Week 7 Section Worksheet: Homogeneous Coordinates and Projective Completion

The purpose of this worksheet is to make sure you have a solid understanding of homogeneous coordinates and to fill in some of the gaps from Section 11 of W7L1.

Notation

Let F be $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, or any other field (if you know what that means, no problem if not!). Recall that $\mathbb{P}^2(F)$ is defined as follows. We consider $F^3 \setminus \{(0,0,0)\}$ and define

$$(x, y, z) \sim (x', y', z')$$

if there exists $\lambda \in F^\times$ such that $(x', y', z') = \lambda(x, y, z)$. The equivalence class of (x, y, z) is denoted by

$$[x : y : z].$$

The points of $\mathbb{P}^2(F)$ are the equivalence classes $[x : y : z]$. The lines of $\mathbb{P}^2(F)$ are subsets of the form

$$\{[x : y : z] \in \mathbb{P}^2(F) : ax + by + cz = 0\},$$

where $(a, b, c) \neq (0, 0, 0)$. Incidence is the usual relation of belonging.

Problem: $\mathbb{P}^2(F)$ as the projective completion of F^2

The goal of this problem is to prove that $\mathbb{P}^2(F)$ is isomorphic to the projective completion of the affine plane F^2.

Part 1: Homogeneous coordinates

(a) Show that $\sim$ is an equivalence relation on $F^3 \setminus \{(0,0,0)\}$.
Hint: Check reflexivity, symmetry, and transitivity.

(b) For each of the following, answer YES or NO. Briefly justify your answer.

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(i) Is $[1:0:0] = [2:0:0]$ in $\mathbb{P}^2(\mathbb{Q})$?
(ii) Is $[1:0:0] = [1:1:1]$ in $\mathbb{P}^2(\mathbb{Q})$?
(iii) Is $[1:2:3] = [2:4:6]$ in $\mathbb{P}^2(\mathbb{Q})$?
(iv) Is $[1:2:3] = [2:4:5]$ in $\mathbb{P}^2(\mathbb{Q})$?
(v) Is $[1:1:0] = [2:2:0]$ in $\mathbb{P}^2(\mathbb{Q})$?

(c) Let $[x:y:z] \in \mathbb{P}^2(F)$ with $z \neq 0$. Show that

$$[x:y:z] = \left[\frac{x}{z} : \frac{y}{z} : 1\right].$$

Part 2: Lines in $\mathbb{P}^2(F)$

(a) Show that the expression

$$ax + by + cz = 0$$

defines a well-defined condition on projective points. In other words, show that if $[x:y:z] = [x':y':z']$, then $ax + by + cz = 0$ if and only if $ax' + by' + cz' = 0$.

(b) Describe the projective line passing through $[1:0:0]$ and $[0:0:1]$.

Hint: Find a nonzero triple (a, b, c) such that both points satisfy $ax + by + cz = 0$. There will be many solutions; a one-dimensional space of solutions to be more precise.

(c) Describe the projective line passing through $[1:0:0]$ and $[0:1:0]$.

(d) Let $a, b, c \in F$, with $(a, b) \neq (0, 0)$, and consider the affine line

$$\ell = \{(x, y) \in F^2 : ax + by + c = 0\}.$$

We define its *projective closure* as

$$\overline{\ell} := \{[x:y:z] \in \mathbb{P}^2(F) : ax + by + cz = 0\}.$$

Show that $\overline{\ell}$ meets the line at infinity $z = 0$ in exactly one point, namely

$$[-b:a:0].$$

Hint: Substitute $z = 0$ into $ax + by + cz = 0$.

Part 3: The map on points

Let $\mathcal{A} = (F^2, \mathcal{L}_{F^2}, I_{F^2})$ be the affine plane over F, and let $\mathcal{A}^* = (\mathcal{P}^*, \mathcal{L}^*, I^*)$ be its projective completion.

Recall that the points of $\mathcal{A}^*$ are of two types: ordinary points $(x, y) \in F^2$, and points at infinity $[\ell]$, where ℓ is a parallel class of affine lines.

Define a map $f : \mathcal{P}^* \to \mathbb{P}^2(F)$ as follows. If $(x, y) \in F^2$, define $f(x, y) = [x : y : 1]$. If $[\ell]$ is a point at infinity, choose a representative ℓ with equation $ax + by + c = 0$, and define

$$f([\ell]) = [-b : a : 0].$$

To help visualize homogeneous coordinates and the projective completion of F^2, you can click here for the Desmos 3D animation introduced in class.

(a) Show that f is well-defined on points at infinity. In other words, show that if two affine lines are parallel, then they determine the same projective point at infinity.

Hint: Parallel affine lines have equations $ax + by + c = 0$ and $\lambda ax + \lambda by + d = 0$, for some $\lambda \in F^\times$.

(b) Show that f is injective.

Hint: Treat separately the cases of two ordinary points, one ordinary point and one point at infinity, and two points at infinity.

(c) Show that f is surjective.

Hint: If $[x : y : z]$ has $z \neq 0$, scale so that $z = 1$. If $z = 0$, show that $[x : y : 0] = [-b : a : 0]$ for some $(a, b) \neq (0, 0)$.

Part 4: The map on lines

Recall that the lines of $\mathcal{A}^*$ are of two types: ordinary affine lines $\ell \in \mathcal{L}_{F^2}$, and the line at infinity ℓ_∞.

Define a map $g : \mathcal{L}^* \to \mathcal{L}_{\mathbb{P}^2(F)}$ as follows. If $\ell \subseteq F^2$ is the affine line $ax + by + c = 0$, define

$$g(\ell) = \{[x : y : z] \in \mathbb{P}^2(F) : ax + by + cz = 0\}.$$

For the line at infinity, define

$$g(\ell_\infty) = \{[x : y : z] \in \mathbb{P}^2(F) : z = 0\}.$$

(a) Show that g is injective.

Hint: If two affine lines have the same projective closure, compare their affine points with $z = 1$. Also, no affine line maps to $z = 0$.

(b) Show that g is surjective.

Hint: Let L be a projective line $Ax+By+Cz=0$. If $(A,B) \neq (0,0)$, show that L is the image of the affine line $Ax+By+C=0$. If $(A,B)=(0,0)$, what is L?

Part 5: Incidence

Finally, show that the maps f and g preserve incidence. In other words, show that for every point $p \in \mathcal{P}^*$ and every line $\ell \in \mathcal{L}^*$,

$$p \, I^* \, \ell \qquad \Longleftrightarrow \qquad f(p) \in g(\ell).$$

(a) Check this when $p=(x,y)$ is an ordinary affine point and ℓ is an ordinary affine line.

(b) Check this when $p=[\ell]$ is a point at infinity and m is an ordinary affine line.

Hint: Write ℓ as $ax+by+c=0$ and m as $a'x+b'y+c'=0$. The point at infinity $[\ell]$ is incident with m exactly when m is parallel to ℓ.

(c) Check this when $p=[\ell]$ is a point at infinity and the line is ℓ_∞.

(d) Conclude that $\mathcal{A}^*$ and $\mathbb{P}^2(F)$ are isomorphic as incidence geometries.