This transformation can be applied to every vector of $\mathbb{R}^2$. We will look at what it does to five vectors.



Even after looking at a handful of vectors it is often difficult to tell what the transformation actually accomplishes. This is why sometimes looking at points instead of vectors can be beneficial. If we consider every point in the left grid below as a tip of a vector, we can apply the transformation to each point to obtain the grid on the right.



Applying T to a grid of points helps us see that the entire plane was *sheared* by the transformation.

We can also analyze the action of T algebraically. Start by finding the image of a generic vector $\begin{bmatrix} x \\ y \end{bmatrix}$.

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 0.5y \\ y \end{bmatrix}$$

We immediately see that the y component of the vector remains unchanged. We also see that the x component increases (or decreases) by an increment that depends on y . When considering T as a transformation acting on points, we see that points located 1 unit above the x -axis, get shifted to the right by 0.5. Points located 2 units above, get shifted to the right by 1. The higher the point, the greater the shift. Points with negative y -coordinates get shifted to the left. In this fashion T shears the entire plane.

Now that we have seen the effect of functions defined via matrix multiplication, we can better appreciate the term *transformation*, as such functions distort the domain and the shapes located in it. The following Exploration will help you visualize this.

Exploration 13 Make your own shape by moving points A, B, C, D, E, F, G in the left pane. (You can also move the entire figure by clicking and dragging the whole polygon.) The images of the points and the polygon under the transformation induced by M are shown on the right.

Access GeoGebra interactives through the online version of this text at

<https://ximera.osu.edu/linearalgebraadv3>.

Try each of the following matrices to determine what each transformation accomplishes. (Type pi into GeoGebra to get π .)

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$M_4 = \begin{bmatrix} \cos(\pi) & -\sin(\pi) \\ \sin(\pi) & \cos(\pi) \end{bmatrix}, \quad M_5 = \begin{bmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{bmatrix}$$

$$M_6 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad M_7 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad M_8 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Match the description of each transformation with the matrix that induces it.

Horizontal Shear: ($M_1 / M_2 / M_3 / M_4 / M_5 / M_6 / M_7 / M_8$)

Rotation by 45° counterclockwise: ($M_1 / M_2 / M_3 / M_4 / M_5 / M_6 / M_7 / M_8$)

Reflection about the x -axis: ($M_1 / M_2 / M_3 / M_4 / M_5 / M_6 / M_7 / M_8$)

Vertical Stretch: ($M_1 / M_2 / M_3 / M_4 / M_5 / M_6 / M_7 / M_8$)

Maps everything to a straight line: ($M_1 / M_2 / M_3 / M_4 / M_5 / M_6 / M_7 / M_8$)

Rotation through a 180° angle: ($M_1 / M_2 / M_3 / M_4 / M_5 / M_6 / M_7 / M_8$)

Horizontal compression: ($M_1 / M_2 / M_3 / M_4 / M_5 / M_6 / M_7 / M_8$)

Reflection about the line $y = x$: ($M_1 / M_2 / M_3 / M_4 / M_5 / M_6 / M_7 / M_8$)

A 2×2 matrix induces a transformation from $\mathbb{R}^2$ into $\mathbb{R}^2$. An $m \times n$ matrix can be multiplied by an $n \times 1$ vector on the right, with the resulting product being an $m \times 1$ vector. Therefore we can use an $m \times n$ matrix A to define a transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ by $T(\mathbf{x}) = A\mathbf{x}$.

Example 5. Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 2 & -2 \end{bmatrix}$. Define a transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$. Find all vectors in the domain that map to $\mathbf{0}$.

Explanation. We need to solve the system $A\mathbf{x} = \mathbf{0}$. We begin by forming an augmented matrix and finding its reduced row echelon form

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 3 & 2 & -2 & 0 \end{array} \right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -1/2 & 0 \\ 0 & 1 & -1/4 & 0 \end{array} \right]$$

There are infinitely many solutions

$$\mathbf{x} = \begin{bmatrix} 1/2 \\ 1/4 \\ 1 \end{bmatrix} t$$

This means that as T transforms $\mathbb{R}^3$ into $\mathbb{R}^2$, all points along the line $x = \frac{1}{2}t; y = \frac{1}{4}t; z = t$ map to the origin.

Linearity of Matrix Transformations

Restating properties ?? and ?? of Section [Matrix Multiplication](#) in terms of matrix-vector multiplication gives us

$$k(A\mathbf{v}) = A(k\mathbf{v}) \quad (4)$$

$$A(\mathbf{v} + \mathbf{w}) = A\mathbf{v} + A\mathbf{w} \quad (5)$$

These two properties of matrix multiplications translate into analogous properties of matrix transformations. Suppose $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a matrix transformation, then for all vectors $\mathbf{v}, \mathbf{w}$ in $\mathbb{R}^n$ and all constants k in $\mathbb{R}$,

$$T(k\mathbf{v}) = kT(\mathbf{v}) \quad (6)$$

$$T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w}) \quad (7)$$

In general, any transformation that satisfies (??) and (??) is called a *linear transformation*. As we have just seen, all matrix transformations are linear. We will study linear transformations in depth later in this chapter.

Where did i Go?

In this section we will look at the images of standard unit vectors under a matrix transformation, and discuss why this information is helpful.

Exploration 14 Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. Find the following products:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix}$$

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a matrix transformation induced by A , then we can say that T maps $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ to the first, second and third columns of A , respectively.

This nice property is not limited to transformations induced by square matrices. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be a linear transformation induced by

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 3 & 5 \end{bmatrix}$$

We will examine the effect of T on the standard unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

$$T(\mathbf{i}) = A\mathbf{i} = \begin{bmatrix} \cdots \\ \cdots \\ \cdots \end{bmatrix}, \quad T(\mathbf{j}) = A\mathbf{j} = \begin{bmatrix} \cdots \\ \cdots \\ \cdots \end{bmatrix}, \quad T(\mathbf{k}) = A\mathbf{k} = \begin{bmatrix} \cdots \\ \cdots \\ \cdots \end{bmatrix}$$

Observe that the image of $\mathbf{i}$ is the first column of A , the image of $\mathbf{j}$ is the second column of A , and the image of $\mathbf{k}$ is the third column.

We formalize our findings in Exploration ?? as follows.

Observation 1. In general, the linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$, induced by an $m \times n$ matrix A maps the standard unit vectors $\mathbf{e}_1 \dots \mathbf{e}_n$ to the columns of A . We summarize this observation by expressing columns of A as images of vectors $\mathbf{e}_1 \dots \mathbf{e}_n$ under T .

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \\ | & | & \cdots & | \end{bmatrix}$$

Why is it that knowing the images of standard unit vectors under a matrix transformation is helpful? Consider the following example.

Example 6. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a matrix transformation such that $T(\mathbf{i}) = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $T(\mathbf{j}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Find $T\left(\begin{bmatrix} -4 \\ 10 \end{bmatrix}\right)$.

Explanation. We will make use of linearity of matrix transformations.

$$T\left(\begin{bmatrix} -4 \\ 10 \end{bmatrix}\right) = T(-4\mathbf{i} + 10\mathbf{j}) = -4T(\mathbf{i}) + 10T(\mathbf{j}) = -4\begin{bmatrix} -2 \\ 3 \end{bmatrix} + 10\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 18 \\ -12 \end{bmatrix}$$

Example ?? illustrates that a matrix transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is *completely determined* by where it maps the standard unit vectors. This is true because we can express every vector $\mathbf{v}$ in $\mathbb{R}^n$ as a linear combination of the standard unit vectors, then use (??) and (??) to find the image of $\mathbf{v}$.

Practice Problems

Problem 15 Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a matrix transformation induced by the matrix A . The GeoGebra window on the left shows the domain of T , with standard unit vectors $\mathbf{i}$ and $\mathbf{j}$, and a vector $\mathbf{x}$. The window on the right shows the codomain of T , with the images of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{x}$ plotted. To use this interactive, you can

- Change the entries of matrix A ;
- Change vector $\mathbf{x}$ by dragging its tip.

Access GeoGebra interactives through the online version of this text at

<https://ximera.osu.edu/linearalgebradv3>.

Choose your matrix A . Visually verify the following claims:

- The image of $\mathbf{i}$ is the first column of matrix A .
- The image of $\mathbf{j}$ is the second column of matrix A .

Let $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Complete the following statement by filling the blanks.

$$T(\mathbf{x}) = T(\dots \mathbf{i} + \dots \mathbf{j}) = \dots T(\mathbf{i}) + \dots T(\mathbf{j})$$

Choose a different matrix A , but keep vector $\mathbf{x}$ the same. Does the above relationship still hold?

Change vector $\mathbf{x}$ by dragging its tip. Observe the image of $\mathbf{x}$ and its relationship to the images of $\mathbf{i}$ and $\mathbf{j}$. Complete the following statement for all vectors $\begin{bmatrix} a \\ b \end{bmatrix}$ in $\mathbb{R}^2$.

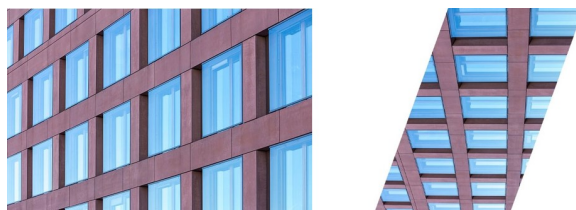
$$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = T(\dots \mathbf{i} + \dots \mathbf{j}) = \dots T(\mathbf{i}) + \dots T(\mathbf{j})$$

Problem 16 Show that a matrix transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ maps $\mathbf{0}$ to $\mathbf{0}$. In other words, $T(\mathbf{0}) = \mathbf{0}$.

Problem 17 Show that a matrix transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ maps a line in $\mathbb{R}^n$ to a line (or the origin) in $\mathbb{R}^m$.

3 Geometric Transformations of the Plane

In Exploration ?? we saw how geometric shapes in the plane can be manipulated with matrix transformations. In this section we will study matrix transformations of the plane in more detail by applying matrix transformations to pixels in photos. We will treat every pixel of a picture as a point or the head of a vector in $\mathbb{R}^2$. When a transformation is applied to each pixel, the output pixel is colored the same color as the input pixel. The figure below shows the result of one such transformation.

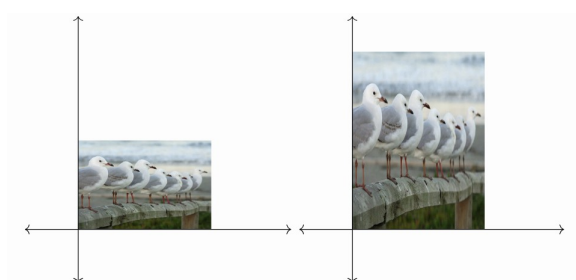


Not all transformations are matrix transformations. Practice Problem ?? offers a cautionary tale of what happens when we try to find a matrix for a transformation when such a matrix does not exist. Luckily, many familiar transformations, including rotations, scalings, reflections, and shears, are matrix transformations. We will focus our attention on those.

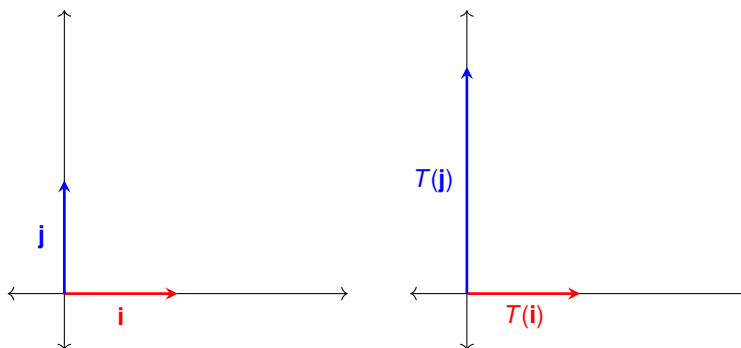
Recall that by Observation ?? of [Matrix Transformations](#), we can find a matrix of a matrix transformation by examining what the transformation does to the standard basis vectors $\mathbf{i}$ and $\mathbf{j}$.

Horizontal and Vertical Scaling

Exploration 18 Let us attempt to find a matrix M for the transformation T that stretches an image vertically by a factor of 2, as shown in the figure below.



Consider what this transformation does to the standard unit vectors. We observe that $T(\mathbf{i}) = \mathbf{i}$ and $T(\mathbf{j}) = 2\mathbf{j}$.



This allows us to construct a candidate for the transformation matrix M , by making the images of $\mathbf{i}$ and $\mathbf{j}$ the columns of M . Thus,

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

We can now check to see what this matrix does to an arbitrary point (a, b) . Treating this point as a vector $\begin{bmatrix} a \\ b \end{bmatrix}$, we compute

$$M \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ 2b \end{bmatrix}$$

Thus, this transformation takes point (a, b) to point $(a, 2b)$. So, the proposed transformation doubles all y -coordinates resulting in a vertical stretch by a factor of 2.

In general, a vertical stretch (or compression) leaves $\mathbf{i}$ unchanged, and scales the vector $\mathbf{j}$ while preserving its vertical direction. Thus, a vertical stretch (or compression) maps $\mathbf{i}$ to $\mathbf{i}$, and maps $\mathbf{j}$ to $k\mathbf{j}$ for some positive number k . Similarly, a horizontal stretch (or compression) maps $\mathbf{i}$ to $k\mathbf{i}$, and maps $\mathbf{j}$ to $\mathbf{j}$.

Formula 1 (Horizontal and Vertical Scaling). A matrix transformation induced by

$$M_v = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}, \quad (8)$$

where $(k > 0)$, scales objects in the plane vertically by a factor of k .

A matrix transformation induced by

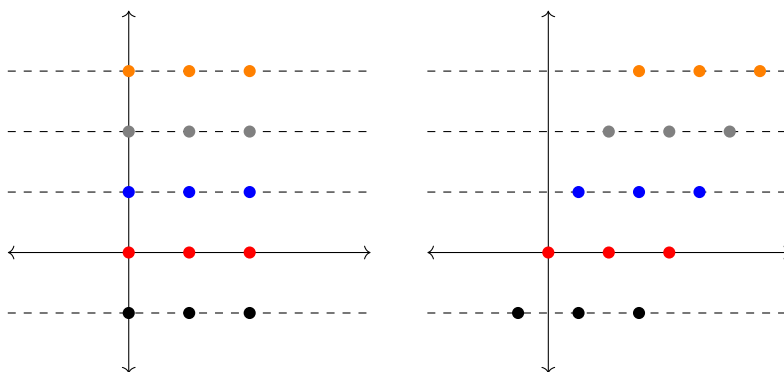
$$M_h = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}, \quad (9)$$

where $(k > 0)$, scales objects in the plane horizontally by a factor of k .

In stating the above formula we stipulated that $k > 0$. If we were to allow k to be zero, what would the resulting transformations accomplish? In what way would the resulting matrices be fundamentally different from matrices M_v and M_h ? What would happen if k were allowed to be negative? (See Practice Problem ??)

Horizontal and Vertical Shears

A *horizontal shear* is a transformation that takes an arbitrary point (a, b) and maps it to the point $(a + kb, b)$. The effect of this transformation is that all points along a fixed horizontal line slide to the left or to the right by a fixed amount. Note that the higher the point (a, b) is above the x -axis, the greater is the magnitude of kb , resulting in a greater amount of horizontal slide.



Adding a scalar multiple of the y component to the x component can be accomplished by matrix multiplication. Observe that

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a + kb \\ b \end{bmatrix}$$

A *vertical shear* is a transformation that takes an arbitrary point (a, b) and maps it to the point $(a, b + ka)$. This too, can be accomplished by matrix multiplication.

$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b + ka \end{bmatrix}$$

Formula 2 (Horizontal and Vertical Shears). A matrix transformation induced by

$$M_{hs} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \tag{10}$$

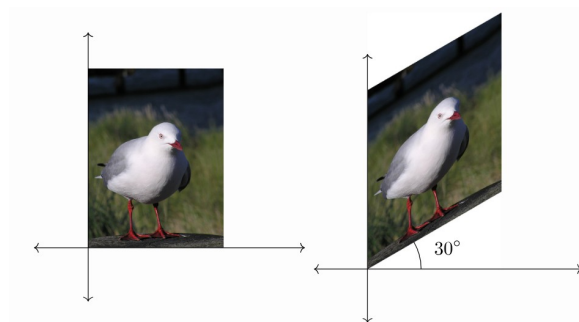
shears the plane horizontally.

A matrix transformation induced by

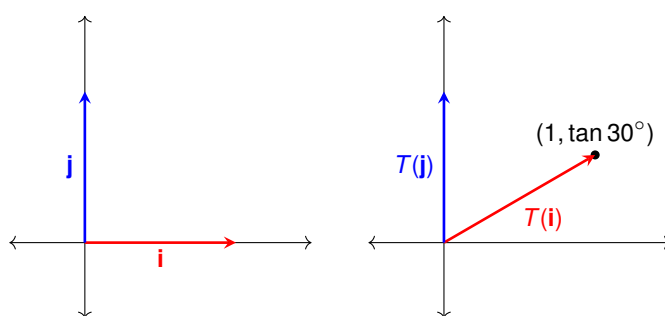
$$M_{vs} = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \tag{11}$$

shears the plane vertically.

Example 7. Find the matrix M_{vs} of a matrix transformation T that *shears* the image of a seagull, as shown in the figure below.



Explanation. Consider what this transformation does to the standard unit vectors.



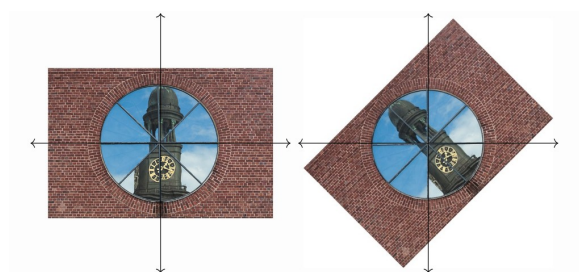
The tip of the vector $\mathbf{i}$ slides up a vertical line and its x -component remains the same. Vector $\mathbf{j}$ stays fixed. We observe that $T(\mathbf{i}) = \begin{bmatrix} 1 \\ \tan 30^\circ \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{3}/3 \end{bmatrix}$ and $T(\mathbf{j}) = \mathbf{j}$. This allows us to construct M_{vs} , by making the images of $\mathbf{i}$ and $\mathbf{j}$ the columns of M_{vs} . Thus,

$$M_{vs} = \begin{bmatrix} 1 & 0 \\ \sqrt{3}/3 & 1 \end{bmatrix}$$

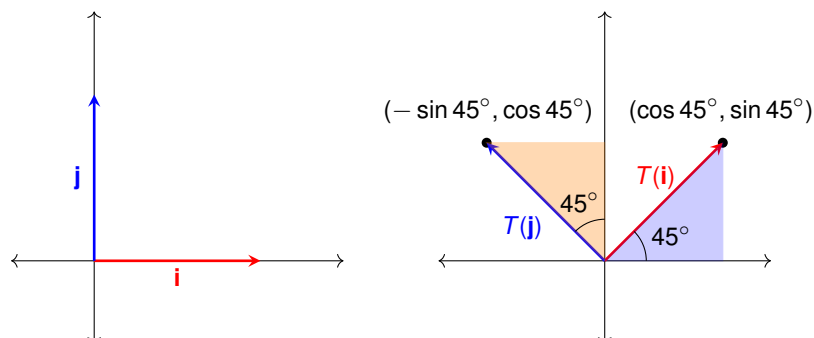
Rotations about the Origin

It turns out that rotations about the origin are also matrix transformations. You will have an opportunity to revisit and prove this claim in [Challenge Problems for Chapter 6](#).

Example 8. Let $R_{45^\circ} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a transformation that rotates $\mathbb{R}^2$ counterclockwise about the origin by 45° . Find a matrix that induces R_{45° .



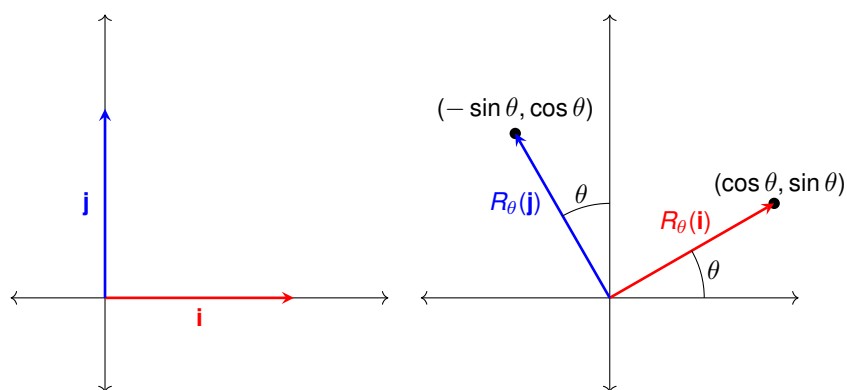
Explanation. Consider the action of R_θ on the standard unit vectors.



We observe that $R_\theta(\mathbf{i}) = \begin{bmatrix} \cos 45^\circ \\ \sin 45^\circ \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$ and $R_\theta(\mathbf{j}) = \begin{bmatrix} -\sin 45^\circ \\ \cos 45^\circ \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$. This allows us to construct the matrix M_θ , by making the images of $\mathbf{i}$ and $\mathbf{j}$ the columns of M_θ . Thus,

$$M_\theta = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

In general, we find the rotation matrix by determining the images of vectors $\mathbf{i}$ and $\mathbf{j}$.



Formula 3 (Counterclockwise Rotation). A matrix transformation induced by

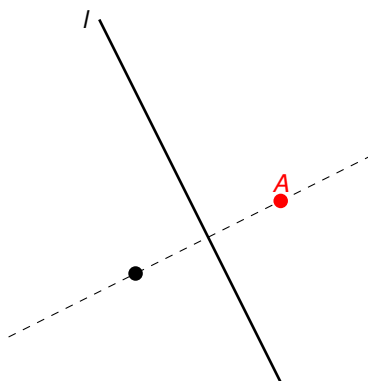
$$M_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \quad (12)$$

rotates the plane counterclockwise through angle θ about the origin.

Reflections about Lines of the Form $y = mx$

When a point is reflected about a line, its image is located on the opposite side of the line and the same distance away from the line as the original point. This is another example of a matrix transformation. (We will prove that this is a matrix transformation in [Challenge Problems for Chapter 6](#).)

For example, the figure below shows the reflection of point A about line l . Note that the reflection lies on a line through A perpendicular to l .



Our task is to find the matrix of a reflection of the plane about an arbitrary line through the origin.

Exploration 19 In this problem we will find the matrix for the reflection about the x and y axes. You can easily do this on your own by finding the images of vectors $\mathbf{i}$ and $\mathbf{j}$.

We will start with the reflection about the x -axis.

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{maps to} \quad \begin{bmatrix} \dots\dots \\ \dots\dots \end{bmatrix}$$

$$\mathbf{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{maps to} \quad \begin{bmatrix} \dots\dots \\ \dots\dots \end{bmatrix}$$

So, the matrix that induces the reflection about the x -axis is

$$\begin{bmatrix} \dots\dots & \dots\dots \\ \dots\dots & \dots\dots \end{bmatrix}$$

Next, we will consider the reflection about the y -axis.

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{maps to} \quad \begin{bmatrix} \dots\dots \\ \dots\dots \end{bmatrix}$$

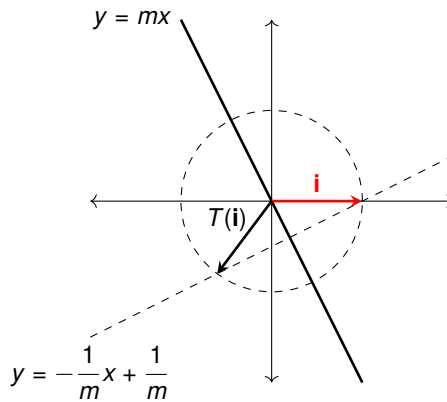
$$\mathbf{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{maps to} \quad \begin{bmatrix} \dots\dots \\ \dots\dots \end{bmatrix}$$

Thus, the matrix that induces the reflection about the y -axis is

$$\begin{bmatrix} \dots & \dots \\ \dots & \dots \end{bmatrix}$$

Now we will turn our attention to transformations that reflect the plane about the line $y = mx$. We will assume that $m \neq 0$.

Consider the vector $\mathbf{i}$ and its reflection.



Observe that the head of the image vector, $T(\mathbf{i})$, will lie on the line that passes through $(1, 0)$ and is perpendicular to the line $y = mx$. The equation of this line is given by

$$y = -\frac{1}{m}x + \frac{1}{m} \quad (13)$$

The head of $T(\mathbf{i})$ will also lie on the circle with equation

$$x^2 + y^2 = 1$$

To find the image of $\mathbf{i}$ we need to determine where the line $y = -\frac{1}{m}x + \frac{1}{m}$ intersects the circle. Substitution gives us

$$x^2 + \left(-\frac{1}{m}x + \frac{1}{m}\right)^2 = 1$$

After a little algebra we get

$$\left(1 + \frac{1}{m^2}\right)x^2 - \left(\frac{2}{m^2}\right)x + \left(\frac{1}{m^2} - 1\right) = 0$$

The quadratic formula yields

$$x = 1 \quad \text{and} \quad x = \frac{1 - m^2}{m^2 + 1}$$

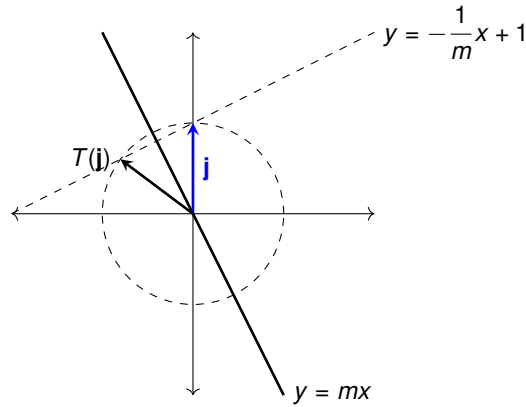
The solution $x = 1$ corresponds to the head of the vector $\mathbf{i}$. So, the x -component of $T(\mathbf{i})$ is $x = \frac{1 - m^2}{m^2 + 1}$. We find the y -component of $T(\mathbf{i})$ by substituting $x = \frac{1 - m^2}{m^2 + 1}$ into Equation ??.

$$y = -\frac{1}{m} \left(\frac{1 - m^2}{m^2 + 1} \right) + \frac{1}{m} = \frac{2m}{m^2 + 1}$$

Thus, the image of $\mathbf{i}$ under this reflection is given by

$$T(\mathbf{i}) = \begin{bmatrix} \frac{1 - m^2}{m^2 + 1} \\ \frac{2m}{m^2 + 1} \end{bmatrix}$$

Next we need to find the image of $\mathbf{j}$. The head of $T(\mathbf{j})$ is located at one of the intersections of line $y = -\frac{1}{m}x + 1$ and the circle $x^2 + y^2 = 1$.



We leave it to the reader to verify that

$$T(\mathbf{j}) = \begin{bmatrix} \frac{2m}{m^2 + 1} \\ \frac{m^2 - 1}{m^2 + 1} \end{bmatrix} \quad (14)$$

This reflection is induced by the matrix

$$M_{y=mx} = \begin{bmatrix} \frac{1 - m^2}{m^2 + 1} & \frac{2m}{m^2 + 1} \\ \frac{2m}{m^2 + 1} & \frac{m^2 - 1}{m^2 + 1} \end{bmatrix} = \frac{1}{1 + m^2} \begin{bmatrix} 1 - m^2 & 2m \\ 2m & m^2 - 1 \end{bmatrix}$$

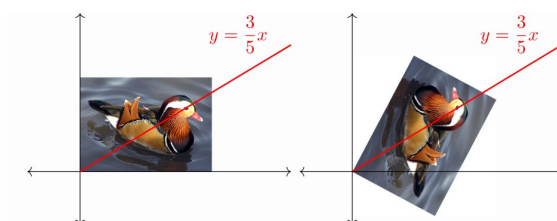
Formula 4 (Reflection about the line $y = mx$). A matrix transformation induced by

$$M_{y=mx} = \frac{1}{1 + m^2} \begin{bmatrix} 1 - m^2 & 2m \\ 2m & m^2 - 1 \end{bmatrix} \quad (15)$$

reflects the plane about the line $y = mx$.

Note that when $m = 0$, expression (??) is consistent with the reflection matrix you found in Exploration ??.

Example 9. Find matrix M that reflects the image of the duck about the line $y = \frac{3}{5}x$.



Explanation.

$$M = \frac{1}{1+9/25} \begin{bmatrix} 1-9/25 & 6/5 \\ 6/5 & 9/25-1 \end{bmatrix} = \frac{25}{34} \begin{bmatrix} 16/25 & 6/5 \\ 6/5 & -16/25 \end{bmatrix} = \begin{bmatrix} 8/17 & 15/17 \\ 15/17 & -8/17 \end{bmatrix}$$

Note that the eye of the duck in Example ?? is located on the line $y = \frac{3}{5}x$. The reflection leaves the eye fixed in place. The eye is an example of a *fixed point*. In Practice Problem ?? you will be asked to show that every point along the line $y = \frac{3}{5}x$ is a fixed point.

Composition of Linear Transformations

If a matrix transformation is followed by another matrix transformation, the resulting transformation can be represented as a product of the two matrices that induce the individual transformations. Thus, if T_1 is induced by M_1 and T_2 is induced by M_2 , then $T = T_2 \circ T_1$ is induced by $M = M_2 M_1$.

$$\mathbb{R}^2 \xrightarrow{T_1} \mathbb{R}^2 \xrightarrow{T_2} \mathbb{R}^2$$

$\curvearrowright$
 T

Remember that matrix multiplication is not commutative, so the order in which the matrices are multiplied is of utmost importance.

Consider the following example which incorporates a reflection as well as a rotation of vectors.

Example 10. 1. Find the matrix of the linear transformation which is obtained by first rotating all vectors counter-clockwise by an angle of $\pi/6$ and then reflecting across the x -axis.

2. Multiply the vector $\mathbf{v} = \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}$ by the matrix you found in part (a) to illustrate how the transformation works.

Explanation. 1. Using Formula ??, the matrix of the transformation which involves rotating through an angle of $\pi/6$ is

$$M_1 = \begin{bmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} \end{bmatrix}$$

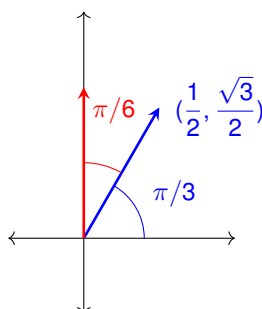
As we learned in Exploration ??, the matrix for the transformation which reflects all vectors about the x-axis is

$$M_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Therefore, the matrix of the linear transformation which first rotates through $\pi/6$ and then reflects through the x-axis is given by

$$M = M_2 M_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \end{bmatrix}$$

2. From unit circle trigonometry, we know that $\mathbf{v}$ makes a $\pi/3$ angle with the x-axis. If we rotate $\mathbf{v}$ counterclockwise by $\pi/6$, it will point straight up.



Following the rotation with reflection across the x-axis will make it point straight down. Sure enough,

$$M\mathbf{v} = \begin{bmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \end{bmatrix} \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Exploration 20 Use the GeoGebra interactive below to decompose a matrix into a product of two matrices corresponding to the basic transformations we discussed above: scalings, rotations, shears and reflections.

Consider the matrix $M = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$. This matrix induces a transformation that can be broken into two parts: (1) a reflection followed by (2) a shear. Find matrices M_1 and M_2 that induce the reflection and the shear respectively. Verify that the product of the two matrices is equal to M (be careful about the order of the product!).

$$M = M_2 M_1 = \begin{bmatrix} \dots & \dots \\ \dots & \dots \end{bmatrix} \begin{bmatrix} \dots & \dots \\ \dots & \dots \end{bmatrix}$$

Access GeoGebra interactives through the online version of this text at

<https://ximera.osu.edu/linearalgebradv3/LinearAlgebraInteractiveIntro>.

Let $M = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. Use the GeoGebra interactive above to visually examine the transformation induced by M .

What sequence of transformations is equivalent to the transformation induced by M ?

Multiple Choice:

1. Rotation by 180 degrees.
2. Reflection about the x -axis, followed by a reflection about the y -axis.
3. Reflection about the y -axis, followed by a reflection about the x -axis.
4. All of the above.

Let $M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Use the GeoGebra interactive above to visually examine the transformation induced by M . What sequence of transformations is equivalent to the transformation induced by M ?

Multiple Choice:

1. Reflection about the x -axis, followed by a reflection about the line $y = -x$.
2. Reflection about the y -axis, followed by a reflection about the x -axis.
3. All of the above.
4. None of the above.

Example 11. In this problem we will consider compositions of two reflections and use geometry to illustrate non-commutativity of matrix multiplication. Let

$$T_{y=-2x} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

be a reflection about the line $y = -2x$. Let

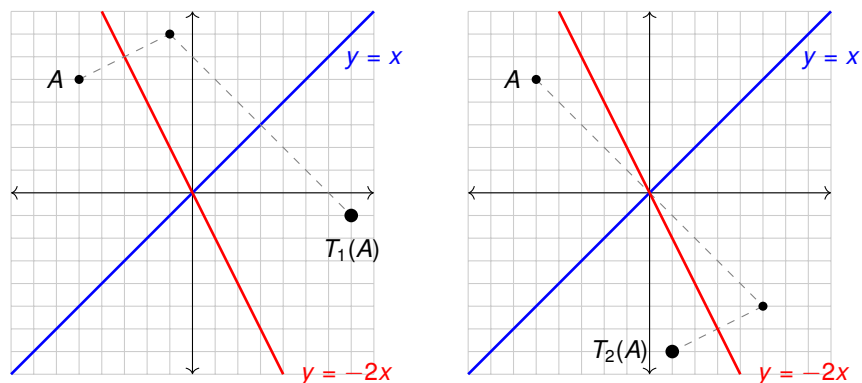
$$T_{y=x} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

be a reflection about the line $y = x$. We will denote the standard matrices for these transformations by $M_{y=-2x}$ and $M_{y=x}$, and use geometry to demonstrate that $M_{y=x}M_{y=-2x} \neq M_{y=-2x}M_{y=x}$.

To do this, consider transformations $T_1 = T_{y=x} \circ T_{y=-2x}$ and $T_2 = T_{y=-2x} \circ T_{y=x}$. Transformation T_1 is induced by $M_{y=x}M_{y=-2x}$, and T_2 is induced by $M_{y=-2x}M_{y=x}$.

The figure on the left illustrates the action of T_1 on a single point A . First, A is reflected about the line $y = -2x$, then A is reflected about the line $y = x$.

The figure on the right shows the action of T_2 on the same point A . The point is first reflected about the line $y = x$, followed by a reflection about the line $y = -2x$. The final images of point A under T_1 and T_2 are clearly different.



Since $T_1 \neq T_2$, we conclude that $M_{y=x}M_{y=-2x} \neq M_{y=-2x}M_{y=x}$.

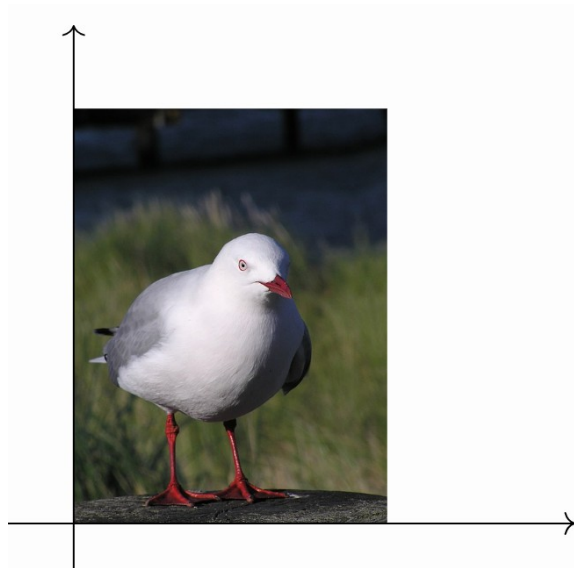
Example 12. Some pairs of matrices do commute. For example, geometry makes it is easy to see that two rotation matrices commute.

Practice Problems

Problem 21 Consider matrices M_v and M_h in (??) and (??).

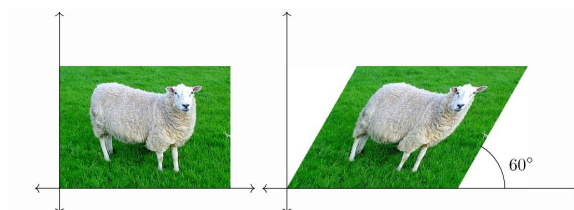
1. If we were to allow k to be zero, what would the resulting transformations accomplish?
2. If $k = 0$, in what way would the resulting matrices be fundamentally different from matrices M_v and M_h ($k \neq 0$)?
3. Do M_v and M_h ($k \neq 0$) have inverses? What about M_v and M_h ($k = 0$)?
4. What would happen if we allowed k to be negative?

Problem 22 Find a matrix M that would double the length of a photo horizontally, and triple the height of the photo.



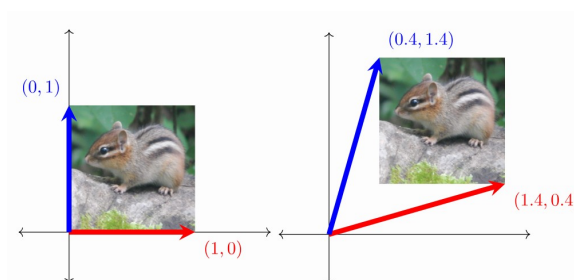
$$M = \begin{bmatrix} \dots & \dots \\ \dots & \dots \end{bmatrix}$$

Problem 23 (Sheared Sheep) Find a matrix that induces the transformation shown in the figure.



$$M = \begin{bmatrix} \cdots & \cdots \\ \cdots & \cdots \end{bmatrix}$$

Problem 24 Suppose a 1 by 1 photo of a chipmunk was shifted as shown in the figure.



Suppose we tried to construct a standard matrix M for this transformation by making the images of $\mathbf{i}$ and $\mathbf{j}$ the columns of M . We would obtain

$$M = \begin{bmatrix} 1.4 & 0.4 \\ 0.4 & 1.4 \end{bmatrix}$$

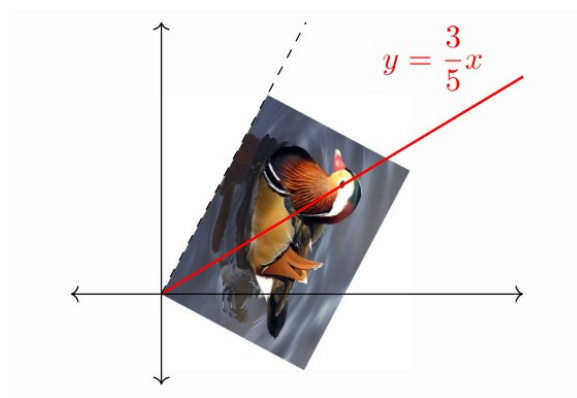
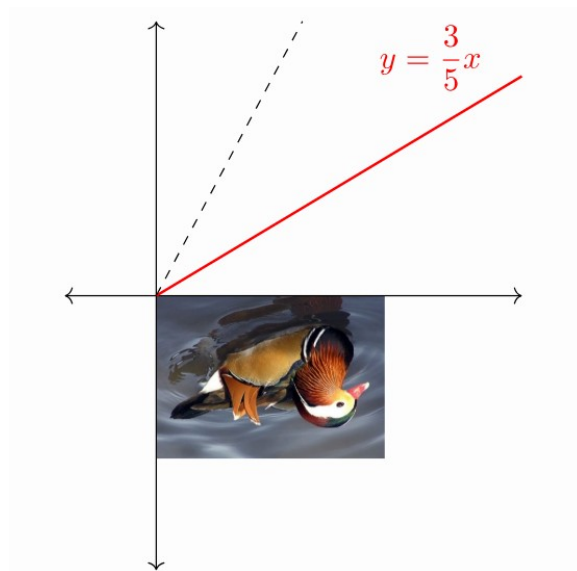
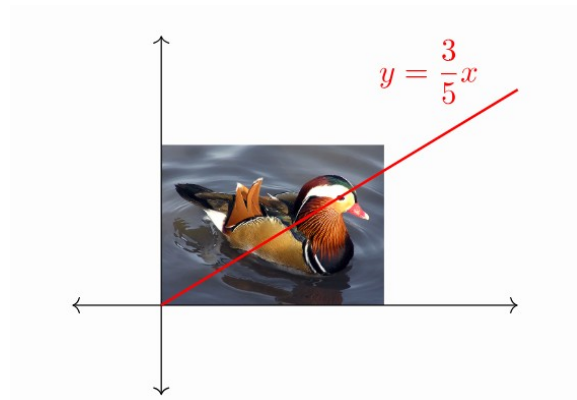
Does this matrix describe the transformation? If so, prove it. If not, explain why not.

Problem 25 A transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that shifts all points in the plane horizontally or vertically by a fixed amount is called a translation. Is T a matrix transformation? Prove your claim.

Problem 26 A reflection about the line $y = mx$ followed by another reflection about the same line, returns all points to their original position. Prove this using matrix multiplication.

Problem 27 Prove that every point along the line $y = \frac{3}{5}x$ in Example ?? is a fixed point.

Problem 28 The figure below shows a sequence of two matrix transformations that accomplishes a reflection about the line $y = \frac{3}{5}x$. The first transformation is a reflection of the plane about the x -axis. The second transformation is a rotation of the plane about the origin. Find matrices that induce the two transformations and verify that their product (in the correct order) is the reflection matrix of Example ??.



Example Source

Example ?? was adapted from Example 5.27 of Ken Kuttler's *A First Course in Linear Algebra*. (CC-BY)

Ken Kuttler, *A First Course in Linear Algebra*, Lyryx 2017, Open Edition, p. 290.

Photo Credits

The following images are courtesy of [Wikimedia Commons](#)

Adrian Pingstone, *A male Mandarin Duck at Slimbridge Wildfowl and Wetlands Centre, Gloucestershire, England*. Public Domain

Ansgar Koreng, *Facade of ARD-Hauptstadtstudio in Berlin-Mitte*. CC-BY 4.0

Christoph Braun, *Reflection of St. Michaelis Church in a window of St. Ansgar in Hamburg, Germany*. Public Domain.

Daniel Gammert, *Red-billed Gulls Chroicocephalus novaehollandiae scopulinus. Brighton Beach, New Zealand*. Public Domain

I, Tony Wills, *Red billed gull in Wellington Harbour, Wellington, New Zealand*. CC-BY

Jackhynes, *Lley sheep taken with a Sony Digital Camera at 3.2 megapixels in Devon, UK*. Public Domain

Erik Davis, *Chipmunk*. CC-BY