Week 4 Section Worksheet

A worksheet on incidence geometry and parallel properties.

We recall the axioms of incidence geometry:

- **(I1)** Given two distinct points, there is a unique line incident with both of them.
- **(I2)** Given a line, there exist two distinct points incident with it.
- **(I3)** There exist three non-collinear points.

Problem 1 *Assume only the incidence axioms **I1** and **I3**.*

Show that the following statements can already be deduced:

(i) For every point, there exists at least one line not passing through it.

(ii) For every point P, there exist at least two distinct lines passing through P.

Problem 2 *In each of the following models, determine which of the incidence axioms **I1**, **I2**, and **I3** are satisfied and which are not. Then determine whether the model has the* elliptic, Euclidean, *or* hyperbolic *parallel property.*

(i) "Points" are lines in Euclidean three-dimensional space, "lines" are planes in Euclidean three-dimensional space, and "incidence" is the usual relation of a line lying in a plane.

(ii) Same as in (i), except that we restrict ourselves to lines and planes passing through a fixed ordinary point O.

(iii) "Points" are points on the unit sphere in $\mathbb{R}^3$, "lines" are great circles (that is, intersections of the sphere with planes in $\mathbb{R}^3$ passing through 0), and "incidence" is the standard one: a point is incident to a line if it lies on it.

Problem 3 *(i1) Can you find a model with only 2 points satisfying **I1** and **I2**?*

(i2) Do you think it is possible to find two such models which are not isomorphic? Prove your claim.

*(ii1) Can you find a model with only 3 points satisfying **I1**, **I2**, and **I3**?*

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(ii2) Do you think it is possible to find two such models which are not isomorphic? Prove your claim.

Problem 4 *Show that axioms **I1**, **I2**, and **I3** are independent. In other words, for each axiom, find a model satisfying the other two but not that one.*

Problem 5 ***If time permits.***

*(i) Find a model of incidence geometry (i.e. satisfying axioms **I1**, **I2**, and **I3**) which does not satisfy any of the elliptic, Euclidean, or hyperbolic parallel properties.*

Hint. *There is such a model with 5 points.*

(ii) How many non-isomorphic models of incidence geometry with 4 points can you find? For each model, determine which parallel property (elliptic, Euclidean, hyperbolic), or lack of parallel property, it satisfies.

(iii) Same question with 5 points.