Homework 1

Homework 1 for MATH 128A.

MATH 128A: Homework 1 **Due date: Thursday, April 9th**

Instructions. You will submit your solutions on Gradescope, which you can access through our Canvas course page by clicking on the assignment.

You are welcome to submit your homework either handwritten or in LaTeX. If you choose to submit handwritten work, please use pen rather than pencil, and make sure that both your handwriting and your scans are as readable as possible in order to facilitate grading.

Please also feel free to attend office hours if you would like hints on the homework or if you would like to discuss more advanced material.

You will see that our Ximera course page proposes you to see the answers, but this will only become available once I publish the solution.

Exercise 1 ***Multiple Choice (10 points).*** *Which of the following is* not *one of the six parts of a proposition discussed in class?*

Multiple Choice:

(a) *Enunciation*

(b) *Exposition*

(c) *Construction*

(d) *Corollary* ✓

Answer. *The correct answer is* Corollary. *The six parts discussed in class were:*

Enunciation, Exposition, Specification, Construction, Proof, Conclusion.

Exercise 2 ***Multiple Choice (10 points).*** *According to the discussion in class, what is the main gap in Euclid's proof of Proposition I.1?*

Multiple Choice:

(a) *Euclid never uses Common Notion 1.*

Author(s):

(b) *Euclid does not justify why the two circles intersect.* ✓

(c) *Euclid assumes that all triangles are equilateral.*

(d) *Euclid forgets to draw the segment AB.*

Answer. *The correct answer is that* Euclid does not justify why the two circles intersect. *His construction of an equilateral triangle on a given segment AB requires the two circles centered at A and B with radius AB to meet, but this is not guaranteed by the postulates stated so far.*

Exercise 3 ***Multiple Choice (10 points).*** *Which of the following was presented in class as a primitive geometric form?*

Multiple Choice:

(a) *Circle*

(b) *Triangle*

(c) *Between* ✓

(d) *Angle*

Answer. *The correct answer is* Between. *In class, primitive notions included objects such as points and lines, together with primitive relations such as betweenness.*

Problem 4 ***Construction of Proposition I.3 (20 points).***

Using GeoGebra, GeoGebra Classic, or pen and paper, perform the construction of Proposition I.3.

If you use pen and paper, include a drawing of your construction with your homework.

If you submit your homework in LaTeX, you may either include an image of your construction (for example, as a `.png` *file), or share your construction using the sharing option provided by GeoGebra. In that case, your answer may simply be a link to your construction.*

Answer. *https://www.geogebra.org/classic/mdyp9mnc*

Remark. *Observe how, when you move the initial points A, B, C, and D, the output (named f on my solution) moves with it. In general, when using GeoGebra, your inputs have the color blue. Everything else is something constructed from this initial data, and takes the color grey. If you construction is correct, your final construction point should move when you move your initial points. If it does not, please reach out to me as there is likely a gap in your understanding of GeoGebra.*

Answer. We choose Proposition I.5.

Enunciation. In isosceles triangles the angles at the base equal one another, and, if the equal straight lines are produced further, then the angles under the base equal one another.

Exposition. Let ABC be an isosceles triangle having the side AB equal to the side AC, and let the straight lines BD and CE be produced further in a straight line with AB and AC.

Specification. I say that the angle ABC equals the angle ACB, and the angle CBD equals the angle BCE.

Construction. Take an arbitrary point F on BD. Cut off AG from AE, the greater, equal to AF, the less, by Proposition I.3, and join the straight lines FC and GB.

Proof. Since AF equals AG, and AB equals AC, therefore the two sides FA and AC equal the two sides GA and AB, respectively, and they contain a common angle, the angle FAG.

Therefore the base FC equals the base GB, the triangle AFC equals the triangle AGB, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides, that is, the angle ACF equals the angle ABG, and the angle AFC equals the angle AGB, by Proposition I.4.

Since the whole AF equals the whole AG, and in these AB equals AC, therefore the remainder BF equals the remainder CG, by Common Notion 3.

But FC was also proved equal to GB, therefore the two sides BF and FC equal the two sides CG and GB respectively, and the angle BFC equals the angle CGB, while the base BC is common to them. Therefore the triangle BFC also equals the triangle CGB, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides. Therefore the angle FBC equals the angle GCB, and the angle BCF equals the angle CBG, by Proposition I.4.

Accordingly, since the whole angle ABG was proved equal to the angle ACF, and in these the angle CBG equals the angle BCF, the remaining angle ABC equals the remaining angle ACB, by Common Notion 3, and they are at the base of the triangle ABC. But the angle FBC was also proved equal to the angle GCB, and they are under the base.

Conclusion. Therefore in isosceles triangles the angles at the base equal one another, and, if the equal straight lines are produced further, then the angles under the base equal one another.

Comment on rigor. I do not see an actual logical gap in Euclid's proof of Proposition I.5. Rather, the interesting point is that a modern reader may

be tempted to derive the equality of the exterior angles from the equality of the interior base angles by subtracting from straight angles. However, Euclid cannot argue this way here, because the relevant result about adjacent angles on a straight line only appears later in Proposition I.13, and using it here would be circular. So the proof is not missing a step; instead, it is carefully written to avoid relying on facts that have not yet been established.

Problem 5 ***Axiomatic theories and models (30 points total).***

(a) ***(15 points)*** *In your own words, what is the difference between an* axiomatic theory *and a* model *for an axiomatic theory?*

Answer. *An axiomatic theory is a collection of primitive terms together with axioms describing how those terms are supposed to behave. A model for an axiomatic theory is a concrete interpretation of those primitive terms in which all the axioms are true.*

For example, a theory might use the primitive words point *and* line *and say that through any two distinct points there passes a line. A model would then specify what the points and lines actually are, and verify that this statement holds in that setting.*

(b) ***(15 points)*** *Can you imagine a model with three elements for the axiomatic theory consisting only of the primitive terms* point, line, *and Postulate 1?*

If so, describe your model clearly by explaining:

(i) *what the points are,*
(ii) *what the lines are,*
(iii) *why Postulate 1 is satisfied.*

Answer. *Yes.*

Let the points be three distinct objects:

$$P, \quad Q, \quad R.$$

Let the lines be the sets

$$\{P, Q\}, \qquad \{P, R\}, \qquad \{Q, R\}.$$

In other words, each line consists of exactly two of the three points.

Then Postulate 1 is satisfied, because given any two distinct points, there is a line containing them:

- *the points P and Q lie on the line $\{P, Q\}$,*
- *the points P and R lie on the line $\{P, R\}$,*
- *the points Q and R lie on the line $\{Q, R\}$.*

So this gives a model with three points and three lines satisfying Postulate 1.