Homework 8

Homework 8 for MATH 128A.

MATH 128A: Homework 8 **Due date: Thursday, May 28**

Instructions. You will submit your solutions on Gradescope, which you can access through our Canvas course page by clicking on the assignment.

You are welcome to submit your homework either handwritten or in LaTeX. If you choose to submit handwritten work, please use pen rather than pencil, and make sure that both your handwriting and your scans are as readable as possible in order to facilitate grading.

I encourage you to attend office hours if you would like hints on the homework or if you would like to discuss more advanced material. Also, you are highly encouraged to work in groups!

In this homework, we study the following incidence geometry $\mathcal{M}_{\mathbb{Z}^2}$. The points are elements of $\mathbb{Z}^2$, i.e. ordered pairs (k, ℓ), where $k, \ell \in \mathbb{Z}$. The lines are the horizontal and vertical edges

$$\{(k,\ell),(k+1,\ell)\} \qquad \text{and} \qquad \{(k,\ell),(k,\ell+1)\},$$

where $k, \ell \in \mathbb{Z}$. Two points are called **adjacent** if they lie on one common line.

Problem 1 ***(40 points) Examples of transformations of the square grid.***

(a) ***(7 points)*** *For $(a, b) \in \mathbb{Z}^2$, define $T_{(a,b)} : \mathbb{Z}^2 \to \mathbb{Z}^2$ by*

$$T_{(a,b)}(k,\ell) = (k+a, \ell+b).$$

Show that $T_{(a,b)}$ is a transformation of $\mathcal{M}_{\mathbb{Z}^2}$.

(b) ***(7 points)*** *Show that the inverse of a translation is a translation. More precisely, show that*

$$T_{(a,b)}^{-1} = T_{(-a,-b)}.$$

(c) ***(7 points)*** *Define*

$$R_x(k,\ell) = (-k,\ell), \qquad R_y(k,\ell) = (k,-\ell).$$

Show that R_x and R_y are transformations of $\mathcal{M}_{\mathbb{Z}^2}$.

Author(s):

(d) ***(6 points)*** *Show that each of the transformations*

$$\text{id}, \qquad R_x, \qquad R_y, \qquad R_x \circ R_y$$

is its own inverse.

(e) ***(7 points)*** *Define $S(k, \ell) = (\ell, k)$. Show that S is a transformation of $\mathcal{M}_{\mathbb{Z}^2}$.*

(f) ***(6 points)*** *Show that S is its own inverse, i.e.*

$$S^{-1} = S.$$

Problem 2 ***(35 points) Normalizing a transformation.***

Let F be a transformation of $\mathcal{M}_{\mathbb{Z}^2}$.

(a) ***(7 points)*** *Let $p \in \mathbb{Z}^2$. Show that F sends the four points adjacent to p to the four points adjacent to $F(p)$.*

Hint: Each point has exactly four adjacent points. Since F is injective and sends lines to lines, what happens to the four neighbors of p?

(b) ***(7 points)*** *Show that, by composing F on the left with a translation, we may assume that*

$$F(0, 0) = (0, 0).$$

Hint: If $F(0, 0) = (a, b)$, compose F with $T_{(-a,-b)}$.

(c) ***(7 points)*** *Now assume that $F(0, 0) = (0, 0)$. Show that $F(1, 0)$ must be one of*

$$(1, 0), \quad (-1, 0), \quad (0, 1), \quad (0, -1).$$

(d) ***(7 points)*** *Show that, by composing F on the left with transformations chosen from*

$$\text{id}, \quad R_x, \quad R_y, \quad R_x \circ R_y, \quad S, \quad R_x \circ S, \quad R_y \circ S, \quad R_x \circ R_y \circ S,$$

we may assume that

$$F(0, 0) = (0, 0) \qquad \text{and} \qquad F(1, 0) = (1, 0).$$

(e) ***(7 points)*** *Now assume that $F(0, 0) = (0, 0)$ and $F(1, 0) = (1, 0)$. Show that*

$$F(0, 1) = (0, 1) \qquad \text{or} \qquad F(0, 1) = (0, -1).$$

Then explain why, by composing with R_y if necessary, we may assume that

$$F(0, 0) = (0, 0), \qquad F(1, 0) = (1, 0), \qquad F(0, 1) = (0, 1).$$

Hint: First use part (a). To rule out $F(0, 1) = (-1, 0)$, use the fact that $(1, 0)$ and $(0, 1)$ have two common adjacent points, while $(1, 0)$ and $(-1, 0)$ do not.

Problem 3 ***(25 points) The normalized transformation fixes part of the grid.***

Let F be a transformation of $\mathcal{M}_{\mathbb{Z}^2}$. Assume that

$$F(0,0) = (0,0), \qquad F(1,0) = (1,0), \qquad F(0,1) = (0,1).$$

(a) ***(8 points)*** *Show that $F(1,1) = (1,1)$.*

Hint: The point $(1,1)$ is adjacent to both $(1,0)$ and $(0,1)$. It is the unique point, different from $(0,0)$, with this property.

(b) ***(8 points)*** *Show that $F(-1,0) = (-1,0)$ and $F(0,-1) = (0,-1)$.*

Hint: By Problem 2(a), the remaining neighbors of $(0,0)$ must be sent to the remaining neighbors of $(0,0)$. To rule out switching them, use the fact that $(1,0)$ and $(0,-1)$ have two common adjacent points, while $(1,0)$ and $(-1,0)$ do not.

(c) ***(9 points)*** *Show, by induction, that*

$$F(k,0) = (k,0)$$

for every integer $k \geq 0$.

Hint: This is similar to the proof for the integer line from lecture, but there is one extra difficulty. Once you know $F(k-1,0) = (k-1,0)$ and $F(k,0) = (k,0)$, use the line $\{(k,0),(k+1,0)\}$ to restrict the possible values of $F(k+1,0)$. Then use injectivity and compare the number of common adjacent points to rule out the wrong possibilities.

Remark 1. The remaining part of the proof, namely showing that F fixes every point of $\mathbb{Z}^2$, will not be graded. However, I highly recommend thinking about it. A scaffolded version appears in the challenge problems.

Remark 2. The previous problems, together with Challenge Problem (a), prove the following classification.

Every transformation of $\mathcal{M}_{\mathbb{Z}^2}$ can be written as

$$T \circ r \circ s,$$

where T is a translation, $r \in \{\mathrm{id}, R_x, R_y, R_x \circ R_y\}$, and $s \in \{\mathrm{id}, S\}$.

Here is how the conclusion works. Let F be a transformation of $\mathcal{M}_{\mathbb{Z}^2}$. By Problem 2, we can choose a translation T, an element

$$r \in \{\mathrm{id}, R_x, R_y, R_x \circ R_y\},$$

and an element

$$s \in \{\mathrm{id}, S\}$$

such that

$$s \circ r \circ T \circ F$$

sends $(0,0)$ to $(0,0)$, sends $(1,0)$ to $(1,0)$, and sends $(0,1)$ to $(0,1)$.

The challenge problem below completes the proof that any transformation fixing these three points must be the identity. Assuming this result, we get

$$s \circ r \circ T \circ F = \mathrm{id}\,.$$

Now compose on the left by s^{-1}, then by r^{-1}, then by T^{-1}. We obtain

$$F = T^{-1} \circ r^{-1} \circ s^{-1}.$$

By Problem 1, the inverse of a translation is a translation. Also, each element of

$$\{\mathrm{id}, R_x, R_y, R_x \circ R_y\}$$

is its own inverse, and $S^{-1} = S$. Therefore T^{-1} is another translation, r^{-1} is again one of

$$\mathrm{id}, R_x, R_y, R_x \circ R_y,$$

and s^{-1} is again one of

$$\mathrm{id}, S.$$

Thus F can be written in the desired form

$$F = T' \circ r' \circ s',$$

where T' is a translation, $r' \in \{\mathrm{id}, R_x, R_y, R_x \circ R_y\}$, and $s' \in \{\mathrm{id}, S\}$.

In other words, every transformation of the square grid is obtained by combining a translation, possibly a reflection in one or both coordinate axes, and possibly a coordinate swap.

Problem 4 ***Weekly reflection (2 bonus points).***

Please answer in a few complete sentences. Your responses will be ***anonymous****.*

(a) *Have you worked in groups on your homework? Are there any barriers that make it difficult for you to work in groups?*

(b) *Did you enjoy the ungraded quizzes? If so, would you like to see more of them?*

(c) *Would you like to have a recap class, even if it meant that we would cover less new content?*

Problem 5 ***Challenge problems (not graded).***

(a) ***(Completing the proof for the square grid †)*** *Continue Problem 3 and show that the normalized transformation is the identity.*

More precisely, let F be a transformation of $\mathcal{M}_{\mathbb{Z}^2}$, and assume that

$$F(0,0) = (0,0), \qquad F(1,0) = (1,0), \qquad F(0,1) = (0,1).$$

In Problem 3, you proved some first consequences of these assumptions. The goal is to prove that

$$F(k,\ell) = (k,\ell)$$

for every $(k,\ell) \in \mathbb{Z}^2$.

(i) Show, using arguments similar to Problem 3(c), that F fixes the negative part of the horizontal axis and the entire vertical axis.

(ii) Show that $F(k,\ell) = (k,\ell)$ for every $k,\ell \geq 0$.

Suggested strategy: Use induction on $k+\ell$. The induction hypothesis should be: $F(a,b) = (a,b)$ for every point (a,b) with $a,b \geq 0$ and $a+b \leq n$. Then prove the result for a point (k,ℓ) with $k,\ell \geq 0$ and $k+\ell = n+1$.

Hint: If $k,\ell \geq 1$, then (k,ℓ) is the unique point, different from $(k-1,\ell-1)$, adjacent to both $(k-1,\ell)$ and $(k,\ell-1)$.

(iii) Adapt the same argument to the other three quadrants.

(iv) Conclude that $F = \mathrm{id}$.

(b) ***(Higher-dimensional square grids †)*** *Consider the model with point set $\mathbb{Z}^n$, where lines are edges between points that differ by 1 in exactly one coordinate. Describe the transformations of this model.*

(c) ***(Lattice lines in $\mathbb{Z}^2$ †)*** *Consider the model with point set $\mathbb{Z}^2$, where lines are subsets of the form*

$$v + \mathbb{Z}u = \{v + ku : k \in \mathbb{Z}\},$$

where $v \in \mathbb{Z}^2$ and $u = (a,b) \in \mathbb{Z}^2$ satisfies $a\mathbb{Z} + b\mathbb{Z} = \mathbb{Z}$. Equivalently, u is primitive, meaning $\gcd(a,b) = 1$. Can you describe the transformations of this model?

(d) ***(A model with no nontrivial transformations †)*** *Can you construct a model satisfying axioms **I1** and **I2** whose only transformation is the identity?*