Homework 7 Solutions

Solutions to Homework 7.

Problem 1

(a) Let $f : \Sigma \to \Sigma$ and $g : \Sigma \to \Sigma$ be bijections. We show that $f \circ g$ is a bijection.

First, $f \circ g$ is injective. Suppose

$$(f \circ g)(x) = (f \circ g)(y).$$

Then $f(g(x)) = f(g(y))$. Since f is injective, $g(x) = g(y)$. Since g is injective, $x = y$.

Second, $f \circ g$ is surjective. Let $z \in \Sigma$. Since f is surjective, there exists $w \in \Sigma$ such that $f(w) = z$. Since g is surjective, there exists $x \in \Sigma$ such that $g(x) = w$. Then

$$(f \circ g)(x) = f(g(x)) = f(w) = z.$$

Hence $f \circ g$ is bijective.

(b) Let $\ell \in \mathcal{L}$. We prove that $F^{-1}(\ell)$ is a line.

By **I2**, choose two distinct points $p, q \in \ell$. Since F is bijective, the points

$$F^{-1}(p), \qquad F^{-1}(q)$$

are distinct. By **I1**, there is a unique line $m \in \mathcal{L}$ incident with both $F^{-1}(p)$ and $F^{-1}(q)$.

Since F is a transformation, $F(m)$ is a line. It contains

$$F(F^{-1}(p)) = p \qquad \text{and} \qquad F(F^{-1}(q)) = q.$$

The line ℓ also contains p and q. By uniqueness in **I1**,

$$F(m) = \ell.$$

Therefore

$$m = F^{-1}(\ell).$$

Thus $F^{-1}(\ell)$ is a line. Since ℓ was arbitrary, F^{-1} sends lines to lines. Also, F^{-1} is a bijection. Hence F^{-1} is a transformation.

Author(s):

Problem 2

(a) The line $\{2,3,4\}$ is the only line containing three points. If F is a transformation, then $F(\{2,3,4\})$ is a line. Since F is injective, the three points $F(2), F(3), F(4)$ are distinct. Thus $F(\{2,3,4\})$ is a line with three points. The only such line is $\{2,3,4\}$. Therefore

$$F(\{2,3,4\}) = \{2,3,4\}.$$

(b) Since F is a bijection and

$$F(\{2,3,4\}) = \{2,3,4\},$$

it sends the complement of $\{2,3,4\}$ to the complement of $\{2,3,4\}$. Hence

$$F(\{0,1\}) = \{0,1\}.$$

(c) We have $F = f(\sigma) \circ g(\eta)$, so we first apply $g(\eta)$, then $f(\sigma)$. Since $g(\eta)$ exchanges 0 and 1, and $f(\sigma)$ fixes 0 and 1,

$$F(0) = 1, \qquad F(1) = 0.$$

Since $g(\eta)$ fixes $2,3,4$, and $f(\sigma)$ sends $2 \mapsto 3$, $3 \mapsto 4$, and $4 \mapsto 2$,

$$F(2) = 3, \qquad F(3) = 4, \qquad F(4) = 2.$$

(d) By part (b),

$$F(\{0,1\}) = \{0,1\}.$$

If $F(0) = 0$, then injectivity forces

$$F(1) = 1.$$

(e) By part (d), $F(1) = 1$. By part (a), F sends $\{2,3,4\}$ to itself. Since F is bijective, its restriction to $\{2,3,4\}$ is a permutation. Define

$$\sigma(i) = F(i) \qquad \text{for } i \in \{2,3,4\}.$$

Then σ is a permutation of $\{2,3,4\}$, and F agrees with $f(\sigma)$ on all five points. Therefore

$$F = f(\sigma).$$

(f) By part (b),

$$F(\{0,1\}) = \{0,1\}.$$

If $F(0) = 1$, then injectivity forces

$$F(1) = 0.$$

(g) By part (f), $F(1) = 0$. By part (a), F sends $\{2,3,4\}$ to itself. Since F is bijective, its restriction to $\{2,3,4\}$ is a permutation. Define

$$\sigma(i) = F(i) \qquad \text{for } i \in \{2,3,4\}.$$

Let $\eta = (01)$. Then $g(\eta) \circ f(\sigma)$ sends

$$0 \mapsto 1, \qquad 1 \mapsto 0,$$

and for $i \in \{2,3,4\}$,

$$i \mapsto \sigma(i) = F(i).$$

Thus F and $g(\eta) \circ f(\sigma)$ agree on every point, so

$$F = g(\eta) \circ f(\sigma).$$

(h) Let F be a transformation. By part (b),

$$F(\{0,1\}) = \{0,1\}.$$

Hence either $F(0) = 0$ or $F(0) = 1$.

If $F(0) = 0$, then by part (e),

$$F = f(\sigma) = g(\mathrm{id}) \circ f(\sigma)$$

for some permutation σ of $\{2,3,4\}$.

If $F(0) = 1$, then by part (g),

$$F = g(\eta) \circ f(\sigma),$$

where $\eta = (01)$ and σ is a permutation of $\{2,3,4\}$.

Therefore every transformation is of the form

$$g(\eta) \circ f(\sigma),$$

where η is a permutation of $\{0,1\}$ and σ is a permutation of $\{2,3,4\}$.

Conversely, every map of this form is a transformation. Indeed, it sends $\{0,1\}$ to itself, sends $\{2,3,4\}$ to itself, and sends each two-point line of the form

$$\{a,i\}, \qquad a \in \{0,1\}, \quad i \in \{2,3,4\},$$

to another line of the same form:

$$\{\eta(a), \sigma(i)\}.$$

(i) There are

$$2! = 2$$

permutations of $\{0,1\}$, and

$$3! = 6$$

permutations of $\{2,3,4\}$. For each choice of η and σ, the map $g(\eta) \circ f(\sigma)$ is a transformation, and different choices give different maps because the restrictions to $\{0,1\}$ and $\{2,3,4\}$ determine η and σ. Therefore

$$|\operatorname{Aut}(\mathcal{M})| = 2! \cdot 3! = 2 \cdot 6 = 12.$$

Problem 3

Let

$$\Pi = (\mathcal{P}, \mathcal{L}, I)$$

be a projective plane, and let

$$\Pi^* = (\mathcal{P}^*, \mathcal{L}^*, I^*)$$

be the dual incidence geometry.

(a) Let $\ell, m \in \mathcal{P}^*$ be two distinct points of Π^*. By definition, ℓ and m are two distinct lines of Π.

By the elliptic parallel property in Π, the two lines ℓ and m meet. Hence there exists a point $p \in \mathcal{P}$ such that

$$p \, I \, \ell \qquad \text{and} \qquad p \, I \, m.$$

Since points of Π are lines of Π^*, this point p is a line of Π^*. Moreover, by definition of the dual incidence relation,

$$\ell \, I^* \, p \qquad \text{and} \qquad m \, I^* \, p.$$

Thus ℓ and m are incident with at least one common line of Π^*.

We now prove uniqueness. Suppose that $q \in \mathcal{L}^* = \mathcal{P}$ is another line of Π^* incident with both ℓ and m. Then, in Π,

$$q \, I \, \ell \qquad \text{and} \qquad q \, I \, m.$$

If $q \neq p$, then the two distinct points p and q of Π are both incident with the two lines ℓ and m. By axiom **I1** in Π, there is a unique line through p and q, so $\ell = m$, contradicting the assumption that ℓ and m are distinct.

Therefore $q = p$. Hence any two distinct points of Π^* are incident with a unique line of Π^*.

(b) Let $p \in \mathcal{P}$. Choose three non-collinear points A, B, C in Π. Let

$$\ell_{AB}, \qquad \ell_{AC}, \qquad \ell_{BC}$$

be the unique lines through A, B, through A, C, and through B, C, respectively.

We claim that at least one of these three lines is not incident with p. Suppose, for contradiction, that

$$p \, I \, \ell_{AB}, \qquad p \, I \, \ell_{AC}, \qquad p \, I \, \ell_{BC}.$$

Since ℓ_{AB} and ℓ_{AC} both contain A, and since A, B, C are not collinear, the lines ℓ_{AB} and ℓ_{AC} are distinct. Therefore their unique intersection point is A. Since p is incident with both ℓ_{AB} and ℓ_{AC}, we get

$$p = A.$$

But p is also incident with ℓ_{BC}, so A is incident with ℓ_{BC}. This means that A, B, C are all incident with ℓ_{BC}, contradicting the fact that A, B, C are non-collinear.

Therefore at least one of the lines

$$\ell_{AB}, \qquad \ell_{AC}, \qquad \ell_{BC}$$

is not incident with p.

(c) Let $p \in \mathcal{P}$. By part (b), choose a line $\ell \in \mathcal{L}$ such that

$$p \not I \ell.$$

Since Π is a projective plane, the line ℓ contains at least three distinct points. Choose distinct points

$$a, b, c \in \ell.$$

Let

$$\ell_{pa}, \qquad \ell_{pb}, \qquad \ell_{pc}$$

be the unique lines through p, a, through p, b, and through p, c, respectively.

We claim that these three lines are distinct. Suppose, for example, that

$$\ell_{pa} = \ell_{pb}.$$

Then this line contains both a and b. Since ℓ also contains a and b, and since $a \neq b$, axiom **I1** implies

$$\ell_{pa} = \ell.$$

But p is incident with ℓ_{pa}, so this would imply

$$p \, I \, \ell,$$

contradicting the choice of ℓ. Therefore $\ell_{pa} \neq \ell_{pb}$. The same argument shows that

$$\ell_{pa}, \qquad \ell_{pb}, \qquad \ell_{pc}$$

are pairwise distinct.

Each of these three distinct lines is incident with p. Hence every point $p \in \mathcal{P}$ is incident with at least three distinct lines.

(d) Let $p \in \mathcal{L}^*$ be a line of Π^*. By definition, p is a point of Π. By part (c), there are at least three distinct lines of Π incident with p. These lines of Π are points of Π^*, and they are incident with the line p of Π^*. Therefore every line of Π^* is incident with at least three distinct points.

(e) Choose three non-collinear points A, B, C in Π. Let

$$\ell_{AB}, \qquad \ell_{AC}, \qquad \ell_{BC}$$

be the unique lines through A, B, through A, C, and through B, C, respectively. These three lines of Π are three points of Π^*.

We claim that these three points of Π^* are not collinear. Suppose, for contradiction, that they are collinear in Π^*. Then there exists a line of Π^* incident with all three of them. A line of Π^* is a point $p \in \mathcal{P}$. Therefore, in Π, the three lines

$$\ell_{AB}, \qquad \ell_{AC}, \qquad \ell_{BC}$$

are all incident with p.

Since ℓ_{AB} and ℓ_{AC} are distinct and both contain A, their unique intersection point is A. Since p is incident with both ℓ_{AB} and ℓ_{AC}, we get

$$p = A.$$

But p is also incident with ℓ_{BC}, so A is incident with ℓ_{BC}. Hence A, B, C are all incident with ℓ_{BC}, contradicting the fact that A, B, C are non-collinear.

Therefore $\ell_{AB}, \ell_{AC}, \ell_{BC}$ are three non-collinear points of Π^*.

(f) Let $p, q \in \mathcal{L}^*$ be two distinct lines of Π^*. By definition, p and q are two distinct points of Π.

Since Π satisfies axiom **I1**, there exists a line $\ell \in \mathcal{L}$ incident with both p and q. This line ℓ of Π is a point of Π^*.

Since

$$p \, I \, \ell \qquad \text{and} \qquad q \, I \, \ell,$$

the definition of the dual incidence relation gives

$$\ell \, I^* \, p \qquad \text{and} \qquad \ell \, I^* \, q.$$

Thus the two lines p and q of Π^* meet at the point ℓ of Π^*.

Parts (a), (d), (e), and (f) verify the projective plane axioms for Π^*. Therefore Π^* is a projective plane.