Week 10 Section Worksheet: Transformations, Ceva, and Conics

This worksheet explores how transformations can simplify geometric proofs. We prove Ceva's theorem by affine normalization and prove that five points determine a conic by projective normalization.

1. Transformations as a proof strategy

A recurring idea in this course is that transformations allow us to replace a complicated configuration by a simpler one.

For example, in the proof of Pappus' theorem, we may send two projective lines to convenient coordinate axes.

The general strategy is:

(a) Normalize the configuration.

(b) Prove the theorem in the normalized case.

(c) Transform back to the original configuration.

2. Ceva's theorem by affine normalization

We begin with a classical theorem about concurrence in a triangle.

Theorem (Ceva's theorem). Let ABC be a triangle, and let

$$D \in BC, \qquad E \in CA, \qquad F \in AB$$

be points on the three sides, none of them equal to a vertex. Then the three lines

$$AD, \qquad BE, \qquad CF$$

are concurrent if and only if

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$

In this section, we prove Ceva's theorem using transformations. The key idea is that affine transformations preserve:

Author(s):

- collinearity;
- concurrency of lines;
- ratios along a line.

Therefore, instead of proving Ceva's theorem in an arbitrary triangle, we may use an affine transformation to send the triangle to the standard triangle

$$(0,0), \qquad (1,0), \qquad (0,1).$$

The proof then becomes a coordinate calculation.

Problem 1 ***Affine normalization of a triangle.***

Let $A, B, C \in \mathbb{R}^2$ be three non-collinear points.

(a) *Explain why the vectors $B - A$ and $C - A$ form a basis of $\mathbb{R}^2$.*

(b) *Define*

$$\Phi(u, v) = A + u(B - A) + v(C - A).$$

Show that Φ is an affine transformation.

(c) *Compute*

$$\Phi(0,0), \qquad \Phi(1,0), \qquad \Phi(0,1).$$

Conclude that any nondegenerate triangle can be sent by an affine transformation to the standard triangle

$$(0,0), \qquad (1,0), \qquad (0,1).$$

Problem 2 ***Ceva's theorem in the standard triangle.***

We now prove Ceva's theorem for the standard triangle. Let

$$A = (0,0), \qquad B = (1,0), \qquad C = (0,1).$$

Let $D \in BC$, $E \in CA$, and $F \in AB$, with none of D, E, F equal to a vertex. Write

$$D = (1 - t, t), \qquad E = (0, u), \qquad F = (v, 0),$$

where $0 < t, u, v < 1$.

(a) *Show that*

$$\frac{BD}{DC} = \frac{t}{1-t}, \qquad \frac{CE}{EA} = \frac{1-u}{u}, \qquad \frac{AF}{FB} = \frac{v}{1-v}.$$

(b) *Show that the equation of the line AD is*

$$tx - (1-t)y = 0.$$

(c) *Show that the equations of the lines BE and CF are*

$$ux + y - u = 0 \qquad \text{and} \qquad x + vy - v = 0.$$

(d) *Explain why the three lines AD, BE, and CF are concurrent exactly when the system*

$$\begin{cases} tx - (1-t)y = 0, \\ ux + y - u = 0, \\ x + vy - v = 0 \end{cases}$$

has a common solution (x, y).

(e) *Rewrite this system as*

$$\begin{pmatrix} t & -(1-t) & 0 \\ u & 1 & -u \\ 1 & v & -v \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Deduce that concurrency implies

$$\det \begin{pmatrix} t & -(1-t) & 0 \\ u & 1 & -u \\ 1 & v & -v \end{pmatrix} = 0.$$

(f) *Compute this determinant and show that the condition above is equivalent to*

$$u(1-t)(1-v) = tv(1-u).$$

(g) *Conclude that AD, BE, and CF are concurrent if and only if*

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$

Problem 3 ***Ceva's theorem in an arbitrary triangle.***

Explain why the result for the standard triangle implies Ceva's theorem for any nondegenerate triangle.

Your answer should mention:

- *affine transformations;*
- *preservation of collinearity;*
- *preservation of concurrency;*
- *preservation of ratios along a line.*

3. Five points determine a conic

We now move from affine normalization to projective normalization.

Recall that a conic in $\mathbb{P}^2(\mathbb{F})$ is the zero set of a nonzero homogeneous quadratic equation

$$aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ = 0.$$

Two equations which differ by a nonzero scalar define the same conic.

Theorem. Let

$$P_1, P_2, P_3, P_4, P_5 \in \mathbb{P}^2(\mathbb{F})$$

be five points such that no three are collinear. Then there is a unique conic passing through them.

We will prove this theorem by using a projective transformation to simplify four of the points.

Problem 4 ***Normalizing four points.***

Let $P_1, P_2, P_3, P_4 \in \mathbb{P}^2(\mathbb{F})$ be four points such that no three of them are collinear.

(a) *Choose vector representatives*

$$P_i = [v_i], \qquad v_i \in \mathbb{F}^3 \setminus \{0\}.$$

Explain why v_1, v_2, v_3 form a basis of $\mathbb{F}^3$.

(b) *Write*

$$v_4 = \alpha v_1 + \beta v_2 + \gamma v_3.$$

Explain why

$$\alpha, \beta, \gamma \neq 0.$$

(c) *Define a linear map $A : \mathbb{F}^3 \to \mathbb{F}^3$ by*

$$A(\alpha v_1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad A(\beta v_2) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad A(\gamma v_3) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Explain why A is invertible.

(d) *Let $[A]$ be the induced projective transformation of $\mathbb{P}^2(\mathbb{F})$. Show that*

$$[A](P_1) = [1 : 0 : 0], \qquad [A](P_2) = [0 : 1 : 0], \qquad [A](P_3) = [0 : 0 : 1],$$

and

$$[A](P_4) = [1 : 1 : 1].$$

Problem 5 ***Conics through four normalized points.***

Consider the four points

$$[1:0:0], \qquad [0:1:0], \qquad [0:0:1], \qquad [1:1:1]$$

in $\mathbb{P}^2(\mathbb{F})$.

(a) *Let*

$$aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ = 0$$

be a conic passing through

$$[1:0:0], \qquad [0:1:0], \qquad [0:0:1].$$

Show that

$$a = b = c = 0.$$

(b) *Deduce that the equation has the form*

$$dXY + eXZ + fYZ = 0.$$

(c) *Now impose that the conic passes through* $[1:1:1]$. *Show that*

$$d + e + f = 0.$$

Problem 6 ***Adding the fifth point.***

Let

$$P = [u:v:w] \in \mathbb{P}^2(\mathbb{F})$$

be a fifth point, and suppose that the five points

$$[1:0:0], \qquad [0:1:0], \qquad [0:0:1], \qquad [1:1:1], \qquad [u:v:w]$$

are distinct and no three are collinear.

(a) *Explain why*

$$u, v, w \neq 0.$$

(b) *Show that a conic passing through the five points must have equation*

$$dXY + eXZ + fYZ = 0,$$

where

$$d + e + f = 0$$

and

$$duv + euw + fvw = 0.$$

(c) *Explain why the two equations*

$$d + e + f = 0 \qquad \text{and} \qquad duv + euw + fvw = 0$$

are independent.

Hint: If they were dependent, then (uv, uw, vw) *would be proportional to* $(1, 1, 1)$.

(d) *Conclude that the space of triples* (d, e, f) *satisfying these equations is one-dimensional.*

(e) *Explain why this means that there is a unique conic passing through the five points.*

Problem 7 ***The five-point theorem for conics.***

Prove the following theorem.

Theorem. *Let*

$$P_1, P_2, P_3, P_4, P_5 \in \mathbb{P}^2(\mathbb{F})$$

be five points such that no three are collinear. Then there is a unique conic passing through them.

Your proof should use:

- *the projective normalization of four points;*
- *the normalized computation above;*
- *the fact that projective transformations send conics to conics.*