Homework 4 Solutions

Solutions to Homework 4 for MATH 128A.

Problem 1 ***Independence of axiom I1.***

Give an example of a model

$$(\mathcal{P}, \mathcal{L}, \mathcal{R})$$

satisfying axioms ***I2*** *and* ***I3****, but not axiom* ***I1****.*

Solution: *Let*

$$\mathcal{P} = \{A, B, C, D\}$$

and let

$$\mathcal{L} = \{\ell, m\}.$$

Define the incidence relation by

$$R(\ell) = \{A, B\}, \qquad R(m) = \{C, D\}.$$

Axiom ***I2*** *is satisfied because each line contains at least two distinct points.*

Axiom ***I3*** *is also satisfied. For example, the three points A, C, D are not all incident with a common line. Indeed, C and D lie on m, but A does not lie on m. The only other line is ℓ, and C, D do not lie on ℓ.*

However, axiom ***I1*** *is not satisfied. For example, the two distinct points A and C do not have any line incident with both of them. Therefore this model satisfies* ***I2*** *and* ***I3****, but not* ***I1****.*

Problem 2 ***Basic properties of affine planes.***

Consider an affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{R})$, that is, assume axioms ***I1****,* ***I2****, and* ***I3*** *are satisfied, and assume also the Euclidean parallel property:*

Given a line $\ell \in \mathcal{L}$ and a point $p \in \mathcal{P}$ such that p is not incident with ℓ, there exists a unique line $\ell' \in \mathcal{L}$ such that $\ell \parallel \ell'$ and $p \, I \, \ell'$.

Show that if two lines ℓ and m are distinct and not parallel, then there exists a unique point $p \in \mathcal{P}$ such that $p \, I \, \ell$ and $p \, I \, m$.

Solution: *Since ℓ and m are not parallel, by definition they meet. Therefore there exists at least one point $p \in \mathcal{P}$ such that*

$$p \, I \, \ell \qquad \text{and} \qquad p \, I \, m.$$

Author(s):

We now prove uniqueness. Suppose that p and q are two points incident with both ℓ and m. We want to show that $p = q$.

*Assume, for contradiction, that $p \neq q$. Then by axiom **I1**, there exists a unique line incident with both p and q. But both ℓ and m are incident with p and q. Therefore, by uniqueness in axiom **I1**, we must have*

$$\ell = m.$$

This contradicts the assumption that ℓ and m are distinct.

Hence $p = q$. Therefore ℓ and m meet in a unique point.

Problem 3 ***The standard affine plane over a field.***

*Let $\mathbb{F} = \mathbb{R}$ (or $\mathbb{Q}$, or $\mathbb{C}$, or any field of your choice). Consider the affine plane $\mathbb{F}^2$ defined in Week 4 Lecture 2. Prove that this affine plane satisfies axiom **I3**.*

Solution: In the standard affine plane $\mathbb{F}^2$, the points are the elements of $\mathbb{F}^2$, and the lines are the subsets of the form

$$\{(x, y) \in \mathbb{F}^2 : ax + by + c = 0\},$$

where $(a, b, c) \in \mathbb{F}^3$ and

$$(a, b) \neq (0, 0).$$

Axiom **I3** says that there exist three non-collinear points. We claim that the three points

$$P = (0, 0), \qquad Q = (1, 0), \qquad R = (0, 1)$$

are not collinear.

Suppose, for contradiction, that P, Q, R are collinear. Then there exists a line

$$ax + by + c = 0,$$

with $(a, b, c) \in \mathbb{F}^3$ and $(a, b) \neq (0, 0)$, containing all three points.

Since $P = (0, 0)$ lies on the line, we have

$$a \cdot 0 + b \cdot 0 + c = 0,$$

so

$$c = 0.$$

Since $Q = (1, 0)$ lies on the line, we have

$$a \cdot 1 + b \cdot 0 + c = 0,$$

so

$$a + c = 0.$$

Since $c = 0$, this gives

$$a = 0.$$

Since $R = (0, 1)$ lies on the line, we have

$$a \cdot 0 + b \cdot 1 + c = 0,$$

so

$$b + c = 0.$$

Since $c = 0$, this gives

$$b = 0.$$

Therefore

$$a = b = c = 0.$$

In particular,

$$(a, b) = (0, 0),$$

which contradicts the requirement that $(a, b) \neq (0, 0)$.

Hence P, Q, R are not collinear. Therefore the standard affine plane $\mathbb{F}^2$ satisfies axiom **I3**.

Problem 4 ***Non-isomorphic models with three points.***

Find two non-isomorphic models with exactly three points satisfying axioms ***I1*** *and* ***I2****. You may give a picture to define your models, but you also need to prove that your two models are not isomorphic.*

Solution: *Let*

$$\mathcal{P} = \{A, B, C\}.$$

We define two models.

Model 1. *Let*

$$\mathcal{L}_1 = \{\ell\}$$

with

$$R(\ell) = \{A, B, C\}.$$

Thus all three points lie on a single line.

This model satisfies ***I1****: any two distinct points lie on the unique line ℓ. It also satisfies* ***I2****: the line ℓ contains at least two distinct points.*

Model 2. *Let*

$$\mathcal{L}_2 = \{\ell_{AB}, \ell_{AC}, \ell_{BC}\},$$

where

$$R(\ell_{AB}) = \{A, B\}, \qquad R(\ell_{AC}) = \{A, C\}, \qquad R(\ell_{BC}) = \{B, C\}.$$

*This model satisfies **I1**: any two distinct points lie on exactly one of the three lines. It also satisfies **I2**: each line contains exactly two distinct points. We now prove that the two models are not isomorphic.*

Suppose, for contradiction, that the two models are isomorphic. Then there exist two bijections

$$f : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$$

and

$$g : \mathcal{L}_1 \longrightarrow \mathcal{L}_2$$

such that incidence is preserved. In other words, for every point $P \in \mathcal{P}_1$ and every line $r \in \mathcal{L}_1$,

$$P \, I_1 \, r \qquad \Longleftrightarrow \qquad f(P) \, I_2 \, g(r).$$

In particular, an isomorphism includes a bijection

$$g : \mathcal{L}_1 \longrightarrow \mathcal{L}_2$$

between the set of lines of the first model and the set of lines of the second model. Therefore the two models must have the same number of lines.

However, Model 1 has exactly one line:

$$\mathcal{L}_1 = \{\ell\},$$

whereas Model 2 has exactly three lines:

$$\mathcal{L}_2 = \{\ell_{AB}, \ell_{AC}, \ell_{BC}\}.$$

Thus

$$\#\mathcal{L}_1 = 1 \qquad \text{and} \qquad \#\mathcal{L}_2 = 3.$$

This is impossible if there is a bijection

$$g : \mathcal{L}_1 \longrightarrow \mathcal{L}_2.$$

Therefore the two models are not isomorphic.

Problem 5 ***Isomorphisms and parallel properties.***

Let $\mathcal{M} = (\mathcal{P}, \mathcal{L}, \mathcal{R})$ and $\mathcal{M}' = (\mathcal{P}', \mathcal{L}', \mathcal{R}')$ be two isomorphic models of incidence geometry. Show that if $\mathcal{M}$ satisfies the elliptic parallel property, then $\mathcal{M}'$ also satisfies the elliptic parallel property.

Solution: *Let*

$$\varphi : \mathcal{P} \longrightarrow \mathcal{P}', \qquad \Phi : \mathcal{L} \longrightarrow \mathcal{L}'$$

be an isomorphism of incidence geometries. Thus φ and Φ are bijections, and incidence is preserved:

$$P \, I \, \ell \qquad \Longleftrightarrow \qquad \varphi(P) \, I' \, \Phi(\ell).$$

Assume that $\mathcal{M}$ satisfies the elliptic parallel property. This means that any two lines in $\mathcal{M}$ meet.

We want to prove that $\mathcal{M}'$ also satisfies the elliptic parallel property. Let $\ell', m' \in \mathcal{L}'$. Since Φ is a bijection, there exist lines $\ell, m \in \mathcal{L}$ such that

$$\Phi(\ell) = \ell', \qquad \Phi(m) = m'.$$

Since $\mathcal{M}$ satisfies the elliptic parallel property, the lines ℓ and m meet. Therefore there exists a point $P \in \mathcal{P}$ such that

$$P \, I \, \ell \qquad \text{and} \qquad P \, I \, m.$$

By preservation of incidence,

$$\varphi(P) \, I' \, \Phi(\ell) = \ell' \qquad \text{and} \qquad \varphi(P) \, I' \, \Phi(m) = m'.$$

Thus ℓ' and m' meet.

Since ℓ' and m' were arbitrary, every pair of lines in $\mathcal{M}'$ meets. Therefore $\mathcal{M}'$ satisfies the elliptic parallel property.

Problem 6 ***Reflection.***

Reflect on the first four weeks of lecture.

(a) *What have you enjoyed the most so far?*

(b) *Is there something you wish we did more of?*

(c) *How will you prepare for the midterm? If you are completing this homework after the midterm, how did you prepare?*

(d) *Are there any points you find, or found, difficult?*

(e) *What is your plan to improve on these points?*

Please answer in a few complete sentences. Your responses will be anonymous.

Problem 7 ***Challenge problems.***

(a) ***(Classifying three-point models)*** *Show that any model with exactly three points satisfying axioms **I1** and **I2** is necessarily isomorphic to one of the two models you found previously in problem 4.*

(b) ***(Quadrangles in affine planes)*** *Show that any affine plane contains a quadrangle, that is, a set of four points no three of which are collinear.*

(c) ***(Finite affine planes †)*** *Consider an affine plane, and let ℓ and m be two distinct lines. Show that the set of points incident with ℓ is in bijection with the set of points incident with m. Deduce that if one of the lines has n points, where $n \geq 2$ is an integer, then the affine plane has n^2 points. How many lines does the affine plane have? Are there any affine planes with exactly 5 points?*

(d) ***(A minimality question †)*** *Suppose that in a model of incidence geometry every line has at least three distinct points incident with it.*

(i) *What is the least number of points and the least number of lines such a model can have?*

(ii) *Suppose further that the model has the Euclidean parallel property, that is, suppose it is an affine plane. Show that 9 is now the least possible number of points, and 12 is the least possible number of lines.*