Homework 2

Homework 2 for MATH 128A.

MATH 128A: Homework 2 **Due date: Thursday, April 16**

Instructions. You will submit your solutions on Gradescope, which you can access through our Canvas course page by clicking on the assignment.

You are welcome to submit your homework either handwritten or in LaTeX. If you choose to submit handwritten work, please use pen rather than pencil, and make sure that both your handwriting and your scans are as readable as possible in order to facilitate grading.

Please also feel free to attend office hours if you would like hints on the homework or if you would like to discuss more advanced material. Also, you are highly encouraged to work in groups!

You will see that our Ximera course page proposes you to see the answers, but this will only become available once I publish the solution.

Problem 1 ***Parallelograms (45 points total).***

A quadrilateral is a figure determined by four distinct points A, B, C, D such that no three lie on the same straight line, and the segments AB, BC, CD, and DA intersect only at their endpoints.

A quadrilateral $ABCD$ is called a parallelogram *if its opposite sides are parallel, that is,*

$$(AB) \parallel (CD) \qquad \text{and} \qquad (BC) \parallel (AD).$$

Hint: *For these exercises, you should always start with a diagram. Then, you may rely on:*

- *the congruence theorems I.4 (SAS), I.8 (SSS), and I.26 (AAS/ASA),*
- *the angle equality I.15 (vertical angles),*
- *the angle properties of parallel lines I.27, I.28, and I.29.*

How we will grade Problem 1. *We will give some credit for a correct initial diagram, some credit for the method, some credit for the correctness of the argument, and some credit for proving the desired conclusion. In particular, we will look at how well you use the information given in the problem together with postulates and previously proved propositions to deduce new consequences.*

If you use a proposition, make sure to explain why its hypotheses are satisfied before applying it.

Author(s):

(a) ***(15 points)*** *Show that if $ABCD$ is a parallelogram, then*

$$AB = CD \qquad \text{and} \qquad BC = AD.$$

Solution: GeoGebra Link

Draw the diagonal AC.

Since $ABCD$ is a parallelogram, we have $(AB) \parallel (CD)$ and $(BC) \parallel (AD)$. Because the line AC meets the parallel lines (AB) and (CD), Proposition I.29 gives

$$\angle BAC = \angle DCA.$$

Likewise, since the line AC meets the parallel lines (BC) and (AD), Proposition I.29 gives

$$\angle BCA = \angle DAC.$$

Also, the side AC is common to triangles BAC and DCA.

Therefore triangles BAC and DCA have two equal angles and the included side equal, so by Proposition I.26 (ASA) they are congruent. Hence the corresponding sides are equal:

$$AB = CD \qquad \text{and} \qquad BC = AD.$$

This is what we wanted to prove.

(b) ***(15 points)*** *Let $ABCD$ be a parallelogram. Show that the diagonals AC and BD intersect at their midpoint.*

You may assume that AC and BD intersect at a point E, and you need to show

$$AE = EC \qquad \text{and} \qquad BE = ED.$$

Solution: GeoGebra Link

Let E be the intersection point of the diagonals AC and BD.

Consider triangles AEB and CED. Since the diagonals intersect, the angles $\angle AEB$ and $\angle CED$ are vertical angles, so by Proposition I.15,

$$\angle AEB = \angle CED.$$

Since $(AB) \parallel (CD)$ and the line BD meets them, Proposition I.29 gives

$$\angle ABE = \angle CDE.$$

From part (a), opposite sides of a parallelogram are equal, so

$$AB = CD.$$

Therefore triangles AEB and CED have two equal angles and one corresponding side equal, so by Proposition I.26 (AAS) they are congruent. Hence corresponding sides are equal:

$$AE = EC \qquad \text{and} \qquad BE = ED.$$

Thus the diagonals bisect one another, so they intersect at their midpoint.

(c) ***(15 points)*** *(Converse) Let $ABCD$ be a quadrilateral such that*

$$AB = CD \qquad \text{and} \qquad BC = AD.$$

Show that $ABCD$ is a parallelogram.

Solution: GeoGebra Link

Draw the diagonal AC.

In triangles ABC and CDA, we are given

$$AB = CD \qquad \text{and} \qquad BC = AD,$$

and the side AC is common to both triangles. Therefore, by Proposition I.8 (SSS), triangles ABC and CDA are congruent.

Hence corresponding angles are equal:

$$\angle BAC = \angle DCA \qquad \text{and} \qquad \angle BCA = \angle DAC.$$

Now the line AC meets the lines (AB) and (CD), and the alternate interior angles $\angle BAC$ and $\angle DCA$ are equal. Therefore, by Proposition I.27, the lines (AB) and (CD) are parallel.

Similarly, the line AC meets the lines (BC) and (AD), and the alternate interior angles $\angle BCA$ and $\angle DAC$ are equal. Therefore, again by Proposition I.27,

$$(BC) \parallel (AD).$$

Since both pairs of opposite sides are parallel, $ABCD$ is a parallelogram.

Problem 2 ***Reading for the Pythagorean Theorem (25 points).***

Read Propositions I.34, I.35, I.37, I.41, I.46, and I.47 in Book I of Euclid's Elements.

Important. *In these propositions, Euclid uses the word "equal" to refer to* area. *Two figures are said to be equal if they can be subdivided into smaller pieces which are themselves equal.*

Choose three *propositions among*

I.34, I.35, I.37, I.41, I.46, I.47,

and for each of your choices, draw a summary diagram *that helps you remember the statement.*

How we will grade Problem 2. *We will give full credit for a clean diagram with clear markings introducing the relevant notations, together with a brief piece of text stating the hypotheses and the conclusion, in the style used during Week 2 Lecture 1.*

Remark: *You are encouraged to be creative. For example, you may use different colors (or shading) to highlight the different regions in each figure. If you do not want to use color, you may instead distinguish the regions by using different patterns.*

For instance, Proposition I.34 asserts that in a parallelogram $ABCD$, the triangles ABC and CDA are equal, in the sense that they have the same area. You might highlight the region ABC in blue and the region CDA in red, and then summarize the proposition by writing something like

$$ABCD \text{ parallelogram} \overset{I.34}{\Longrightarrow} \text{(blue region)} = \text{(red region)}.$$

If you prefer not to use color, you could instead mark one region with one pattern and the other region with a different pattern, and write the same kind of conclusion.

Solution: *Since this problem asks for diagrams, below I write the short text that can accompany three summary diagrams.*

Choice 1: Proposition I.34.

Hypotheses: *$ABCD$ is a parallelogram, and the diagonal AC is drawn.*

Conclusion: *The diagonal cuts the parallelogram into two equal triangles:*

$$\triangle ABC = \triangle CDA \qquad \text{(equal in area)}.$$

In the diagram, one can shade $\triangle ABC$ one way and $\triangle CDA$ another way. GeoGebra Link

Choice 2: Proposition I.41.

Hypotheses: *A parallelogram and a triangle stand on the same base and lie between the same parallels.*

Conclusion: *The parallelogram has twice the area of the triangle:*

$$\text{area of parallelogram} = 2 \cdot \text{area of triangle}.$$

In the diagram, one can mark the whole parallelogram with one pattern and the triangle with another, and then write that the parallelogram is double the triangle. GeoGebra Link

Choice 3: Proposition I.47.

Hypotheses: *ABC is a right triangle with right angle at A, and squares are constructed on the three sides.*

Conclusion: *The square on the side opposite the right angle equals the sum of the squares on the two legs:*

$$\text{square on } BC = \text{square on } AB + \text{square on } AC.$$

In the diagram, one can draw the three squares and mark the two smaller ones with different patterns, indicating that together they equal the large square. *GeoGebra Link*

Problem 3 ***Straight-edge and compass constructions (30 points total).***

There will be a compass-and-straightedge construction problem on the midterm, to be done by hand (no GeoGebra).

One thing you may have noticed is that Euclid does not hesitate to reuse constructions that were established earlier. This is perfectly natural, but some of these previously established constructions require many steps. There is another kind of exercise, closer in spirit to a puzzle, which asks you to carry out a construction in as few moves as possible using only a straightedge and compass.

We first need to explain how moves are counted.

- *Drawing a new line counts as one move.*
- *Drawing a new circle counts as one move.*
- *Extending a line already drawn on the paper does* not *count as a move.*
- *Choosing points at random on the paper does* not *count as a move.*
- *Naming intersection points does* not *count as a move.*

Thus, only actual uses of the straightedge to draw a new line and actual uses of the compass to draw a new circle count as moves.

The goal in the following exercises is to complete each construction in as few moves as possible. You will still receive full credit if your construction is correct but does not use the minimum possible number of moves.

We will sometimes give a par *value for a construction. This is the typical number of moves an experienced constructor might use. By thinking harder, you may sometimes do better.*

For example, the construction of an equilateral triangle on a given segment AB takes 4 moves in this counting system:

(a) *draw the circle with center A and radius AB;*

(b) *draw the circle with center B and radius BA;*

(c) *let C be an intersection point of the two circles, and draw the segment AC;*

(d) *draw the segment BC.*

Then ABC is the required triangle.

In particular, the segment AB is considered to be already given: you should draw it on the paper, but you should not count it as a move.

How we will grade Problem 3. *We will give points for the following:*

- *all the lines and circles used in the construction are shown;*
- *the diagram has clear markings;*
- *the number of moves is stated;*
- *a short proof is provided;*
- *the construction achieves the required goal.*

Using this convention, do the following constructions in as few moves as possible.

(a) **(10 points)** Given an angle $\angle BAC$, construct its bisector. (par = 4)

Solution: Construction (4 moves). GeoGebra Link

(i) Draw a circle centered at A meeting the rays AB and AC at points D and E.

(ii) Draw a circle centered at D with radius DE.

(iii) Draw a circle centered at E with radius ED.

(iv) Let F be an intersection point of the two circles from steps 2 and 3. Draw the line AF.

Number of moves: 4.

Proof. Since $AD = AE$ (they are radii of the circle centered at A), and since $DF = EF$ (they are radii of the circles centered at D and E), and since AF is common, triangles ADF and AEF are congruent by Proposition I.8. Therefore

$$\angle DAF = \angle FAE.$$

But D lies on the ray AB and E lies on the ray AC, so

$$\angle BAF = \angle FAC.$$

Hence the line AF bisects the angle $\angle BAC$.

(b) **(10 points)** Given a segment AB, construct its midpoint. (par = 3)

Solution: Construction (3 moves). GeoGebra Link

(i) Draw the circle centered at A with radius AB.
(ii) Draw the circle centered at B with radius BA.
(iii) Let C and D be the two intersection points of these circles. Draw the line CD, and let M be the intersection point of CD with AB.

Number of moves: 3.

Proof. Because C and D lie on circles with equal radii centered at A and B, we have

$$AC = CB = BD = DA.$$

In particular, $ACBD$ is a parallelogram thanks to Problem 1 (c). Thus, by Problem 1 (b), its diagonals intersect at their midpoint. Hence, E is the midpoint of AB.

(c) **(10 points)** Given a line ℓ and a point A on ℓ, construct the line perpendicular to ℓ through A. (par = 4; possibly 3 according to Hartshorne)

Solution: Construction (4 moves). GeoGebra Link

(i) Draw a circle centered at A meeting the line ℓ at two points B and C.
(ii) Draw a circle centered at B with radius BC.
(iii) Draw a circle centered at C with radius CB.
(iv) Let D be an intersection point of the circles from steps 2 and 3. Draw the line AD.

Number of moves: 4.

Proof. By Construction, A is the midpoint of BC, so $AB = BC$. Then, D is at the intersection of two circles with equal radii, so $CD = BD$. Finally, the side AD is common to both BAD and CAD. Thus, by the SSS Theorem, $\angle BAD = \angle CAD$, and, since A, B, and C lie on the same line, $\angle BAD$ and $\angle CAD$ are right angles. Hence, since the line (AD) is perpendicular to ℓ.

Problem 4 ***Reflection (2 bonus points).***

Reflect on the first three weeks of the course.

(a) *What did you find most challenging so far?*

(b) *How did you find the pace of the lectures: too fast, too slow, or just right?*

(c) *Is there something you wish we did more of, or less of?*

Please answer in a few complete sentences. Your responses will be anonymous.