Week 2 Section Worksheet

A worksheet on Euclid-style proofs, summary diagrams, and constructions from Book I.

Problem 1 ***Euclid-style proof: diagonals of a rhombus.***

Definition. *A rhombus is a quadrilateral with four equal sides.*

Exercise. *Show that the diagonal of a rhombus meet orthogonal.*

Remark. *At this stage, you should assume that the diagonals meet.*

Suggestion. *Begin by drawing an annotated diagram containing the information you know from the definition of a rhombus.*

Hint. *Consider a rhombus $ABCD$ and assume the diagonals AC and BD intersect in a point E. By the definition of right angles, it is enough to show that*

$$\angle BEA = \angle CEB.$$

Indeed, since A, E, and C lie on the same straight line, equal adjacent angles imply that the diagonals are orthogonal.

Problem 2 ***"Summary diagrams".*** *Read the enunciation of Propositions I.13 through I.21. For each proposition, draw a summary diagram that helps you remember what the proposition says.*

Problem 3 ***Recognizing Euclid's propositions.***

For each of the following statements, identify which proposition of Euclid's Book I *states this result (or can be directly applied).*

(a) *Let ABC be a triangle. Suppose that*

$$AB > AC.$$

Then

$$\angle ACB > \angle ABC.$$

(b) *Let ABC be a triangle. Suppose that*

$$\angle ACB > \angle ABC.$$

Then

$$AB > AC.$$

Author(s):

(c) *Let ABC be a triangle, and let D be a point on BC. Then*

$$AB + AC > BD + DC.$$

(d) *Let ABC be a triangle, and let D be a point on BC extended beyond C. Then*

$$\angle ACD > \angle ABC.$$

Problem 4 ***Further activities.***

(a) *In GeoGebra, GeoGebra Classic, or with pen and paper, carry out the constructions of Propositions I.22 and I.23. You may use the following GeoGebra activity where the necessary preliminary tools have already been created.*

See page 4 once you have tried making Proposition I.22.

Remark. Proposition I.23 uses an alternative version of Proposition I.22 which, given a triangle ABC and a ray $[DE)$, allows you to "move" the triangle ABC so that A is sent to D and B lies on the ray $[DE)$.

More precisely, the construction produces points F and G such that F lies on the ray $[DE)$ and

$$DF = AB, \qquad DG = AC, \qquad FG = BC.$$

Thus the triangle DFG is congruent to the triangle ABC, with the side corresponding to AB placed along the given ray.

I created this tool, and you can play around with it using the corresponding link GeoGebra activity.

As a last exercise, you can try to design this tool yourself. In my construction, I only used Proposition I.2.

(b) *Choose one proposition among Propositions I.13 through I.23. Identify its six parts as clearly as you can:*

Enunciation, Exposition, Specification, Construction, Proof, Conclusion.

If you find any gap, unclear step, or point where Euclid's proof is not fully rigorous from a modern point of view, explain it briefly.

(c) *Suppose again that $ABCD$ is a rhombus and that the diagonals AC and BD meet at a point E. Can you show that E is the midpoint of both diagonals?*

See the next page for a possible approach to the rhombus exercise, after you have spent time thinking about it.

A possible approach to the rhombus exercise

Since $AB = AD$, Proposition I.5 gives

$$\angle ABD = \angle BDA.$$

Since $CB = CD$, Proposition I.5 gives

$$\angle CBD = \angle BDC.$$

Now compare triangles ABD and CBD. We have

$$AB = BC, \qquad AD = DC, \qquad BD = BD.$$

Hence, by Proposition I.8,

$$\angle ABD = \angle CBD.$$

Since E lies on the line BD, this gives

$$\angle ABD = \angle ABE \qquad \text{and} \qquad \angle CBD = \angle CBE.$$

Therefore

$$\angle ABE = \angle CBE.$$

Now compare triangles ABE and CBE. We have

$$BA = BC, \qquad BE = BE, \qquad \angle ABE = \angle CBE.$$

Hence, by Proposition I.4,

$$\angle BEA = \angle CEB.$$

Since A, E, and C lie on the same straight line, it follows that the equal adjacent angles $\angle BEA$ and $\angle CEB$ are right angles. Therefore,

$$BD \perp AC \qquad \text{at } E.$$

GeoGebra note on Proposition I.22

One difficulty with GeoGebra is that Euclid's original constructions are not always well suited for building a custom tool.

Recall the goal of Proposition I.22:

> Given three straight lines equal to three given straight lines, to construct a triangle.

In Euclid's notation, the three given segments are usually called A, B, and C. The goal is to construct a triangle whose three sides are equal to A, B, and C.

Ideally, we would like to create a GeoGebra tool whose only inputs are the three given segments, and which works in any configuration where the triangle inequality is satisfied. In other words, no one segment should be longer than the sum of the other two.

As it stands, Euclid's construction has a few problems from the point of view of GeoGebra.

First, it asks you to create a new point D, which has nothing to do with the initial data you started with, and then to create new objects based on this point. As a result, if you want to turn the construction into a custom tool, you would have to add this extra point to your initial input, even though it is not really part of the given data. This is unnecessary.

Second, the construction does not work well in all configurations. For example, in order to carry out Euclid's construction, you need to choose a point E very far along a ray emanating from D. The reason is that later you must remove from the longer segment DE a segment equal to A using Proposition I.3. But this is already problematic in GeoGebra, because the point E is chosen arbitrarily and is not tied to the initial data A, B, and C. As soon as your given sides become too long relative to your original choice of E, the construction will no longer work properly, and your tool will break.

For GeoGebra, it is therefore better to use a modified construction that depends only on the three given segments and works uniformly whenever the triangle inequality holds.

Suppose the three given segments are AB, CD, and EF.

(a) Create the three segments AB, CD, and EF.

(b) Construct the full line (AB).

(c) Use your Proposition I.2 tool to copy the segment CD starting at the point B.

In GeoGebra, click in the order

$$B,\ C,\ D.$$

This produces a point G such that

$$BG = CD.$$

(d) Draw the circle centered at B with radius BG.

(e) Let H be the intersection of this circle with the line (AB) on the right of B.

Then

$$BH = BG = CD.$$

(f) Use your Proposition I.2 tool again to copy the segment EF starting at the point H.

In GeoGebra, click in the order

$$H,\ E,\ F.$$

This produces a point I such that

$$HI = EF.$$

(g) Draw the circle centered at H with radius HI.

(h) Draw the circle centered at B with radius AB.

(i) Let K be an intersection point of these two circles that does not lie on the line (AB).

(j) Draw the segments BK, KH, and BH.

Then the triangle BKH has the required side lengths:

$$BK = AB, \qquad BH = CD, \qquad KH = EF.$$

So $\triangle BKH$ is the triangle we wanted.

Once the construction is complete, you can create a new GeoGebra tool with:

- inputs: A, B, C, D, E, F;
- outputs: the three sides BK, BH, and KH.

You can then test the tool on many different configurations.

A particularly important thing to observe is what happens when the triangle inequality fails. If one of the three given segments is longer than the sum of the other two, then the two circles in the construction will no longer meet in a point above the line, and the construction breaks down. This is exactly what should happen: in that case, no triangle with those three side lengths exists.

You may use the following GeoGebra activity to see a working version of this Proposition I.22 tool.