Homework 8 Solutions

Solutions to Homework 8 for MATH 128A.

MATH 128A: Homework 8 Solutions

In this homework, we study the following incidence geometry $\mathcal{M}_{\mathbb{Z}^2}$. The points are elements of $\mathbb{Z}^2$, i.e. ordered pairs (k, ℓ), where $k, \ell \in \mathbb{Z}$. The lines are the horizontal and vertical edges

$$\{(k,\ell),(k+1,\ell)\} \qquad \text{and} \qquad \{(k,\ell),(k,\ell+1)\},$$

where $k, \ell \in \mathbb{Z}$. Two points are called **adjacent** if they lie on one common line.

Problem 1 ***(40 points) Examples of transformations of the square grid.***

(a) ***(7 points)*** *For $(a, b) \in \mathbb{Z}^2$, define $T_{(a,b)} : \mathbb{Z}^2 \to \mathbb{Z}^2$ by*

$$T_{(a,b)}(k,\ell) = (k+a, \ell+b).$$

Show that $T_{(a,b)}$ is a transformation of $\mathcal{M}_{\mathbb{Z}^2}$.

(b) ***(7 points)*** *Show that the inverse of a translation is a translation. More precisely, show that*

$$T_{(a,b)}^{-1} = T_{(-a,-b)}.$$

(c) ***(7 points)*** *Define*

$$R_x(k,\ell) = (-k,\ell), \qquad R_y(k,\ell) = (k,-\ell).$$

Show that R_x and R_y are transformations of $\mathcal{M}_{\mathbb{Z}^2}$.

(d) ***(6 points)*** *Show that each of the transformations*

$$\mathrm{id}, \qquad R_x, \qquad R_y, \qquad R_x \circ R_y$$

is its own inverse.

(e) ***(7 points)*** *Define $S(k, \ell) = (\ell, k)$. Show that S is a transformation of $\mathcal{M}_{\mathbb{Z}^2}$.*

(f) ***(6 points)*** *Show that S is its own inverse, i.e.*

$$S^{-1} = S.$$

Author(s):

Solution: **(a)** First, $T_{(a,b)}$ is a bijection. Indeed, for every $(k,\ell) \in \mathbb{Z}^2$,

$$T_{(-a,-b)}(T_{(a,b)}(k,\ell)) = T_{(-a,-b)}(k+a, \ell+b) = (k,\ell),$$

and similarly

$$T_{(a,b)}(T_{(-a,-b)}(k,\ell)) = T_{(a,b)}(k-a, \ell-b) = (k,\ell).$$

Therefore $T_{(a,b)}$ is a bijection, with inverse $T_{(-a,-b)}$.

We now show that $T_{(a,b)}$ sends lines to lines. If

$$L = \{(k,\ell), (k+1,\ell)\}$$

is a horizontal line, then

$$T_{(a,b)}(L) = \{(k+a, \ell+b), (k+1+a, \ell+b)\},$$

which is again a horizontal line. If

$$L = \{(k,\ell), (k,\ell+1)\}$$

is a vertical line, then

$$T_{(a,b)}(L) = \{(k+a, \ell+b), (k+a, \ell+1+b)\},$$

which is again a vertical line. Therefore $T_{(a,b)}$ is a transformation.

(b) See previous question.

(c) We first consider R_x. The map $R_x(k,\ell) = (-k,\ell)$ is a bijection. Indeed, for every $(k,\ell) \in \mathbb{Z}^2$,

$$R_x(R_x(k,\ell)) = R_x(-k,\ell) = (k,\ell).$$

Therefore $R_x^{-1} = R_x$, so R_x is a bijection.

If

$$L = \{(k,\ell), (k+1,\ell)\},$$

then

$$R_x(L) = \{(-k,\ell), (-k-1,\ell)\} = \{(-k-1,\ell), (-k,\ell)\},$$

which is a horizontal line. If

$$L = \{(k,\ell), (k,\ell+1)\},$$

then

$$R_x(L) = \{(-k,\ell), (-k,\ell+1)\},$$

which is a vertical line. Therefore R_x is a transformation.

Similarly, $R_y(k,\ell) = (k,-\ell)$ is a bijection. It sends horizontal lines to horizontal lines and vertical lines to vertical lines:

$$\{(k,\ell), (k+1,\ell)\} \longmapsto \{(k,-\ell), (k+1,-\ell)\},$$

and

$$\{(k,\ell),(k,\ell+1)\} \longmapsto \{(k,-\ell),(k,-\ell-1)\}.$$

Therefore R_y is also a transformation.

(d) See previous question.

(e) The map

$$S(k,\ell) = (\ell,k)$$

is a bijection. Indeed, for every $(k,\ell) \in \mathbb{Z}^2$,

$$S(S(k,\ell)) = S(\ell,k) = (k,\ell).$$

Therefore $S^{-1} = S$, so S is a bijection.

If

$$L = \{(k,\ell),(k+1,\ell)\}$$

is horizontal, then

$$S(L) = \{(\ell,k),(\ell,k+1)\},$$

which is vertical. If

$$L = \{(k,\ell),(k,\ell+1)\}$$

is vertical, then

$$S(L) = \{(\ell,k),(\ell+1,k)\},$$

which is horizontal. Hence S sends lines to lines. Therefore S is a transformation.

(f) See previous question.

Problem 2 ***(35 points) Normalizing a transformation.***

Let F be a transformation of $\mathcal{M}_{\mathbb{Z}^2}$.

(a) ***(7 points)*** *Let $p \in \mathbb{Z}^2$. Show that F sends the four points adjacent to p to the four points adjacent to $F(p)$.*

(b) ***(7 points)*** *Show that, by composing F on the left with a translation, we may assume that*

$$F(0,0) = (0,0).$$

(c) ***(7 points)*** *Now assume that $F(0,0) = (0,0)$. Show that $F(1,0)$ must be one of*

$$(1,0), \quad (-1,0), \quad (0,1), \quad (0,-1).$$

(d) ***(7 points)*** *Show that, by composing F on the left with transformations chosen from*

$$\text{id}, \quad R_x, \quad R_y, \quad R_x \circ R_y, \quad S, \quad R_x \circ S, \quad R_y \circ S, \quad R_x \circ R_y \circ S,$$

we may assume that

$$F(0,0) = (0,0) \qquad \textit{and} \qquad F(1,0) = (1,0).$$

(e) ***(7 points)*** *Now assume that* $F(0,0) = (0,0)$ *and* $F(1,0) = (1,0)$. *Show that*

$$F(0,1) = (0,1) \qquad \text{or} \qquad F(0,1) = (0,-1).$$

Then explain why, by composing with R_y *if necessary, we may assume that*

$$F(0,0) = (0,0), \qquad F(1,0) = (1,0), \qquad F(0,1) = (0,1).$$

Solution: **(a)** Let $p \in \mathbb{Z}^2$. If q is adjacent to p, then $\{p,q\}$ is a line. Since F is a transformation,

$$F(\{p,q\}) = \{F(p), F(q)\}$$

is a line. Therefore $F(q)$ is adjacent to $F(p)$.

The point p has exactly four adjacent points. Since F is injective, the images of these four points are four distinct adjacent points of $F(p)$. But $F(p)$ also has exactly four adjacent points. Therefore F sends the four points adjacent to p exactly to the four points adjacent to $F(p)$.

We will also use the following consequence. If two points p, q have r common adjacent points, then $F(p)$ and $F(q)$ also have r common adjacent points. Indeed, by the previous paragraph, F sends the set of neighbors of p bijectively to the set of neighbors of $F(p)$, and similarly for q.

(b) Suppose

$$F(0,0) = (a,b).$$

Define

$$G := T_{(-a,-b)} \circ F.$$

Since translations are transformations and the composition of transformations is a transformation, G is a transformation. Moreover,

$$G(0,0) = T_{(-a,-b)}(F(0,0)) = T_{(-a,-b)}(a,b) = (0,0).$$

Thus, by composing F on the left with a translation, we may assume that

$$F(0,0) = (0,0).$$

(c) Assume

$$F(0,0) = (0,0).$$

Since $(1,0)$ is adjacent to $(0,0)$, part (a) implies that $F(1,0)$ is adjacent to $F(0,0) = (0,0)$. The four points adjacent to $(0,0)$ are

$$(1,0), \quad (-1,0), \quad (0,1), \quad (0,-1).$$

Therefore

$$F(1,0) \in \{(1,0), (-1,0), (0,1), (0,-1)\}.$$

(d) By part (b), after composing on the left with a translation, we may assume that

$$F(0,0) = (0,0).$$

By part (c),

$$F(1,0) \in \{(1,0), (-1,0), (0,1), (0,-1)\}.$$

We now compose on the left with one of the listed transformations, all of which fix $(0,0)$, to send $F(1,0)$ to $(1,0)$.

If $F(1,0) = (1,0)$, use id. If $F(1,0) = (-1,0)$, use R_x. If $F(1,0) = (0,1)$, use S. If $F(1,0) = (0,-1)$, use $R_x \circ S$, since

$$(R_x \circ S)(0,-1) = R_x(-1,0) = (1,0).$$

Therefore, after composing F on the left with transformations from the given list, we may assume that

$$F(0,0) = (0,0) \qquad \text{and} \qquad F(1,0) = (1,0).$$

(e) Assume

$$F(0,0) = (0,0), \qquad F(1,0) = (1,0).$$

Since $(0,1)$ is adjacent to $(0,0)$, part (a) implies that $F(0,1)$ is adjacent to $(0,0)$. Hence

$$F(0,1) \in \{(1,0), (-1,0), (0,1), (0,-1)\}.$$

Since $F(1,0) = (1,0)$ and F is injective, we cannot have

$$F(0,1) = (1,0).$$

We now rule out $F(0,1) = (-1,0)$. The points $(1,0)$ and $(0,1)$ have two common adjacent points:

$$(0,0) \qquad \text{and} \qquad (1,1).$$

By part (a), $F(1,0)$ and $F(0,1)$ must also have two common adjacent points. But if $F(0,1) = (-1,0)$, then $F(1,0) = (1,0)$, and the points $(1,0)$ and $(-1,0)$ have only one common adjacent point, namely $(0,0)$. This is impossible.

Therefore

$$F(0,1) = (0,1) \qquad \text{or} \qquad F(0,1) = (0,-1).$$

If $F(0,1) = (0,1)$, we do nothing. If $F(0,1) = (0,-1)$, we compose on the left with R_y. Since

$$R_y(0,0) = (0,0), \qquad R_y(1,0) = (1,0), \qquad R_y(0,-1) = (0,1),$$

we may assume

$$F(0,0) = (0,0), \qquad F(1,0) = (1,0), \qquad F(0,1) = (0,1).$$

Problem 3 ***(25 points) The normalized transformation fixes part of the grid.***

Let F be a transformation of $\mathcal{M}_{\mathbb{Z}^2}$. Assume that

$$F(0,0) = (0,0), \qquad F(1,0) = (1,0), \qquad F(0,1) = (0,1).$$

(a) ***(8 points)*** *Show that $F(1,1) = (1,1)$.*

(b) ***(8 points)*** *Show that $F(-1,0) = (-1,0)$ and $F(0,-1) = (0,-1)$.*

(c) ***(9 points)*** *Show, by induction, that*

$$F(k,0) = (k,0)$$

for every integer $k \geq 0$.

Solution: Assume throughout that

$$F(0,0) = (0,0), \qquad F(1,0) = (1,0), \qquad F(0,1) = (0,1).$$

(a) The point $(1,1)$ is adjacent to both $(1,0)$ and $(0,1)$. Therefore $F(1,1)$ is adjacent to both

$$F(1,0) = (1,0) \qquad \text{and} \qquad F(0,1) = (0,1).$$

The common adjacent points of $(1,0)$ and $(0,1)$ are exactly

$$(0,0) \qquad \text{and} \qquad (1,1).$$

Hence

$$F(1,1) = (0,0) \qquad \text{or} \qquad F(1,1) = (1,1).$$

Since $F(0,0) = (0,0)$ and F is injective, we cannot have $F(1,1) = (0,0)$. Therefore

$$F(1,1) = (1,1).$$

(b) By Problem 2(a), F sends the four neighbors of $(0,0)$ to the four neighbors of $(0,0)$. Since

$$F(1,0) = (1,0) \qquad \text{and} \qquad F(0,1) = (0,1),$$

the remaining two neighbors must be sent to the remaining two neighbors:

$$F(\{(-1,0),(0,-1)\}) = \{(-1,0),(0,-1)\}.$$

Thus either

$$F(-1,0) = (-1,0), \quad F(0,-1) = (0,-1),$$

or

$$F(-1,0) = (0,-1), \quad F(0,-1) = (-1,0).$$

We rule out the second possibility. The points $(1,0)$ and $(0,-1)$ have two common adjacent points:

$$(0,0) \qquad \text{and} \qquad (1,-1).$$

Therefore their images must also have two common adjacent points. If $F(0,-1) = (-1,0)$, then the images would be

$$F(1,0) = (1,0) \qquad \text{and} \qquad F(0,-1) = (-1,0).$$

But $(1,0)$ and $(-1,0)$ have only one common adjacent point, namely $(0,0)$. This is impossible.

Therefore

$$F(-1,0) = (-1,0) \qquad \text{and} \qquad F(0,-1) = (0,-1).$$

(c) We prove by induction that

$$F(k,0) = (k,0)$$

for every integer $k \geq 0$.

The base cases $k = 0$ and $k = 1$ hold by assumption.

Assume that $n \geq 1$, and suppose that

$$F(k,0) = (k,0)$$

for every $0 \leq k \leq n$. We prove that

$$F(n+1,0) = (n+1,0).$$

Since $(n+1,0)$ is adjacent to $(n,0)$, its image $F(n+1,0)$ is adjacent to

$$F(n,0) = (n,0).$$

Hence

$$F(n+1,0) \in \{(n-1,0),(n+1,0),(n,1),(n,-1)\}.$$

The first possibility is impossible, because $F(n-1,0) = (n-1,0)$ by the induction hypothesis and F is injective.

We now rule out the possibilities $(n,1)$ and $(n,-1)$. The original points

$$(n-1,0) \qquad \text{and} \qquad (n+1,0)$$

have exactly one common adjacent point, namely $(n,0)$. Therefore their images must also have exactly one common adjacent point.

If

$$F(n+1,0) = (n,1),$$

then the two image points

$$F(n-1,0) = (n-1,0) \qquad \text{and} \qquad F(n+1,0) = (n,1)$$

have two common adjacent points:

$$(n,0) \qquad \text{and} \qquad (n-1,1).$$

This is impossible.

Similarly, if

$$F(n+1,0) = (n,-1),$$

then the two image points

$$(n-1,0) \qquad \text{and} \qquad (n,-1)$$

have two common adjacent points:

$$(n,0) \qquad \text{and} \qquad (n-1,-1).$$

This is also impossible.

Therefore the only remaining possibility is

$$F(n+1,0) = (n+1,0).$$

By induction,

$$F(k,0) = (k,0)$$

for every $k \geq 0$.