Homework 7

Homework 7 for MATH 128A.

MATH 128A: Homework 7 **Due date: Thursday, May 21**

Instructions. You will submit your solutions on Gradescope, which you can access through our Canvas course page by clicking on the assignment.

You are welcome to submit your homework either handwritten or in LaTeX. If you choose to submit handwritten work, please use pen rather than pencil, and make sure that both your handwriting and your scans are as readable as possible in order to facilitate grading.

I encourage you to attend office hours if you would like hints on the homework or if you would like to discuss more advanced material. Also, you are highly encouraged to work in groups!

Problem 1 *(20 points) Warm-up: transformations and inverses.*

(a) ***(10 points)*** *Let Σ be a set. Show that if $f : \Sigma \to \Sigma$ and $g : \Sigma \to \Sigma$ are bijections, then $f \circ g : \Sigma \to \Sigma$ is also a bijection.*

(b) ***(10 points)*** *Let $\mathcal{M} = (\mathcal{P}, \mathcal{L}, \mathcal{R})$ be a model satisfying axioms **I1** and **I2**. Assume that $F : \mathcal{P} \to \mathcal{P}$ is a transformation of $\mathcal{M}$, i.e., F is a bijection such that, for every line $\ell \in \mathcal{L}$, the image $F(\ell)$ is also a line in $\mathcal{L}$.*

Show that the inverse map $F^{-1} : \mathcal{P} \to \mathcal{P}$ is also a transformation.

Hint: Let $\ell \in \mathcal{L}$. You need to show that $F^{-1}(\ell)$ is a line. Choose two distinct points $p, q \in \ell$, and use the fact that F is a transformation.

Problem 2 *(40 points) Transformations of a five-point model.*

Consider the incidence geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, I)$, where $\mathcal{P} = \{0, 1, 2, 3, 4\}$, and where the lines are

$$\{0,1\}, \quad \{2,3,4\}, \quad \{0,2\}, \quad \{0,3\}, \quad \{0,4\}, \quad \{1,2\}, \quad \{1,3\}, \quad \{1,4\}.$$

Equivalently, you can view the model using the following GeoGebra diagram:

GeoGebradiagram.

*Recall that a **transformation** of $\mathcal{M}$ is a bijection $F : \mathcal{P} \to \mathcal{P}$ such that, for every line $\ell \in \mathcal{L}$, the image $F(\ell)$ is also a line in $\mathcal{L}$. The goal of this problem is to determine all transformations of $\mathcal{M}$.*

Author(s):

(a) ***(4 points)*** *Explain why any transformation F must send the line $\{2,3,4\}$ to itself.*

Hint: This line is special: it is the only line containing three points. What happens to the image of a three-point line under a bijection?

(b) ***(4 points)*** *Deduce that any transformation F must send the set $\{0,1\}$ to itself.*

Hint: Since F is a bijection, it must send the complement of $\{2,3,4\}$ to the complement of $\{2,3,4\}$.

(c) ***(4 points)*** *Given a permutation σ of $\{2,3,4\}$, define a map*

$$\begin{array}{rcl} f(\sigma):\mathcal{P} & \longrightarrow & \mathcal{P} \\ 0 & \longmapsto & 0 \\ 1 & \longmapsto & 1 \\ 2 & \longmapsto & \sigma(2) \\ 3 & \longmapsto & \sigma(3) \\ 4 & \longmapsto & \sigma(4). \end{array}$$

Given a permutation η of $\{0,1\}$, define a map

$$\begin{array}{rcl} g(\eta):\mathcal{P} & \longrightarrow & \mathcal{P} \\ 0 & \longmapsto & \eta(0) \\ 1 & \longmapsto & \eta(1) \\ 2 & \longmapsto & 2 \\ 3 & \longmapsto & 3 \\ 4 & \longmapsto & 4. \end{array}$$

Let $\sigma = (234)$, meaning that $\sigma(2) = 3$, $\sigma(3) = 4$, and $\sigma(4) = 2$. Let $\eta = (01)$, meaning that $\eta(0) = 1$ and $\eta(1) = 0$. Define $F = f(\sigma) \circ g(\eta)$. Compute $F(0)$, $F(1)$, $F(2)$, $F(3)$, and $F(4)$.

(d) ***(4 points)*** *Let F be a transformation of $\mathcal{M}$, and assume that $F(0) = 0$. Show that $F(1) = 1$.*

(e) ***(5 points)*** *Let F be a transformation of $\mathcal{M}$, and assume that $F(0) = 0$. Show that there exists a permutation σ of $\{2,3,4\}$ such that $F = f(\sigma)$.*

(f) ***(4 points)*** *Let F be a transformation of $\mathcal{M}$, and assume that $F(0) = 1$. Show that $F(1) = 0$.*

(g) ***(5 points)*** *Let F be a transformation of $\mathcal{M}$, and assume that $F(0) = 1$. Let $\eta = (01)$ be the permutation of $\{0,1\}$ that exchanges 0 and 1. Show that there exists a permutation σ of $\{2,3,4\}$ such that $F = g(\eta) \circ f(\sigma)$.*

(h) ***(5 points)*** *Conclude that every transformation of $\mathcal{M}$ is of the form $g(\eta) \circ f(\sigma)$, where σ is a permutation of $\{2,3,4\}$ and η is a permutation of $\{0,1\}$.*

(i) ***(5 points)*** *Deduce that* $|\operatorname{Aut}(\mathcal{M})| = 12$*, i.e., there are exactly* 12 *transformations.*

Problem 3 ***(40 points) The dual of a projective plane.***

Let $\Pi = (\mathcal{P}, \mathcal{L}, I)$ *be a projective plane. We define a new incidence geometry*

$$\Pi^* = (\mathcal{P}^*, \mathcal{L}^*, I^*)$$

as follows. The points of Π^* *are the lines of* Π*, so* $\mathcal{P}^* := \mathcal{L}$*. The lines of* Π^* *are the points of* Π*, so* $\mathcal{L}^* := \mathcal{P}$*. Finally, incidence is defined by reversing the original incidence relation: if* $\ell \in \mathcal{P}^* = \mathcal{L}$ *and* $p \in \mathcal{L}^* = \mathcal{P}$*, then*

$$\ell \, I^* \, p \qquad \Longleftrightarrow \qquad p \, I \, \ell.$$

The goal of this problem is to prove that Π^* *is also a projective plane.*

(a) ***(7 points)*** *Show that any two distinct points of* Π^* *are incident with a unique line of* Π^**.*

Hint: A point of Π^* *is a line of* Π*. Use the fact that two distinct lines in a projective plane meet in a unique point.*

(b) ***(6 points)*** *Let* $p \in \mathcal{P}$*. Show that there exists a line* $\ell \in \mathcal{L}$ *such that* $p \not I \, \ell$*.*

(c) ***(7 points)*** *Use the previous part to show that every point of* Π *is incident with at least three distinct lines of* Π*.*

Hint: Choose a line ℓ *not incident with* p*, choose three distinct points on* ℓ*, and join them to* p*.*

(d) ***(6 points)*** *Deduce that every line of* Π^* *is incident with at least three distinct points of* Π^**.*

(e) ***(7 points)*** *Show that there exist three points of* Π^* *which are not collinear in* Π^**.*

Hint: In terms of the original projective plane Π*, this means finding three lines which are not all incident with one common point.*

(f) ***(7 points)*** *Show that any two distinct lines of* Π^* *meet.*

Hint: A line of Π^* *is a point of* Π*. Use the projective plane axiom saying that any two distinct points of* Π *are incident with a unique line.*

Conclude that Π^* *is a projective plane. This projective plane is called the* ***dual projective plane*** *of* Π*.*

Problem 4 ***Weekly reflection (2 bonus points).***

Please answer in a few complete sentences. Your responses will be ***anonymous****.*

(a) *How confident do you feel with the concepts from the past week? If you feel confident, would you be interested in trying some of the challenge problems? If you do not feel confident yet, what is one strategy you could use to improve your understanding? For example, would you consider talking to a classmate, coming to office hours, or trying extra practice problems?*

(b) *Have you used AI tools at all during this class? If yes, how have you used them? If you use AI tools, what do you do to make sure homework and worksheets are still helping you practice and prepare for the final exam? If you are not using AI, explain why you are not using it.*

Problem 5 ***Challenge problems (not graded).***

(a) ***(Every projective plane contains an affine plane †)*** *Let $\Pi = (\mathcal{P}, \mathcal{L}, I)$ be a projective plane. Fix a line $\ell \in \mathcal{L}$. Define*

$$\mathcal{P}' = \{p \in \mathcal{P} : p \not I\, \ell\} \qquad \text{and} \qquad \mathcal{L}' = \mathcal{L} \setminus \{\ell\}.$$

We define incidence I' by restriction of the old incidence relation: if $p' \in \mathcal{P}'$ and $m' \in \mathcal{L}'$, then

$$p'\, I'\, m' \qquad \Longleftrightarrow \qquad p'\, I\, m'.$$

Show that $(\mathcal{P}', \mathcal{L}', I')$ is an affine plane.

(b) ***(Automorphism groups from two special lines †)*** *Consider the following incidence geometry. There are distinct points*

$$A_1, \ldots, A_n, B_1, \ldots, B_m.$$

The points $A_1, \ldots, A_n$ lie on one line ℓ, and the points $B_1, \ldots, B_m$ lie on another line m. In addition, for each pair (i, j), we declare

$$\{A_i, B_j\}$$

to be a line.

Describe the automorphism group of this model.

(i) *What happens when $n \neq m$?*

(ii) *What happens when $n = m$?*

(iii) *Can you find a model whose automorphism group has cardinality 144?*

(iv) *How much can you generalize this construction?*