Homework 5

Homework 5 for MATH 128A.

MATH 128A: Homework 5 **Due date: Thursday, May 7**

Instructions. You will submit your solutions on Gradescope, which you can access through our Canvas course page by clicking on the assignment.

You are welcome to submit your homework either handwritten or in LaTeX. If you choose to submit handwritten work, please use pen rather than pencil, and make sure that both your handwriting and your scans are as readable as possible in order to facilitate grading.

I encourage you to attend office hours if you would like hints on the homework or if you would like to discuss more advanced material. Also, you are highly encouraged to work in groups!

Problem 1 *(20 points) A transitivity property for parallel lines.*

Let $\alpha = (\mathcal{P}, \mathcal{L}, \mathcal{R})$ be an ***affine plane*** *and recall that two lines are called parallel if they do not meet.*

Let $\ell, m, n \in \mathcal{L}$ be three lines and assume that $\ell \parallel m$ and $m \parallel n$. Show that

$$\ell = n \qquad \text{or} \qquad \ell \parallel n.$$

In other words, show that if two lines are both parallel to the same line m, then they are either equal or parallel to each other.

Hint. *Use the Euclidean parallel property.*

Problem 2 *(80 points) Affine planes and bijections between points on lines.*

Let $\alpha = (\mathcal{P}, \mathcal{L}, \mathcal{R})$ be an ***affine plane*** *and recall that, for a line $\ell \in \mathcal{L}$, we defined*

$$R(\ell) = \{P \in \mathcal{P} : P \,\mathrm{I}\, \ell\}$$

as the set of points incident with ℓ. In class, we started proving that if ℓ and m are any two lines in an affine plane, then there exists a bijection

$$f : R(\ell) \longrightarrow R(m).$$

The goal of this problem is to complete the proof. In what follows, we assume that ℓ and m are ***distinct lines****.*

Author(s):

Recap from W5L1: *In class, we chose two points $a \in \mathcal{P}$ and $b \in \mathcal{P}$ such that*

$$a \in R(\ell), \quad a \notin R(m), \quad b \in R(m), \quad b \notin R(\ell).$$

Then, we introduced n, the unique line through a and b and showed that

$$R(n) \cap R(\ell) = \{a\} \qquad \text{and} \qquad R(n) \cap R(m) = \{b\}.$$

We defined a map

$$f : R(\ell) \longrightarrow R(m)$$

as follows.

- *First, define*
$$f(a) = b.$$
- *Now let $x \in R(\ell)$ with $x \neq a$. By the Euclidean parallel property, there exists a unique line n_x such that*
$$x \,\mathrm{I}\, n_x \qquad \text{and} \qquad n_x \parallel n.$$
The line n_x is distinct and not parallel to m (see the notes of W5L1 for more details), so n_x and m meet in a unique point. We define $f(x)$ to be this unique point:
$$R(n_x) \cap R(m) = \{f(x)\}.$$

Thus

$$f(x) = \begin{cases} b, & \text{if } x = a, \\ \text{the unique point in } R(n_x) \cap R(m), & \text{if } x \neq a. \end{cases}$$

Important: *In the problem, you may use anything from the recap and anything we proved during W5L1.*

(1) ***Injectivity.*** *Let $x_1, x_2 \in R(\ell)$, and assume*

$$\boxed{f(x_1) = f(x_2)}$$

We want to show that $x_1 = x_2$.

- *(a) (10 points) Assume that $x_1 = a$. Show that $x_2 = a$.*
Hint: Use the fact that $f(a) = b$, and that $R(n) \cap R(m) = \{b\}$.
- *(b) (10 points) Now assume that $x_1 \neq a$. Show that $x_2 \neq a$ and conclude that $x_1 = x_2$.*

(2) ***Surjectivity.***

Let $y \in R(m)$. We want to show that there exists $x \in R(\ell)$ such that $f(x) = y$.

(a) *(10 points) First assume that $y = b$. Find $x \in R(\ell)$ such that $f(x) = y$.*

From now on, we assume that $y \neq b$.

(b) *(10 points) Explain why $y \notin R(n)$.*

(c) *(10 points) By the Euclidean parallel property, there exists a unique line n_y such that*

$$y \,\mathrm{I}\, n_y \qquad \text{and} \qquad n_y \parallel n.$$

Explain why n_y is distinct and not parallel to ℓ.

Hint: You may proceed by contradiction and use Problem 1.

(d) *(10 points) Since n_y is distinct and not parallel to ℓ, the lines n_y and ℓ meet in a unique point. Let x be this point:*

$$R(n_y) \cap R(\ell) = \{x\}.$$

Show that $x \neq a$.

(e) *(10 points) Show that*

$$n_y = n_x.$$

Hint: Use the Euclidean parallel property.

(f) *(10 points) Conclude that*

$$f(x) = y.$$

Conclusion. *Since f is both injective and surjective, f is a bijection. Therefore $R(\ell)$ and $R(m)$ have the same number of points.*

Problem 3 ***Reflection on the midterm exam (2 bonus points).***

- *How did your midterm exam go? (Did you lose time on certain questions? Did you find one question particularly challenging?)*
- *If you could go back, is there something you would have done differently during the exam or while preparing for the exam?*
- *What did you learn from this experience, and how will you use it to prepare more effectively for the final exam?*

Please answer in a few complete sentences. Your responses will be anonymous.

Comment. *Since there was only one lecture last week, this week's homework is a little shorter. You can use this as an opportunity to catch up with the material and submit regrade requests for past homework assignments.*