Homework 3 Solutions

Solutions to Homework 3 for MATH 128A.

Problem 1 ***Proofs in circle geometry.***

(a) ***Solution:*** GeoGebra Link *Draw the diagonals AC and BD.*

By Proposition III.21, angles standing on the same chord are equal. In particular,

$$\angle DAC = \angle DBC \qquad \text{and} \qquad \angle CAB = \angle CDB.$$

Thus,

$$\angle DAB = \angle DAC + \angle CAB = \angle DBC + \angle CDB.$$

In triangle BCD, Proposition I.32 gives

$$\angle DBC + \angle BCD + \angle CDB = 2 \text{ right angles.}$$

Therefore,

$$\angle DAB + \angle BCD = 2 \text{ right angles.}$$

(b) ***Solution:*** GeoGebra Link *Let D be the intersection of the angle bisectors of $\angle A$ and $\angle B$.*

Construct perpendiculars:

$$DE \perp AB, \quad DF \perp BC, \quad DG \perp AC.$$

Since AD bisects $\angle A$,

$$\angle EAD = \angle DAG,$$

and both $\angle AED$ and $\angle AGD$ are right angles. Hence, by AAS,

$$\triangle AED \cong \triangle AGD,$$

so $DE = DG$.

Similarly, since BD bisects $\angle B$,

$$\triangle BED \cong \triangle BFD,$$

so $DE = DF$.

Thus $DF = DG$.

Author(s):

Now consider right triangles DCF and DCG. We have

$$DF = DG, \quad DC = DC.$$

By RASS,

$$\triangle DCF \cong \triangle DCG,$$

so

$$\angle FCD = \angle DCG.$$

Hence DC bisects $\angle C$.

(c) ***Solution:*** [GeoGebra Link] *Let two circles be tangent externally at A. Let B, C be points on the first circle, and let the lines (AB) and (AC) meet the second circle again at D and E, respectively.*

Let O_1 and O_2 be the centers of the two circles. Since the circles are tangent at A, the points O_1, A, O_2 are collinear. Hence the line through A perpendicular to O_1A (equivalently, to O_2A) is tangent to both circles at A.

Let t denote this common tangent at A.

By Proposition III.32 (the angle between a tangent and a chord equals the angle in the opposite arc), applied to the first circle, we obtain

$$\angle(t, AB) = \angle ACB.$$

Applying Proposition III.32 to the second circle, we obtain

$$\angle(t, AD) = \angle AED.$$

Since AB and AD lie on the same straight line, the angles $\angle(t, AB)$ and $\angle(t, AD)$ are equal. Hence

$$\angle ACB = \angle AED.$$

Since AC is a transversal intersecting the lines (BC) and (DE), this implies

$$BC \parallel DE.$$

Problem 2 [GeoGebra Link] ***Construction.***

Solution: *To construct the circumscribed circle of triangle ABC:*

(a) *Draw a circle centered at A with radius AB, and a circle centered at B with radius BA. Let E and F be their intersection points. Draw the line EF (this is the perpendicular bisector of AB).*

(b) *Draw a circle centered at A with radius AC, and a circle centered at C with radius CA. Let G and H be their intersection points. Draw the line GH (this is the perpendicular bisector of AC).*

(c) *Let D be the intersection point of the lines EF and GH.*

(d) *Draw the circle centered at D with radius DA.*

Number of moves:

- *4 circles (two for each perpendicular bisector),*
- *2 lines (the two perpendicular bisectors),*
- *1 circle (center D, radius DA).*

Total: 7 moves.

Problem 3 ***Reading: the construction of the pentagon.***

(a) ***Solution:*** GeoGebra Link *Following Hartshorne Problem 4.1, we construct an isosceles triangle ABC such that*

$$AB = AC \qquad \text{and} \qquad \angle B = \angle C = 2\angle A.$$

(b) ***Solution:*** ***Proposition II.11 (summary).*** GeoGebra Link

A segment AB is divided at a point H such that

$$AB \cdot BH = AH^2.$$

Proposition III.36 (summary). GeoGebra Link

From a point outside a circle, a tangent and a secant are drawn. Then

$$(\text{tangent})^2 = (\text{whole secant})(\text{external part}).$$

(c) ***Solution:*** GeoGebra Link *Following Hartshorne Problem 4.2, we construct five equally spaced points on a circle and join consecutive points to obtain a regular pentagon.*