
Week 8

June 5, 2026

Contents

1 Introduction to Linear Transformations

We start by reviewing the definition of a function.

Definition 1. Let V and W be sets. A *function* f from V into W , denoted by

$$f : V \rightarrow W$$

assigns to each element x of V , a single element $y = f(x)$ of W .

The set V is called the *domain* of f , and the set W is called the *codomain*.

If $y = f(x)$, we say that x *maps to* y , and y is the *image* of x .

The collection of images of all points of V is called the *image* of V under f , or the *image* of f . (It is also known as the *range* of f .)

In algebra and calculus you worked with functions $f : \mathbb{R} \rightarrow \mathbb{R}$ whose domain and codomain were each the set of all real numbers. In linear algebra, we call our functions *transformations*. The domain and codomain of a transformation are *vector spaces*.

Exploration 1 In this exercise we will introduce a very special type of transformation by contrasting the effects of two transformations on vectors of $\mathbb{R}^2$. We will see that some transformations have “nice” properties, while others do not. Define T_1 and T_2 as follows:

$$\begin{aligned} T_1 : \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ T_1 \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) &= \begin{bmatrix} x - y \\ x \end{bmatrix} \\ T_2 : \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ T_2 \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) &= \begin{bmatrix} -x + y + 1 \\ y - 2 \end{bmatrix} \end{aligned}$$

Each of these transformations takes a vector in $\mathbb{R}^2$, and maps it to another vector in $\mathbb{R}^2$. To see if you understand how these transformations are defined, see if you can determine what these transformations do to the vector $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$.

$$T_1 \left(\begin{bmatrix} 4 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} \dots \dots \\ \dots \dots \end{bmatrix} \quad \text{and} \quad T_2 \left(\begin{bmatrix} 4 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} \dots \dots \\ \dots \dots \end{bmatrix}.$$

Now, let's take the vector $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ and multiply it by a scalar, say 7.

$$7 \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 28 \\ 21 \end{bmatrix}$$

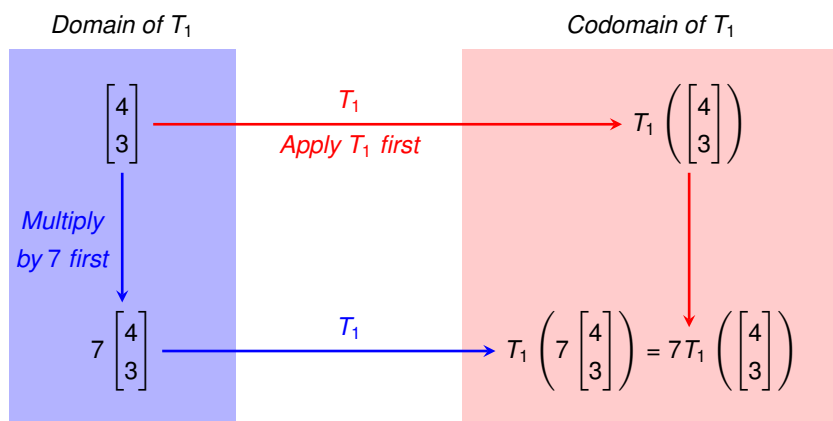
Now let's compare how T_1 and T_2 "handle" this product. Starting with T_1 , we compute:

$$T_1 \left(7 \begin{bmatrix} 4 \\ 3 \end{bmatrix} \right) = T_1 \left(\begin{bmatrix} 28 \\ 21 \end{bmatrix} \right) = \begin{bmatrix} 7 \\ 28 \end{bmatrix}$$

Observe that multiplying the original vector by 7, then applying T_1 , has the same effect as applying T_1 to the original vector, then multiplying the image by 7. In other words,

$$T_1 \left(7 \begin{bmatrix} 4 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} 7 \\ 28 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 4 \end{bmatrix} = 7 T_1 \left(\begin{bmatrix} 4 \\ 3 \end{bmatrix} \right)$$

Diagrammatically, this can be represented as follows.



You should try to verify that this property does not hold for transformation T_2 . In other words,

$$T_2 \left(7 \begin{bmatrix} 4 \\ 3 \end{bmatrix} \right) \neq 7 T_2 \left(\begin{bmatrix} 4 \\ 3 \end{bmatrix} \right)$$

There is nothing special about the number 7, and it is not hard to prove that for any scalar k and vector $\mathbf{u}$ of $\mathbb{R}^2$, T_1 satisfies

$$kT_1(\mathbf{u}) = T_1(k\mathbf{u}). \quad (1)$$

It turns out that T_1 satisfies another important property. For all vectors $\mathbf{u}$ and $\mathbf{v}$ of $\mathbb{R}^2$ we have:

$$T_1(\mathbf{u} + \mathbf{v}) = T_1(\mathbf{u}) + T_1(\mathbf{v}) \quad (2)$$

We leave it to the reader to illustrate this property with a specific example (see Practice Problem ??). We will show that T_1 satisfies (??) in general.

Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, then

$$\begin{aligned} T_1(\mathbf{u} + \mathbf{v}) &= T_1\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = T_1\left(\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}\right) = \begin{bmatrix} u_1 + v_1 - u_2 - v_2 \\ u_1 + v_1 \end{bmatrix} \\ &= \begin{bmatrix} u_1 - u_2 \\ u_1 \end{bmatrix} + \begin{bmatrix} v_1 - v_2 \\ v_1 \end{bmatrix} = T_1\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) + T_1\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) \\ &= T_1(\mathbf{u}) + T_1(\mathbf{v}) \end{aligned}$$

It turns out that T_2 fails to satisfy this property. Can you prove that this is the case? Remember that to prove that a property DOES NOT hold, it suffices to find a counter-example. See if you can find vectors $\mathbf{u}$ and $\mathbf{v}$ such that

$$T_2(\mathbf{u} + \mathbf{v}) \neq T_2(\mathbf{u}) + T_2(\mathbf{v}). \quad (3)$$

(See Practice Problem ??.)

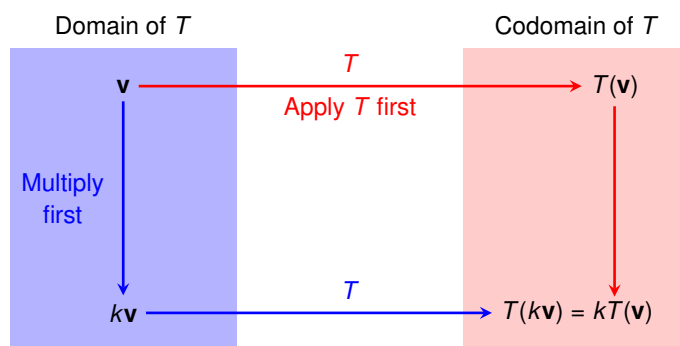
Transformations satisfying (??) and (??), like T_1 , are called *linear transformations*. Transformations like T_2 are not linear. You have already encountered several linear transformations in the form of matrix transformations in sections [Matrix Transformations](#) and [Geometric Transformations of the Plane](#).

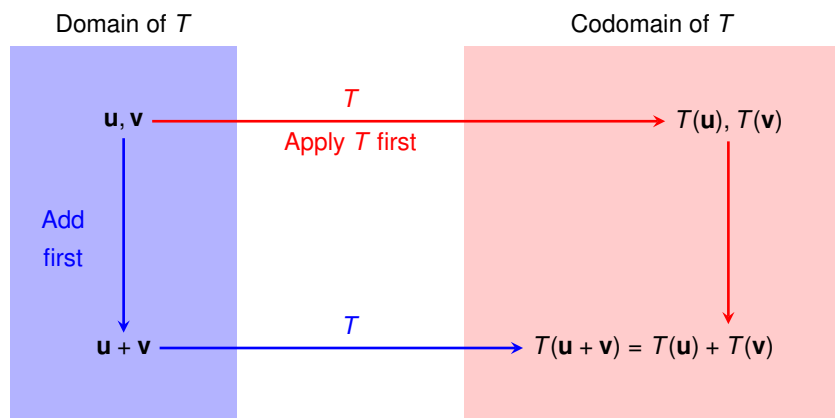
Definition 2. A transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is called a *linear transformation* if the following are true for all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, and scalars k .

$$T(k\mathbf{u}) = kT(\mathbf{u}) \quad (4)$$

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \quad (5)$$

Equations (??) and (??) of the above definition can be illustrated diagrammatically as follows.





Remark 1. Properties (??) and (??) are often combined into a single property.

$$T(k_1 \mathbf{u} + k_2 \mathbf{v}) = k_1 T(\mathbf{u}) + k_2 T(\mathbf{v}) \quad (6)$$

Example 1. Suppose $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is a linear transformation such that

$$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

Find each of the following:

1. $T\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = T\left(2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \end{bmatrix}\right)$
2. $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

Explanation. ?? Because T is a linear transformation, it satisfies (??). We compute:

$$\begin{aligned} T\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) &= T\left(2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) \\ &= 2T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) - T\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) \\ &= 2 \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -2 \\ 0 \\ 6 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 6 \end{bmatrix} \end{aligned}$$

?? Observe that $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. By (??) we have:

$$\begin{aligned} T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) &= T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) \\ &= T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) + T\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) \\ &= \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \end{aligned}$$

Observation 1. In Example ?? we were given the images of two vectors, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$, under a linear transformation T . Based on this information, we were able to determine the images of two additional vectors: $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The reason we were able to determine $T\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right)$ and $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ is because $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ can be written as unique linear combinations of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

Can every vector of $\mathbb{R}^2$ be written as a linear combination of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$?

Multiple Choice:

1. Yes
2. No

Is the information provided in Example ?? sufficient to determine the image of every vector in $\mathbb{R}^2$ under T ?

Multiple Choice:

1. Yes
2. No

Example 2. Suppose $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a transformation such that

$$T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad T\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

Determine whether T is a linear transformation.

Explanation. Observe that

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

If T were a linear transformation, then we would have:

$$T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$$

But according to the given,

$$T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

Since $\begin{bmatrix} 6 \\ 5 \end{bmatrix} \neq \begin{bmatrix} 2 \\ -4 \end{bmatrix}$ we conclude that transformation T is not linear.

In Exploration ?? we introduced a transformation T_2 which turned out to be non-linear. It took some work to show that T_2 is not linear. The following theorem would have made our work easier.

Theorem 1. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Then

1. $T(\mathbf{0}) = \mathbf{0}$. In other words, linear transformations map the zero vector to the zero vector.
2. T maps any line in $\mathbb{R}^n$ to a line (or the zero vector) in $\mathbb{R}^m$.

Proof To prove part ??, let $\mathbf{v}$ be any vector in $\mathbb{R}^n$. By linearity of T , we have:

$$T(\mathbf{0}) = T(\mathbf{v} - \mathbf{v}) = T(\mathbf{v}) - T(\mathbf{v}) = \mathbf{0}$$

Part ?? will become evident after we prove Theorem ?? in [Standard Matrix of a Linear Transformation](#) and combine it with Practice Problem ??. ■

Example 3. Use Theorem ?? to show that transformation T_2 of Exploration ?? is not linear.

Explanation. Recall that $T_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ was defined by

$$T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x + y + 1 \\ y - 2 \end{bmatrix}$$

We evaluate T_2 at $\mathbf{0}$:

$$T_2(\mathbf{0}) = \begin{bmatrix} -0 + 0 + 1 \\ 0 - 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \neq \mathbf{0}$$

Since $T_2(\mathbf{0}) \neq \mathbf{0}$, T_2 is not linear.

Linear Transformations Induced by Matrices

Recall that a transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ defined by $T(\mathbf{v}) = A\mathbf{v}$, where A is some $m \times n$ matrix, is called a matrix transformation (or transformation induced by A). As we had discovered in [Matrix Transformations](#), all matrix transformations are linear. We now formalize this result as a theorem.

Theorem 2. Let A be an $m \times n$ matrix. Define $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ by $T(\mathbf{v}) = A\mathbf{v}$. Then T is a linear transformation.

Proof Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$, and let k be a scalar. By properties of matrix multiplication we have:

$$T(\mathbf{u} + \mathbf{v}) = A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = T(\mathbf{u}) + T(\mathbf{v})$$

$$T(k\mathbf{u}) = A(k\mathbf{u}) = kA\mathbf{u} = kT(\mathbf{u})$$

Therefore T is a linear transformation. ■

Example 4. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation induced by

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 4 \\ 0 & 1 \end{bmatrix}$$

1. Find n and m .
2. Find the image of T .

Explanation. ?? A is a 3×2 matrix, so for the expression $T(\mathbf{x}) = A\mathbf{x}$ to make sense, $\mathbf{x}$ has to be a 2×1 vector. Thus, the domain of T is $\mathbb{R}^2$ ($n = 2$). The product $A\mathbf{x}$ is a 3×1 vector. The codomain of T is $\mathbb{R}^3$ ($m = 3$).

?? By Definition ??, the image of T consists of images of all individual vectors in $\mathbb{R}^2$ under T . Every vector $\mathbf{v}$ in $\mathbb{R}^2$ can be written as $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ for some real numbers a and b . Consider the image of $\mathbf{v}$

$$T(\mathbf{v}) = T(a\mathbf{i} + b\mathbf{j}) = aT(\mathbf{i}) + bT(\mathbf{j}) = a \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}$$

This shows that the range, or the image, of T consists of all linear combinations of the columns of A . In other words,

the image of T is the span of vectors $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}$. The two vectors are not scalar multiples of each other, therefore they span a plane in $\mathbb{R}^3$.

Example 5. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation induced by

$$A = \begin{bmatrix} -2 & 1 & 3 \\ 4 & -2 & -6 \end{bmatrix}$$

1. Find n and m .
2. Find and draw the image of T .

Explanation. ??

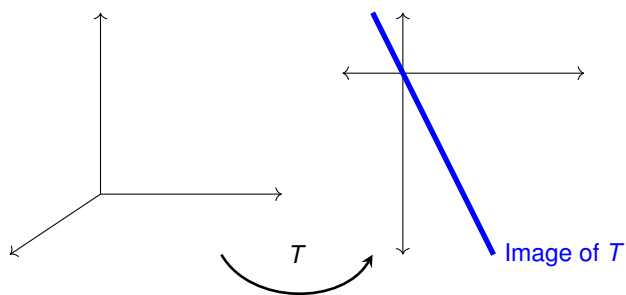
$$n = \dots, \quad m = \dots$$

?? To find the image of T , we will take a slightly different approach from what we did in Example ????.

Let $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be an arbitrary vector of $\mathbb{R}^3$. The image of $\mathbf{v}$ is given by

$$\begin{aligned} T(\mathbf{v}) &= \begin{bmatrix} -2 & 1 & 3 \\ 4 & -2 & -6 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = a \begin{bmatrix} -2 \\ 4 \end{bmatrix} + b \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c \begin{bmatrix} 3 \\ -6 \end{bmatrix} \\ &= (a(-2) + b + c(3)) \begin{bmatrix} 1 \\ -2 \end{bmatrix} \end{aligned}$$

This shows that the image of every vector in $\mathbb{R}^3$ is a scalar multiple of $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$. This means that the image of T is a line in $\mathbb{R}^2$.



Linear Transformations of Subspaces of $\mathbb{R}^n$

Definition ?? defines a linear transformation as a map from $\mathbb{R}^n$ into $\mathbb{R}^m$. We will now make this definition more general by allowing the domain and the codomain of the transformation to be subspaces of $\mathbb{R}^n$ and $\mathbb{R}^m$. Eventually, a linear transformation will be defined as a mapping between vector spaces.

Definition 3. Let V and W be subspaces of $\mathbb{R}^n$ and $\mathbb{R}^m$. A transformation $T : V \rightarrow W$ is called a *linear transformation* if the following are true for all vectors $\mathbf{u}$ and $\mathbf{v}$ in V , and scalars k .

$$T(k\mathbf{u}) = kT(\mathbf{u})$$

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$

Example 6. Let V be a subspace of $\mathbb{R}^3$ consisting of all vectors in the xy -plane. Let W be a subspace of $\mathbb{R}^3$ consisting of all vectors along the z -axis. (Do a quick verification that V and W are subspaces of $\mathbb{R}^3$.) Define a

transformation $T : V \rightarrow W$ by

$$T \left(\begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ a + b \end{bmatrix}$$

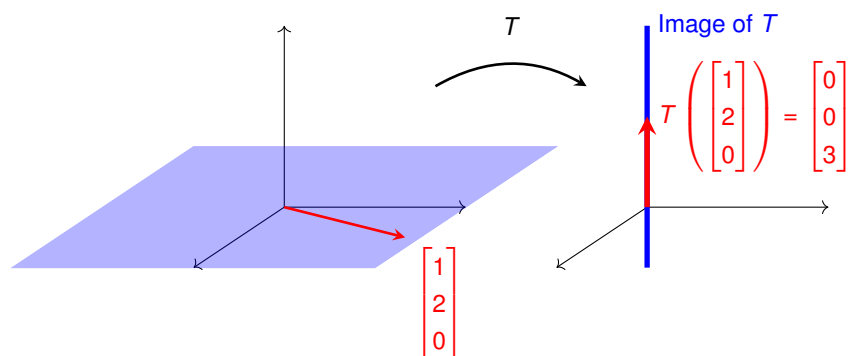
Show that T is a linear transformation, and describe its action geometrically.

Explanation. Consider two arbitrary elements $\begin{bmatrix} a_1 \\ b_1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} a_2 \\ b_2 \\ 0 \end{bmatrix}$ of V .

$$\begin{aligned} T \left(\begin{bmatrix} a_1 \\ b_1 \\ 0 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \\ 0 \end{bmatrix} \right) &= T \left(\begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \\ 0 \end{bmatrix} \right) \\ &= \begin{bmatrix} 0 \\ 0 \\ (a_1 + a_2) + (b_1 + b_2) \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ (a_1 + b_1) + (a_2 + b_2) \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ a_1 + b_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ a_2 + b_2 \end{bmatrix} \\ &= T \left(\begin{bmatrix} a_1 \\ b_1 \\ 0 \end{bmatrix} \right) + T \left(\begin{bmatrix} a_2 \\ b_2 \\ 0 \end{bmatrix} \right) \end{aligned}$$

Verification of the fact that $T \left(k \begin{bmatrix} a_1 \\ b_1 \\ 0 \end{bmatrix} \right) = kT \left(\begin{bmatrix} a_1 \\ b_1 \\ 0 \end{bmatrix} \right)$ is similar, and we omit the details.

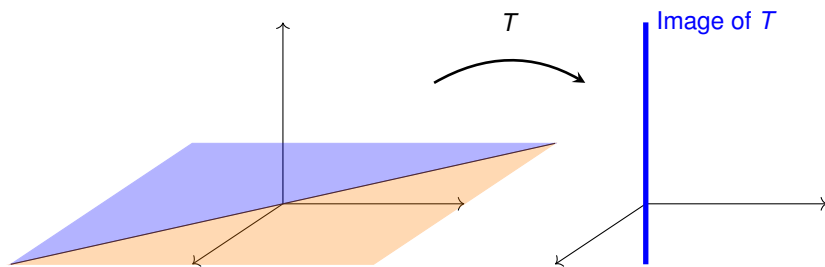
We have shown that T is a linear transformation. T maps all vectors in the xy -plane to the z -axis. The following diagram helps us visualize the action of T on a specific vector.



We can investigate further. Recall that T is defined by

$$T \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ a+b \end{bmatrix}$$

Complete each of the following statements referring to the diagram below.



The image of the line $y = -x$ under T is (a line/ a plane/ the zero vector/ all of $\mathbb{R}^3$)

The image of the orange part of the domain (the front triangle) is (the positive z-axis/ the negative z-axis/ the zero vector/ the entire z-axis)

The image of the purple part of the domain (the back triangle) is (the positive z-axis/ the negative z-axis/ the zero vector/ the entire z-axis)

We conclude this section by introducing two simple but important transformations.

Definition 4. The *identity transformation* on V , denoted by id_V , is a transformation that maps each element of V to itself.

In other words,

$$\text{id}_V : V \rightarrow V$$

is a transformation such that

$$\text{id}_V(\mathbf{v}) = \mathbf{v} \quad \text{for all } \mathbf{v} \in V$$

Definition 5. The *zero transformation*, Z , maps every element of the domain to the zero vector.

In other words,

$$Z : V \rightarrow W$$

is a transformation such that

$$Z(\mathbf{v}) = \mathbf{0} \quad \text{for all } \mathbf{v} \in V$$

Theorem 3. The identity transformation is linear.

Proof Left to the reader. (See Practice Problem ??) ■

Theorem 4. The zero transformation is linear.

Proof Left to the reader. (See Practice Problem ??) ■

Practice Problems

Problem 2 Define T_1 as follows

$$T_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$T_1 \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x - y \\ x \end{bmatrix}$$

(See Exploration ??) Use vectors $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ to illustrate that $T_1(\mathbf{u} + \mathbf{v}) = T_1(\mathbf{u}) + T_1(\mathbf{v})$.

Problem 3 Define T_2 as follows

$$T_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$T_2 \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} -x + y + 1 \\ y - 2 \end{bmatrix}$$

(See Exploration ??)

Use a counter-example to prove that $T_2(\mathbf{u} + \mathbf{v}) \neq T_2(\mathbf{u}) + T_2(\mathbf{v})$

Problem 4 Suppose $T : \mathbb{R}^{10} \rightarrow \mathbb{R}^2$ is a linear transformation such that $T(\mathbf{u}) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $T(\mathbf{v}) = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$. Find the image of $3\mathbf{u} - \mathbf{v}$.

$$T(3\mathbf{u} - \mathbf{v}) = \begin{bmatrix} \dots \dots \\ \dots \dots \end{bmatrix}$$

Problem 5 Let $\mathbf{u}$ be a fixed non-zero vector. Define $T_{\mathbf{u}} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, by $T_{\mathbf{u}}(\mathbf{x}) = \mathbf{u} - \mathbf{x}$.

1. Describe the effect of this transformation by sketching $\mathbf{x}$ and $T_{\mathbf{u}}(\mathbf{x})$ for at least four vectors $\mathbf{x}$ and a fixed vector $\mathbf{u}$ of your choice.
2. Is $T_{\mathbf{u}}$ a linear transformation?

Problem 6 Define $P_{xy} : \mathbb{R}^3 \rightarrow \mathbb{R}^2$, by $P_{xy} \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x \\ y \end{bmatrix}$. This transformation is called an orthogonal projection onto the xy -plane. Show that P_{xy} is a linear transformation.

Problem 7 Suppose a linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ maps $\mathbf{i}$ to $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{j}$ to $\begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}$, and $\mathbf{k}$ to $\begin{bmatrix} 3 \\ 0 \\ -5 \end{bmatrix}$. Find the image of $\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ under T .

$$T\left(\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix}$$

Problem 8 Prove that the identity transformation is linear. (Theorem ??)

Problem 9 Prove that the zero transformation is linear. (Theorem ??)

Problem 10 Find the domain and the codomain of the linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ induced by A ; find and draw the image of T .

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$$

Domain: $\mathbb{R}^n$, where $n = \dots$

Codomain: $\mathbb{R}^m$, where $m = \dots$

Problem 11 Find the domain and the codomain of the linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ induced by A ; find and draw the image of T .

$$A = \begin{bmatrix} 3 & -1 \\ -3 & 1 \end{bmatrix}$$

Exercise Source

Practice Problem ?? is adopted from Problem 5.1.3 of Ken Kuttler's *A First Course in Linear Algebra*. (CC-BY)

Ken Kuttler, *A First Course in Linear Algebra*, Lyryx 2017, Open Edition, p. 272.

2 Standard Matrix of a Linear Transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$

In [Matrix Transformations](#) and [Introduction to Linear Transformations](#) we learned several important properties of matrix transformations of $\mathbb{R}^n$ and subspaces of $\mathbb{R}^n$. Let's summarize the main points.

Summary 1. For a matrix transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$, induced by an $m \times n$ matrix A we have the following results:

- T is linear. (Theorem ??) This means that

$$T(k_1 \mathbf{u} + k_2 \mathbf{v}) = k_1 T(\mathbf{u}) + k_2 T(\mathbf{v})$$

for vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ and scalars k_1 and k_2 in $\mathbb{R}$.

- Columns of A are the images of the standard unit vectors of $\mathbb{R}^n$ under T . (Observation ??)

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & \dots & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} | & | & \dots & | \\ T(\mathbf{e}_1) & T(\mathbf{e}_2) & \dots & T(\mathbf{e}_n) \\ | & | & \dots & | \end{bmatrix}$$

- The action of T on all of the elements of $\mathbb{R}^n$ is completely determined by where T maps the standard unit vectors. (See Examples ?? and ??)

The last point in the summary is so important that it is worth illustrating again.

Example 7. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be a linear transformation. Suppose that the only information we have about this transformation is that $T(\mathbf{i}) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, $T(\mathbf{j}) = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$ and $T(\mathbf{k}) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Is this information sufficient to determine the

image of $\mathbf{w} = \begin{bmatrix} 1 \\ -3 \\ 6 \end{bmatrix}$?

Explanation. Observe that

$$\mathbf{w} = \mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$$

We find $T(\mathbf{w})$ by using the fact that T is linear.

$$T(\mathbf{w}) = T(\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}) = T(\mathbf{i}) - 3T(\mathbf{j}) + 6T(\mathbf{k}) = \begin{bmatrix} 3 \\ -1 \end{bmatrix} - 3 \begin{bmatrix} 0 \\ 4 \end{bmatrix} + 6 \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -9 \\ -7 \end{bmatrix}$$

Because of properties of linear transformations, the information about the images of the standard unit vectors proved to be sufficient for us to determine the image of $\mathbf{w}$.

In Example ??, there was nothing special about the vector $\mathbf{w}$. Any vector $\mathbf{x}$ of $\mathbb{R}^n$ can be written as a unique linear combination of the standard unit vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$. Therefore, the image of any vector $\mathbf{x}$ under a linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is uniquely determined by the images of $\mathbf{e}_1, \dots, \mathbf{e}_n$. Knowing $T(\mathbf{e}_1), \dots, T(\mathbf{e}_n)$ allows us to construct a matrix A , with $T(\mathbf{e}_1), \dots, T(\mathbf{e}_n)$ as columns, that induces transformation T . We formalize this idea in a theorem.

Theorem 5. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Then T is a matrix transformation with

$$A = \begin{bmatrix} | & | & & | \\ T(\mathbf{e}_1) & T(\mathbf{e}_2) & \dots & T(\mathbf{e}_n) \\ | & | & & | \end{bmatrix}$$

as a matrix that induces T .

Proof Observe that

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \dots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + \dots + x_n \mathbf{e}_n$$

Because T is linear, we have

$$\begin{aligned} T(\mathbf{x}) &= T(x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + \dots + x_n \mathbf{e}_n) = x_1 T(\mathbf{e}_1) + x_2 T(\mathbf{e}_2) + \dots + x_n T(\mathbf{e}_n) \\ &= \begin{bmatrix} | & | & & | \\ T(\mathbf{e}_1) & T(\mathbf{e}_2) & \dots & T(\mathbf{e}_n) \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A\mathbf{x} \end{aligned}$$

Thus, for every $\mathbf{x}$ in $\mathbb{R}^n$, we have $T(\mathbf{x}) = A\mathbf{x}$. ■

Theorem ?? shows that every matrix transformation is linear. Theorem ?? states that every linear transformation from $\mathbb{R}^n$ into $\mathbb{R}^m$ is a matrix transformation. We combine these results in a corollary.

Corollary 1. A transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation if and only if it is a matrix transformation.

The results of this section rely on the fact that every vector of $\mathbb{R}^n$ can be written as a unique linear combination of the standard unit vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$. These vectors form the *standard basis* for $\mathbb{R}^n$. We will see in [Matrix of a Linear Transformation with Respect to Arbitrary Bases](#) that the matrix used to represent a linear transformation depends on a choice of basis. Because we are using the standard basis, it is natural to name the matrix in Theorem ?? accordingly.

Definition 6. The matrix in Theorem ?? is known as the *standard matrix of the linear transformation* T .

Example 8. The standard matrix of a linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ such that $T(\mathbf{i}) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $T(\mathbf{j}) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and

$T(\mathbf{k}) = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$ is

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & 4 \end{bmatrix}$$

Example 9. Find the standard matrix of a linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $T(\mathbf{i}) = 2\mathbf{i}$ and $T(\mathbf{j}) = 2\mathbf{j}$.

Explanation. We use the images of $\mathbf{i}$ and $\mathbf{j}$ as columns of the matrix. The standard matrix of T is

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Example 10. Find the standard matrix of a linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ if $T\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 1 \\ 13 \\ -1 \end{bmatrix}$ and $T\left(\begin{bmatrix} -2 \\ 0 \end{bmatrix}\right) =$

$$\begin{bmatrix} -2 \\ 0 \\ -8 \\ 2 \end{bmatrix}.$$

Explanation. In this example we are not given the images of the standard basis vectors $\mathbf{i}$ and $\mathbf{j}$. However, we can find the images of $\mathbf{i}$ and $\mathbf{j}$ by expressing $\mathbf{i}$ and $\mathbf{j}$ as linear combinations of $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 0 \end{bmatrix}$, then apply the fact that T is linear.

Let's start with the easy one.

$$\mathbf{i} = -\frac{1}{2} \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

Therefore, by linearity of T , we have:

$$T(\mathbf{i}) = T\left(-\frac{1}{2} \begin{bmatrix} -2 \\ 0 \end{bmatrix}\right) = -\frac{1}{2} T\left(\begin{bmatrix} -2 \\ 0 \end{bmatrix}\right) = -\frac{1}{2} \begin{bmatrix} -2 \\ 0 \\ -8 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 4 \\ -1 \end{bmatrix}$$

This gives us the first column of the standard matrix for T .

You can solve the vector equation

$$a \begin{bmatrix} 3 \\ 1 \end{bmatrix} + b \begin{bmatrix} -2 \\ 0 \end{bmatrix} = \mathbf{j}$$

to express $\mathbf{j}$ as a linear combination of $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 0 \end{bmatrix}$ as follows:

$$\mathbf{j} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \frac{3}{2} \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

By linearity of T ,

$$\begin{aligned} T(\mathbf{j}) &= T\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \frac{3}{2}\begin{bmatrix} -2 \\ 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) + \frac{3}{2}T\left(\begin{bmatrix} -2 \\ 0 \end{bmatrix}\right) \\ &= \begin{bmatrix} 6 \\ 1 \\ 13 \\ -1 \end{bmatrix} + \frac{3}{2}\begin{bmatrix} -2 \\ 0 \\ -8 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 2 \end{bmatrix} \end{aligned}$$

This gives us the second column of the standard matrix. Putting all of the information together, we get the following standard matrix for T :

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 4 & 1 \\ -1 & 2 \end{bmatrix}$$

Practice Problems

Problem 12 Suppose that a linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is such that $T(\mathbf{i}) = \begin{bmatrix} -4 \\ 2 \\ 1 \end{bmatrix}$ and $T(\mathbf{j}) = \begin{bmatrix} 0 \\ -1 \\ 5 \end{bmatrix}$. Find

$$T\left(\begin{bmatrix} 4 \\ -1 \end{bmatrix}\right).$$

$$T\left(\begin{bmatrix} 4 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix}$$

Problem 13 Suppose that a linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is such that $T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) =$

$$\begin{bmatrix} 0 \\ 6 \\ 4 \end{bmatrix}. \text{ Find the standard matrix } A \text{ of } T.$$

$$A = \begin{bmatrix} \dots & \dots \\ \dots & \dots \\ \dots & \dots \end{bmatrix}$$

Problem 14 Find the standard matrix A of the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ if T doubles the x component of every vector and triples the y component.

$$A = \begin{bmatrix} \dots & \dots \\ \dots & \dots \end{bmatrix}$$

Problem 15 Find the standard matrix A of the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ if T reverses the direction of each vector.

$$A = \begin{bmatrix} \cdots & \cdots \\ \cdots & \cdots \end{bmatrix}$$

Problem 16 Find the standard matrix A of the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ if T doubles the length of each vector.

$$A = \begin{bmatrix} \cdots & \cdots \\ \cdots & \cdots \end{bmatrix}$$

Problem 17 Find the standard matrix A of the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ if T projects each vector onto the x -axis. (e.g. $T \left(\begin{bmatrix} 4 \\ 5 \end{bmatrix} \right) = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$)

$$A = \begin{bmatrix} \cdots & \cdots \\ \cdots & \cdots \end{bmatrix}$$

Problem 18 Find the standard matrix A of the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ if T projects each vector onto the y -axis. (e.g. $T \left(\begin{bmatrix} 4 \\ 5 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$)

$$A = \begin{bmatrix} \cdots & \cdots \\ \cdots & \cdots \end{bmatrix}$$

3 Image and Kernel of a Linear Transformation

Note to Student: In this section we will often use U , V and W to denote subspaces of $\mathbb{R}^n$, or any other finite-dimensional vector space, such as those we study in [Vector Spaces](#).

The Image of a Linear Transformation

Definition 7. Let V and W be vector spaces, and let $T : V \rightarrow W$ be a linear transformation. The *image* of T , denoted by $\text{im}(T)$, is the set

$$\text{im}(T) = \{T(\mathbf{v}) : \mathbf{v} \in V\}$$

In other words, the image of T consists of individual images of all vectors of V .

Example 11. Consider the linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ with standard matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$$

1. Find $\text{im}(T)$.
2. Illustrate the action of T with a sketch.

Explanation. ?? Let $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ then

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = a \begin{bmatrix} 1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 2 \\ 4 \end{bmatrix} + c \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Thus, every element of the image can be written as a linear combination of the columns of A . We conclude that

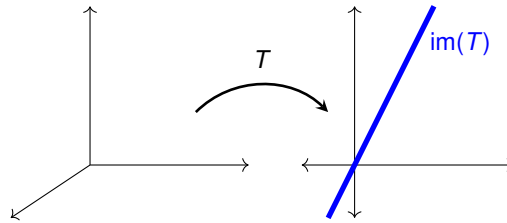
$$\text{im}(T) = \text{span} \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right) = \text{col}(A)$$

Every column of A is a scalar multiple of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Thus,

$$\text{im}(T) = \text{span} \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right) = \text{span} \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$$

The image of T is a line in $\mathbb{R}^2$ determined by the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

?? The action of T can be illustrated with a sketch.



In Example ?? we observed that the image of the linear transformation was equal to the column space of its standard matrix. In general, it is easy to see that if $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation with standard matrix A then the following relationship holds:

$$\text{im}(T) = \text{col}(A)$$

In addition, by Theorem ??, we know that

$$\dim(\text{im}(T)) = \dim(\text{col}(A)) = \text{rank}(A)$$

Example 12. Let $T : \mathbb{R}^5 \rightarrow \mathbb{R}^4$ be a linear transformation with standard matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & -1 & 0 \\ -1 & 3 & 1 & 0 & -1 \\ 3 & 0 & 0 & 3 & 6 \\ 1 & -1 & 1 & -2 & -1 \end{bmatrix}$$

Find $\text{im}(T)$ and $\dim(\text{im}(T))$.

Explanation. As in Example ??, the image of T is given by

$$\text{im}(T) = \text{span} \left(\begin{bmatrix} 1 \\ -1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 6 \\ -1 \end{bmatrix} \right) = \text{col}(A)$$

This time it is harder to detect the vectors that can be eliminated from the spanning set without affecting the span. We have to rely on the reduced row-echelon form of A .

$$\begin{bmatrix} 1 & 2 & 2 & -1 & 0 \\ -1 & 3 & 1 & 0 & -1 \\ 3 & 0 & 0 & 3 & 6 \\ 1 & -1 & 1 & -2 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We can see that $\text{rank}(A) = 3$, so $\dim(\text{im}(T)) = 3$.

To identify vectors that span $\text{im}(T)$, we turn to Procedure ?. We identify the first three columns as pivot columns. These columns are linearly independent and span $\text{col}(A)$. Therefore,

$$\text{im}(T) = \text{col}(A) = \text{span} \left(\begin{bmatrix} 1 \\ -1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right)$$

By Theorem ?? and Definition ??, we know that for an $m \times n$ matrix A , $\text{col}(A)$ is a subspace of $\mathbb{R}^m$. However, when vector spaces other than $\mathbb{R}^m$ are involved, it is not yet clear that $\text{im}(T)$ is a subspace of the codomain. The following theorem resolves this issue.

Theorem 6. Let $T : V \rightarrow W$ be a linear transformation. Then $\text{im}(T)$ is a subspace of W .

Proof To show that $\text{im}(T)$ is a subspace, we need to show that $\text{im}(T)$ is closed under addition and scalar multiplication.

Suppose $\mathbf{w}_1$ and $\mathbf{w}_2$ are in $\text{im}(T)$. Then there are vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in V such that $T(\mathbf{v}_1) = \mathbf{w}_1$ and $T(\mathbf{v}_2) = \mathbf{w}_2$. Then

$$\mathbf{w}_1 + \mathbf{w}_2 = T(\mathbf{v}_1) + T(\mathbf{v}_2) = T(\mathbf{v}_1 + \mathbf{v}_2)$$

This shows that $\mathbf{w}_1 + \mathbf{w}_2$ is in $\text{im}(T)$.

For any scalar a , we have:

$$a\mathbf{w}_1 = aT(\mathbf{v}_1) = T(a\mathbf{v}_1)$$

This shows that $a\mathbf{w}_1$ is in $\text{im}(T)$. ■

We can now define the rank of a linear transformation.

Definition 8. The *rank* of a linear transformation $T : V \rightarrow W$, is the dimension of the image of T .

$$\text{rank}(T) = \dim(\text{im}(T))$$

This definition gives us the following relationship between the rank of a linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and the rank of the standard matrix A associated with it.

Formula 1.

$$\text{rank}(A) = \dim(\text{col}(A)) = \dim(\text{im}(T)) = \text{rank}(T)$$

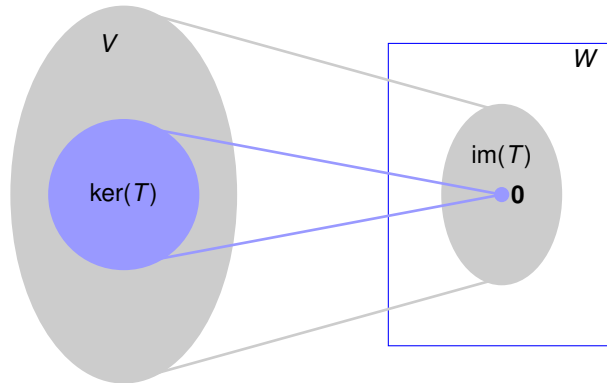
The Kernel of a Linear Transformation

Definition 9. Let V and W be vector spaces, and let $T : V \rightarrow W$ be a linear transformation. The *kernel* of T , denoted by $\ker(T)$, is the set

$$\ker(T) = \{\mathbf{v} : T(\mathbf{v}) = \mathbf{0}\}$$

In other words, the kernel of T consists of all vectors of V that map to $\mathbf{0}$ in W .

It is important to pay attention to the locations of the kernel and the image. We already proved that $\text{im}(T)$ is a subspace of the codomain. In contrast, $\ker(T)$ is located in the domain. (We will prove shortly that it is a subspace of the domain.)



Example 13. Let $T : \mathbb{R}^5 \rightarrow \mathbb{R}^4$ be a linear transformation with standard matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & -1 & 0 \\ -1 & 3 & 1 & 0 & -1 \\ 3 & 0 & 0 & 3 & 6 \\ 1 & -1 & 1 & -2 & -1 \end{bmatrix}$$

1. Find $\ker(T)$
2. Is $\ker(T)$ a subspace of $\mathbb{R}^5$? If so, find $\dim(\ker(T))$.

Explanation. ?? To find the kernel of T , we need to find all vectors of $\mathbb{R}^5$ that map to $\mathbf{0}$ in $\mathbb{R}^4$. This amounts to solving the equation $A\mathbf{x} = \mathbf{0}$.

Gauss-Jordan elimination yields:

$$\begin{bmatrix} 1 & 2 & 2 & -1 & 0 \\ -1 & 3 & 1 & 0 & -1 \\ 3 & 0 & 0 & 3 & 6 \\ 1 & -1 & 1 & -2 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, the kernel of T consists of all elements of the form:

$$\begin{bmatrix} -1 \\ -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -2 \\ -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} t$$

We conclude that

$$\ker(T) = \text{span} \left(\begin{bmatrix} -1 \\ -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right)$$

?? Since $\ker(T)$ is the span of two vectors of $\mathbb{R}^5$, we know that $\ker(T)$ is a subspace of $\mathbb{R}^5$. (See Theorem ??.) Observe that the two vectors in the spanning set are linearly independent. (How can we see this without performing computations?) Therefore $\dim(\ker(T)) = 2$.

Recall that the *null space* of a matrix A is defined to be set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$. This means that if $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation with standard matrix A then

$$\ker(T) = \text{null}(A)$$

We know that $\text{null}(A)$ of an $m \times n$ matrix is a subspace of $\mathbb{R}^n$. (See Theorem ??.) We conclude this section by showing that even when vector spaces other than $\mathbb{R}^n$ are involved, the kernel of a linear transformation is a subspace of the domain of the transformation.

Theorem 7. Let $T : V \rightarrow W$ be a linear transformation, then $\ker(T)$ is a subspace of V .

Proof To show that $\ker(T)$ is a subspace, we need to show that $\ker(T)$ is closed under addition and scalar multiplication.

Suppose that $\mathbf{v}_1$ and $\mathbf{v}_2$ are in $\ker(T)$. Then,

$$T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2) = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

This shows that $\mathbf{v}_1 + \mathbf{v}_2$ is in $\ker(T)$.

For any scalar a we have:

$$T(a\mathbf{v}_1) = aT(\mathbf{v}_1) = a\mathbf{0} = \mathbf{0}$$

This shows that $a\mathbf{v}_1$ is in $\ker(T)$. ■

Definition 10. The *nullity* of a linear transformation $T : V \rightarrow W$, is the dimension of the kernel of T .

$$\text{nullity}(T) = \dim(\ker(T))$$

This definition gives us the following relationship between nullity of a linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and the nullity of the standard matrix A associated with it.

Formula 2.

$$\text{nullity}(A) = \dim(\text{null}(A)) = \dim(\ker(T)) = \text{nullity}(T)$$

Rank-Nullity Theorem for Linear Transformations

In Examples ?? and ??, we found the image and the kernel of the linear transformation $T : \mathbb{R}^5 \rightarrow \mathbb{R}^4$ with standard matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & -1 & 0 \\ -1 & 3 & 1 & 0 & -1 \\ 3 & 0 & 0 & 3 & 6 \\ 1 & -1 & 1 & -2 & -1 \end{bmatrix}$$

We also found that

$$\text{rank}(T) = \dim(\text{im}(T)) = \dim(\text{col}(A)) = \text{rank}(A) = 3$$

and

$$\text{nullity}(T) = \dim(\ker(T)) = \dim(\text{null}(A)) = \text{nullity}(A) = 2$$

Because of Rank-Nullity Theorem for matrices (Theorem ??), it is not surprising that

$$\text{rank}(T) + \text{nullity}(T) = 3 + 2 = 5 = \dim(\mathbb{R}^5)$$

The following theorem is a generalization of this result.

Theorem 8. Let $T : V \rightarrow W$ be a linear transformation. Suppose $\dim(V) = n$, then

$$\text{rank}(T) + \text{nullity}(T) = n$$

Proof By Theorem ??, $\text{im}(T)$ is a subspace of W . There exists a basis for $\text{im}(T)$ of the form $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_r)\}$. By Theorem ??, $\ker(T)$ is a subspace of V . Let $\{\mathbf{u}_1, \dots, \mathbf{u}_s\}$ be a basis for $\ker(T)$.

We will show that $\{\mathbf{u}_1, \dots, \mathbf{u}_s, \mathbf{v}_1, \dots, \mathbf{v}_r\}$ is a basis for V .

For any vector $\mathbf{v}$ in V , we have:

$$T(\mathbf{v}) = c_1 T(\mathbf{v}_1) + \dots + c_r T(\mathbf{v}_r)$$

for some scalars c_i ($1 \leq i \leq r$). Thus,

$$T(\mathbf{v}) - (c_1 T(\mathbf{v}_1) + \dots + c_r T(\mathbf{v}_r)) = \mathbf{0}$$

By linearity,

$$T(\mathbf{v} - (c_1 \mathbf{v}_1 + \dots + c_r \mathbf{v}_r)) = \mathbf{0}$$

Therefore $\mathbf{v} - (c_1 \mathbf{v}_1 + \dots + c_r \mathbf{v}_r)$ is in $\ker(T)$.

Hence there are scalars a_i ($1 \leq i \leq s$) such that

$$\mathbf{v} - (c_1 \mathbf{v}_1 + \dots + c_r \mathbf{v}_r) = a_1 \mathbf{u}_1 + \dots + a_s \mathbf{u}_s$$

Thus,

$$\mathbf{v} = (c_1 \mathbf{v}_1 + \dots + c_r \mathbf{v}_r) + (a_1 \mathbf{u}_1 + \dots + a_s \mathbf{u}_s)$$

We conclude that

$$V = \text{span}(\mathbf{u}_1, \dots, \mathbf{u}_s, \mathbf{v}_1, \dots, \mathbf{v}_r)$$

Now we need to show that $\{\mathbf{u}_1, \dots, \mathbf{u}_s, \mathbf{v}_1, \dots, \mathbf{v}_r\}$ is linearly independent.

Suppose

$$c_1 \mathbf{v}_1 + \dots + c_r \mathbf{v}_r + a_1 \mathbf{u}_1 + \dots + a_s \mathbf{u}_s = \mathbf{0} \tag{7}$$

Applying T to both sides, we get

$$T(c_1 \mathbf{v}_1 + \dots + c_r \mathbf{v}_r + a_1 \mathbf{u}_1 + \dots + a_s \mathbf{u}_s) = T(\mathbf{0})$$

$$c_1 T(\mathbf{v}_1) + \dots + c_r T(\mathbf{v}_r) + a_1 T(\mathbf{u}_1) + \dots + a_s T(\mathbf{u}_s) = \mathbf{0}$$

But $T(\mathbf{u}_i) = \mathbf{0}$ for $1 \leq i \leq s$, thus

$$c_1 T(\mathbf{v}_1) + \dots + c_r T(\mathbf{v}_r) = \mathbf{0}$$

Since $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_r)\}$ is linearly independent, it follows that each $c_i = 0$.

But then Equation (??) implies that $a_1 \mathbf{u}_1 + \dots + a_s \mathbf{u}_s = \mathbf{0}$. Because $\{\mathbf{u}_1, \dots, \mathbf{u}_s\}$ is linearly independent, it follows that each $a_i = 0$.

We conclude that $\{\mathbf{u}_1, \dots, \mathbf{u}_s, \mathbf{v}_1, \dots, \mathbf{v}_r\}$ is a basis for V . Thus,

$$\dim(\ker(T)) + \dim(\operatorname{im}(T)) = s + r = n$$

■

Practice Problems

Problem 19 Describe the image and find the rank of the linear transformation $T : \mathbb{R}^5 \rightarrow \mathbb{R}^2$ induced by $A = \begin{bmatrix} 3 & 2 & 4 & 7 & 1 \\ -1 & -9 & 7 & 6 & 8 \end{bmatrix}$.

Multiple Choice:

1. $\operatorname{im}(T) = \mathbb{R}^2$
2. $\operatorname{im}(T)$ is a line in $\mathbb{R}^2$
3. $\operatorname{im}(T) = \{\mathbf{0}\}$
4. $\operatorname{im}(T) = \mathbb{R}^5$
5. $\operatorname{im}(T)$ is a plane in $\mathbb{R}^5$

$\operatorname{rank}(T) = \dots$

Problem 20 Describe the image and find the rank of the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ induced by $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$

Multiple Choice:

1. $\operatorname{im}(T) = \mathbb{R}^3$
2. $\operatorname{im}(T)$ is a line in $\mathbb{R}^2$
3. $\operatorname{im}(T)$ is a line in $\mathbb{R}^3$
4. $\operatorname{im}(T) = \{\mathbf{0}\}$
5. $\operatorname{im}(T)$ is a plane in $\mathbb{R}^3$

$$\text{rank}(T) = \dots$$

Problem 21 Suppose linear transformations $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $S : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ are such that $\text{im}(T) = \text{im}(S) = \text{span} \left(\begin{bmatrix} 1 \\ -3 \end{bmatrix} \right)$. Does this mean that T and S are the same transformation? Justify your claim.

Problem 22 Describe the kernel and find the nullity of the linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ induced by $A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 1 & -3 \end{bmatrix}$.

Multiple Choice:

1. $\ker(T) = \mathbb{R}^3$
2. $\ker(T) = \{\mathbf{0}\}$
3. $\ker(T) = \mathbb{R}^2$
4. $\ker(T)$ is a plane in $\mathbb{R}^3$
5. $\ker(T)$ is a line in $\mathbb{R}^3$

$$\text{nullity}(T) = \dots$$

Problem 23 Describe the kernel and find the nullity of the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ induced by $A = \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}$.

Multiple Choice:

1. $\ker(T) = \mathbb{R}^2$
2. $\ker(T) = \{\mathbf{0}\}$
3. $\ker(T)$ is a line in $\mathbb{R}^2$

$$\text{nullity}(T) = \dots$$

Problem 24 Describe the kernel and find the nullity of the linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^5$ induced by $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & -1 \\ 1 & 2 & -1 \\ 1 & 2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$

Multiple Choice:

1. $\ker(T)$ is a plane in $\mathbb{R}^3$
2. $\ker(T)$ is a line in $\mathbb{R}^3$
3. $\ker(T)$ is a line in $\mathbb{R}^5$
4. $\ker(T) = \mathbb{R}^3$
5. $\ker(T) = \{\mathbf{0}\}$

$\text{nullity}(T) = \dots$

Problem 25 Suppose a linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is such that $\text{im}(T)$ is a plane in $\mathbb{R}^3$. Then

$$\text{rank}(T) = \dots$$

$$\text{nullity}(T) = \dots$$

Problem 26 Suppose a linear transformation $T : \mathbb{R}^5 \rightarrow \mathbb{R}^5$ is such that $T(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v}$ in $\mathbb{R}^5$. Then

$$\text{rank}(T) = \dots$$

$$\text{nullity}(T) = \dots$$

Problem 27 Let $T : \mathbb{R}^6 \rightarrow \mathbb{R}^4$ be a linear transformation with standard matrix

$$A = \begin{bmatrix} 2 & -1 & 1 & -2 & 1 & 1 \\ 1 & 2 & 3 & 6 & -4 & 1 \\ 0 & 2 & 2 & 4 & -2 & -1 \\ 1 & 3 & 2 & 6 & -3 & 2 \end{bmatrix}$$

Find $\text{im}(T)$ and $\ker(T)$ if the reduced row-echelon form of A is

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 28 Let $V = \text{span} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$, and let $T : V \rightarrow \mathbb{R}^2$ be a linear transformation defined by $T(\mathbf{v}) = 2\mathbf{v}$. Find $\text{im}(T)$ and $\ker(T)$.

Problem 29 Suppose a linear transformation T is induced by a 4×6 matrix A . Let S be a linear transformation induced by A^T . Find $\text{nullity}(S)$, if $\text{nullity}(T) = 3$. Prove your claim.