Homework 5 Solutions

Solutions to Homework 5 for MATH 128A.

Problem 1 ***A transitivity property for parallel lines.***

Let $\alpha = (\mathcal{P}, \mathcal{L}, \mathcal{R})$ be an ***affine plane*** *and recall that two lines are called parallel if they do not meet.*

Let $\ell, m, n \in \mathcal{L}$ be three lines and assume that $\ell \parallel m$ and $m \parallel n$. Show that

$$\ell = n \qquad \text{or} \qquad \ell \parallel n.$$

In other words, show that if two lines are both parallel to the same line m, then they are either equal or parallel to each other.

Solution: *Assume that $\ell \parallel m$ and $m \parallel n$. We want to show that $\ell = n$ or $\ell \parallel n$.*

We distinguish two cases. First, if $\ell \parallel n$, then we are done.

Now assume that ℓ is not parallel to n. Then ℓ and n meet, so there exists a point $x \in \mathcal{P}$ such that $x \,\mathrm{I}\, \ell$ and $x \,\mathrm{I}\, n$. Since $\ell \parallel m$, we have $x \not\mathrm{I}\, m$. Thus x is a point not incident with m. Moreover, both ℓ and n are lines through x parallel to m. By the Euclidean parallel property, we conclude $\ell = n$.

Hence, in all cases, $\ell = n$ or $\ell \parallel n$.

Problem 2 ***Affine planes and bijections between points on lines.***

Let $\alpha = (\mathcal{P}, \mathcal{L}, \mathcal{R})$ be an ***affine plane*** *and recall that, for a line $\ell \in \mathcal{L}$, we defined*

$$R(\ell) = \{P \in \mathcal{P} : P \,\mathrm{I}\, \ell\}$$

as the set of points incident with ℓ. In class, we started proving that if ℓ and m are any two lines in an affine plane, then there exists a bijection

$$f : R(\ell) \longrightarrow R(m).$$

The goal of this problem is to complete the proof. In what follows, we assume that ℓ and m are ***distinct lines****.*

Solution: *We use the notation and construction from the problem statement. Recall that $a \in R(\ell)$, $a \notin R(m)$, $b \in R(m)$, $b \notin R(\ell)$, and n is the unique line through a and b. We also know that $R(n) \cap R(\ell) = \{a\}$ and $R(n) \cap R(m) = \{b\}$.*

The map $f : R(\ell) \longrightarrow R(m)$ is defined by $f(a) = b$, and, if $x \in R(\ell)$ with $x \neq a$, then n_x is the unique line through x parallel to n, and $f(x)$ is the unique point in $R(n_x) \cap R(m)$. We prove that f is injective and surjective.

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Injectivity. *Let $x_1, x_2 \in R(\ell)$, and assume that $f(x_1) = f(x_2)$. We want to prove that $x_1 = x_2$.*

First assume that $x_1 = a$. Then $f(x_1) = f(a) = b$. Since $f(x_1) = f(x_2)$, it follows that $f(x_2) = b$. We claim that $x_2 = a$.

Suppose, for contradiction, that $x_2 \neq a$. Then n_{x_2} is defined, and $f(x_2)$ is the unique point in $R(n_{x_2}) \cap R(m)$. Since $f(x_2) = b$, we have $b \,\mathrm{I}\, n_{x_2}$. But $b \,\mathrm{I}\, n$, and $n_{x_2} \parallel n$. Therefore n_{x_2} and n meet at b, contradicting the fact that $n_{x_2} \parallel n$. Hence $x_2 = a$. Therefore, if $x_1 = a$, then $x_2 = a$, and so $x_1 = x_2$.

Now assume that $x_1 \neq a$. We first show that $x_2 \neq a$. Indeed, if $x_2 = a$, then by the previous case, applied with the roles of x_1 and x_2 reversed, we would have $x_1 = a$. This contradicts our assumption that $x_1 \neq a$. Therefore $x_2 \neq a$.

Since both $x_1 \neq a$ and $x_2 \neq a$, the lines n_{x_1} and n_{x_2} are defined. Let $y = f(x_1) = f(x_2)$. Then $y \,\mathrm{I}\, n_{x_1}$ and $y \,\mathrm{I}\, n_{x_2}$. Also, $n_{x_1} \parallel n$ and $n_{x_2} \parallel n$. By the first problem, we have $n_{x_1} = n_{x_2}$.

Observe that n_{x_1} is distinct from ℓ (since $a \not\mathrm{I}\, n_{x_1}$) and not parallel to ℓ (since $x_1 \,\mathrm{I}\, \ell$). Thus n_{x_1} intersects ℓ in a unique point x_1. For the same reason, n_{x_2} intersects ℓ in a unique point x_2. Since $n_{x_1} = n_{x_2}$, we conclude that $x_1 = x_2$.

Surjectivity. *Let $y \in R(m)$. We want to show that there exists $x \in R(\ell)$ such that $f(x) = y$.*

First assume that $y = b$. Then we can take $x = a$, since $f(a) = b = y$.

From now on, assume that $y \neq b$. Since $y \in R(m)$, and since $R(n) \cap R(m) = \{b\}$, we have $y \not\mathrm{I}\, n$. By the Euclidean parallel property, there exists a unique line n_y such that $y \,\mathrm{I}\, n_y$ and $n_y \parallel n$.

We claim that n_y is distinct and not parallel to ℓ. First, $n_y \neq \ell$. Indeed, if $n_y = \ell$, then n_y and n would meet at a, because $a \,\mathrm{I}\, \ell$ and $a \,\mathrm{I}\, n$. This contradicts $n_y \parallel n$. Therefore $n_y \neq \ell$.

Next, suppose for contradiction that $n_y \parallel \ell$. Since $n_y \parallel n$, Problem 1 implies that $\ell = n$ or $\ell \parallel n$. But $\ell \neq n$ (since $b \not\mathrm{I}\, \ell$), and ℓ meets n at a. Thus ℓ is not parallel to n. This is a contradiction. Hence n_y is not parallel to ℓ.

Since n_y is distinct and not parallel to ℓ, the lines n_y and ℓ meet in a unique point. Let x be this point, so $R(n_y) \cap R(\ell) = \{x\}$.

We claim that $x \neq a$. Suppose, for contradiction, that $x = a$. Then $a \,\mathrm{I}\, n_y$. But $a \,\mathrm{I}\, n$, and $n_y \parallel n$. This contradicts the fact that parallel lines do not meet. Therefore $x \neq a$.

Since $x \in R(\ell)$ and $x \neq a$, the line n_x is defined as the unique line through x parallel to n. But n_y also passes through x, because $x \,\mathrm{I}\, n_y$, and $n_y \parallel n$. By uniqueness in the Euclidean parallel property, we get $n_y = n_x$.

Now $y \,\mathrm{I}\, n_y$, $y \in R(m)$, and $n_y = n_x$. Hence $y \in R(n_x) \cap R(m)$. By definition, $f(x)$ is the unique point in $R(n_x) \cap R(m)$. Therefore $f(x) = y$. Thus f is

surjective.

Since f is both injective and surjective, f is a bijection.