Week 6 Section Worksheet: Counting in Affine and Projective Planes

In this worksheet, we study several counting arguments in affine and projective planes. We first prove that any two lines in a projective plane have the same number of points. We then deduce the standard formula for the number of points in a finite projective plane. Finally, we prove that in a finite affine plane of order n, every point lies on exactly $n+1$ lines.

Notation

Let $\mathcal{M} = (\mathcal{P}, \mathcal{L}, I)$ be an incidence geometry. If $\ell \in \mathcal{L}$, we write

$$R(\ell) = \{p \in \mathcal{P} : p \, I \, \ell\}$$

for the set of points incident with ℓ.

Recall that in a projective plane, any two distinct lines meet in a unique point.

Problem 1: A bijection between two projective lines

Let $\mathcal{M} = (\mathcal{P}, \mathcal{L}, I)$ be a projective plane. The goal of this exercise is to prove the following statement:

If ℓ and m are distinct lines, then there is a bijective map

$$f : R(\ell) \longrightarrow R(m).$$

Below, we introduce a map. Your goal will be to prove that this map is bijective.

Assume that

$$\ell \neq m.$$

By yesterday's activity, there exists a point $P \in \mathcal{P}$ such that

$$P \not I \, \ell \qquad \text{and} \qquad P \not I \, m.$$

Author(s): Ximera Development Team

Let $x \in R(\ell)$. By axiom **I1**, there exists a unique line $n_x \in \mathcal{L}$ incident with both x and P:

$$x \, I \, n_x \qquad \text{and} \qquad P \, I \, n_x.$$

Since $P \not I \, m$, the line n_x is not equal to m.

Thus, by the projective plane axiom that any two distinct lines meet in a unique point, n_x and m meet in a unique point. Define

$$f(x) := \text{the unique point in } R(n_x) \cap R(m).$$

This defines a map

$$f : R(\ell) \longrightarrow R(m).$$

(a) Show that f is injective. That is, show that if

$$f(x_1) = f(x_2),$$

then

$$x_1 = x_2.$$

(b) Show that f is surjective. That is, show that for every $y \in R(m)$, there exists $x \in R(\ell)$ such that

$$f(x) = y.$$

Problem 2: Counting points in a finite projective plane

Let $\mathcal{M} = (\mathcal{P}, \mathcal{L}, I)$ be a projective plane. Suppose that there exists a line

$$\ell \in \mathcal{L}$$

such that $R(\ell)$ contains exactly $n + 1$ points.

The goal of this exercise is to prove that

$$|\mathcal{P}| = n^2 + n + 1.$$

If you want to challenge yourself, I recommend trying this on your own first! Then, you may look at the questions below for hints.

(a) Let $p \in \mathcal{P}$ be a point such that

$$p \not I \, \ell.$$

Explain why such a point exists.

(b) For each point $x \in R(\ell)$, let r_x be the unique line incident with both p and x:

$$p \, I \, r_x \qquad \text{and} \qquad x \, I \, r_x.$$

Show that if $x_1, x_2 \in R(\ell)$ and $x_1 \neq x_2$, then

$$r_{x_1} \neq r_{x_2}.$$

(c) Show that every line incident with p is equal to r_x for some $x \in R(\ell)$.

(d) Deduce that exactly $n + 1$ lines are incident with p.

(e) By Problem 1, every line has the same number of points. Deduce that every line contains exactly $n + 1$ points.

(f) For each $x \in R(\ell)$, explain why

$$R(r_x) \setminus \{p\}$$

has exactly n points.

(g) Show that the sets

$$R(r_x) \setminus \{p\}, \qquad x \in R(\ell),$$

form a partition of

$$\mathcal{P} \setminus \{p\}.$$

(h) Conclude that

$$|\mathcal{P}| = 1 + (n+1)n = n^2 + n + 1.$$

Problem 3: Counting lines through a point in an affine plane

Let $\mathcal{A} = (\mathcal{P}, \mathcal{L}, I)$ be a finite affine plane of order n. This means that every line is incident with exactly n points. You may also use the fact that

$$|\mathcal{P}| = n^2.$$

Fix a point $p \in \mathcal{P}$. The goal of this exercise is to prove that exactly $n + 1$ lines are incident with p.

Let

$$r_1, \ldots, r_k$$

be the distinct lines incident with p.

(a) Show that every point $q \in \mathcal{P} \setminus \{p\}$ lies on one of the lines

$$r_1, \ldots, r_k.$$

(b) Show that if $i \neq j$, then

$$\big(R(r_i) \setminus \{p\}\big) \cap \big(R(r_j) \setminus \{p\}\big) = \varnothing.$$

(c) Deduce that the sets

$$R(r_1) \setminus \{p\}, \ldots, R(r_k) \setminus \{p\}$$

form a partition of $\mathcal{P} \setminus \{p\}$.

(d) Explain why each set

$$R(r_i) \setminus \{p\}$$

has exactly $n - 1$ points.

(e) Deduce that

$$n^2 - 1 = k(n - 1).$$

(f) Conclude that

$$k = n + 1.$$

Therefore exactly $n + 1$ lines are incident with p.

Bonus exercises

The following questions are optional challenge problems.

(a) Show that, up to isomorphism, there exists a unique affine plane with 4 points, i.e. a unique affine plane of order 2.

(b) Show that, up to isomorphism, there exists a unique affine plane with 9 points, i.e. a unique affine plane of order 3.

(c) Construct an affine plane of order 4 using a diagram. That is, construct an affine plane with 16 points.

(d) Construct an affine plane of order 5 using a diagram. That is, construct an affine plane with 25 points.

(e) Can there exist an affine plane of order 6, i.e. an affine plane with 36 points?

For this last question, look up the *Bruck–Ryser–Chowla Theorem* and think about how it can be used.