Homework 3

Homework 3 for MATH 128A.

MATH 128A: Homework 3 **Due date: Thursday, April 23**

Instructions. You will submit your solutions on Gradescope, which you can access through our Canvas course page by clicking on the assignment.

You are welcome to submit your homework either handwritten or in LaTeX. If you choose to submit handwritten work, please use pen rather than pencil, and make sure that both your handwriting and your scans are as readable as possible in order to facilitate grading.

Please also feel free to attend office hours if you would like hints on the homework or if you would like to discuss more advanced material. Also, you are highly encouraged to work in groups!

You will see that our Ximera course page proposes you to see the answers, but this will only become available once I publish the solution.

Problem 1 ***Proofs in circle geometry (60 points total).***

Hint: *For these exercises, you should always start with a diagram. Then, you may rely on:*

- *the congruence theorems I.4 (SAS), I.8 (SSS), I.26 (ASA/AAS), and RASS,*
- *Proposition I.32 on the angle-sum in a triangle,*
- *the angle properties of circles discussed in class, especially III.21 and III.32,*
- *the fact that, if two circles are tangent, then their centers and point of tangency are aligned (III.11 and III.12).*

How we will grade Problem 1. *We will give some credit for a correct initial diagram, some credit for the method, some credit for the correctness of the argument, and some credit for proving the desired conclusion. In particular, we will look at how well you use the information given in the problem together with previously proved propositions to deduce new consequences.*

If you use a proposition, make sure to explain why its hypotheses are satisfied before applying it.

Author(s):

(a) ***(20 points)*** *Show that the sum of opposite angles of a quadrilateral inscribed in a circle is equal to two right angles.*

Idea. *Draw a circle, place four points A, B, C, and D on the circumference of the circle so that $ABCD$ is a quadrilateral.*

Goal. *Show that $\angle A + \angle C = 2$ right angles.*

Hint. *Draw the diagonals AC and BD, and play with the following properties:*

- *the sum of the angles in a triangle is two right angles (Proposition I.32),*
- *given a chord and two points on the circumference on the side of the center, you obtain equal angles (Proposition III.21).*

(b) ***(20 points)*** *Show that all the angle bisectors in a triangle intersect at a common point.*

Idea. *Draw a triangle ABC and draw the angle bisectors at $\angle A$ and $\angle B$, and let D be their intersection point (you may assume without proof that this intersection point exists and is inside the triangle).*

Goal. *Show that DC bisects $\angle C$.*

Hint. *Construct the perpendicular lines through D to:*

- *AB and call E the intersection point,*
- *BC and call F the intersection point,*
- *AC and call G the intersection point.*

Using AAS, show that

$$FD = ED \qquad \text{and} \qquad ED = GD.$$

Deduce that

$$FD = GD.$$

Then use RASS to show that DC bisects $\angle C$.

Remark. *If you draw the circle with center D and radius DE, you obtain the inscribed circle to the triangle ABC. You can try (though I won't grade it), to draw the inscribed circle to a given triangle in as little moves as possible using only a straightedge and a compass, without lifting the compass. (A typical number of moves is* par $= 13$.*)*

(c) ***(20 points)*** *Assume two circles are tangent externally at a point A. Fix points B and C on one of the circles, create the lines (AB) and (AC), and call D and E their respective intersection points with the other circle.*

Show that

$$(BC) \parallel (DE).$$

Hint. *Use Proposition III.32 and the fact that the centers and tangency point are aligned.*

Problem 2 ***Construction (10 points).***

Using only a straightedge and compass, and without lifting the compass, construct the circumscribed circle of a given triangle. (A typical number of moves is par = 7.*)*

Idea. *Draw a triangle ABC. You need to create a point D such that the circle with center D and radius DA also passes by B, and C.*

Hint. *Remember how we found the center of a circle in class using III.1.*

How we will grade Problem 2. *We will give points for the following:*

- *all the lines and circles used in the construction are shown;*
- *the diagram has clear markings;*
- *the number of moves is stated;*
- ***no proof required this time;***
- *the construction achieves the required goal.*

Problem 3 ***Reading: the construction of the pentagon (30 points total).***

Instructions. *Read the attached excerpt from Chapter 1.4 of Hartshorne on the construction of the regular pentagon.*

A remark on this problem. *The construction of the regular pentagon is one of the most beautiful constructions in Euclid's* Elements*, and appears in Book IV. Up to a few additional results, namely Propositions II.11 and III.36–III.37, you now have essentially all the tools needed to read Hartshorne's discussion of the pentagon construction (which is attached to this HW in pdf form and also in a screen-readable form on the course website).*

An important part of becoming a mathematician is learning how to read mathematics on your own. This is especially important if you are interested in research, or if you are interested in teaching and need to read references in order to prepare classes. In this problem, you are asked to read the attached extract from Chapter 1.4 of Hartshorne, to carry out the constructions described in Problems 4.1 and 4.2, and also to provide summary diagrams for Proposition II.11 and for either Proposition III.36 or Proposition III.37.

By the end of this problem, you will have constructed a regular pentagon using only a straightedge and compass. While you will not be graded on the proofs of Propositions II.11 and III.36–III.37, I strongly encourage you to read them as

well. As you will see, these proofs take time and rely on several earlier results from Books II and III.

Important. *For these constructions, you are allowed to use Tool I.2. In other words, you are allowed to keep your compass open when you lift it. For example, if a point A and a segment BC are given, you may open your compass to the length BC, lift it, place one end at A, and draw a circle centered at A with radius equal to BC.*

How we will grade Problem 3. *We will give credit for:*

- *clear and labeled constructions,*
- *the presence of the relevant auxiliary lines and circles from the construction,*
- *evidence that you followed the construction described in the reading,*
- *the correctness and clarity of your summary diagrams.*

(a) ***(10 points)*** *Follow the steps described in Problem 4.1 of the reading to create an isosceles triangle whose base angles are twice the summit angle.*

Remark. *This is also Proposition IV.10 in the* Elements.

(b) ***(10 points)*** *As Hartshorne mentions, Propositions II.11 and III.36–III.37 play an important role in Proposition IV.10. Propose a summary diagram for Proposition II.11 and for either Proposition III.36 or Proposition III.37.*

(c) ***(10 points)*** *Follow the steps described by Hartshorne in Problem 4.2 to construct a regular pentagon.*

Problem 4 ***Reflection (2 bonus points).***

Reflect on the course so far.

(a) *What did you find most challenging in the material from the last two weeks?*

(b) *Which type of exercise is helping you most: proofs, constructions, or reading Euclid and Hartshorne?*

(c) *Is there something you wish we did more of, or less of?*

Please answer in a few complete sentences. Your responses will be anonymous.