Week 9 Section Worksheet: Transformations of Infinite Incidence Geometries

The purpose of this worksheet is to classify all transformations of two infinite incidence geometries.

Problem 1: The one-sided infinite line

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. Define $\mathcal{M}_{\mathbb{N}} = (\mathcal{P}_{\mathbb{N}}, \mathcal{L}_{\mathbb{N}})$ by

$$\mathcal{P}_{\mathbb{N}} = \mathbb{N}, \qquad \mathcal{L}_{\mathbb{N}} = \big\{\{n, n+1\} : n \in \mathbb{N}\big\}.$$

(a) Draw $\mathcal{M}_{\mathbb{N}}$. What object do you picture?

(b) Guess all transformations of $\mathcal{M}_{\mathbb{N}}$.

(c) Show that 0 is the unique point incident with exactly one line, and that every other point is incident with exactly two lines.

Hint: Transformations preserve the number of incident lines. Also, 1 *is the unique point adjacent to* 0*.*

(d) Prove by induction that $F(n) = n$ for all $n \in \mathbb{N}$.

(e) What can you conclude?

Problem 2: The two-sided ladder

Define $\mathcal{M}_2 = (\mathcal{P}_2, \mathcal{L}_2)$ by

$$\mathcal{P}_2 = \{(0, i) : i \in \mathbb{Z}\} \cup \{(1, i) : i \in \mathbb{Z}\},$$

and

$$\mathcal{L}_2 = \big\{\{(0, i), (0, i+1)\}, \{(1, i), (1, i+1)\}, \{(0, i), (1, i)\} : i \in \mathbb{Z}\big\}.$$

We call $\{(0, i), (0, i+1)\}$ and $\{(1, i), (1, i+1)\}$ *vertical lines*, and $\{(0, i), (1, i)\}$ *rungs*.

(a) Draw the part of $\mathcal{M}_2$ with $-3 \leq i \leq 3$. What object do you picture?

Author(s): Ximera Development Team

(b) Can you think of some transformations of this model?

(c) Show that the following maps are transformations and determine their inverse.:

$$T_k(\epsilon, i) = (\epsilon, i + k), \qquad S(\epsilon, i) = (1 - \epsilon, i), \qquad R(\epsilon, i) = (\epsilon, -i),$$

where $k \in \mathbb{Z}$, $\epsilon \in \{0, 1\}$, and $i \in \mathbb{Z}$.

(d) A *4-cycle* is a set of four distinct points $\{p_0, p_1, p_2, p_3\}$ which can be ordered so that consecutive points, including p_3, p_0, are joined by lines. Show that every set

$$Q_i = \{(0, i), (0, i + 1), (1, i), (1, i + 1)\}$$

is a 4-cycle, and that every 4-cycle is equal to Q_i for a unique $i \in \mathbb{Z}$.

Hint: Start with one point and list its three neighbours.

(e) Show that each rung belongs to exactly two 4-cycles, while each vertical line belongs to exactly one 4-cycle. Conclude that every transformation sends rungs to rungs and vertical lines to vertical lines.

(f) Let F be any transformation. Since $\{(0, 0), (0, 1)\}$ is vertical, prove that there exist $\epsilon \in \{0, 1\}$, $k \in \mathbb{Z}$, and $\sigma \in \{1, -1\}$ such that

$$F(0, 0) = (\epsilon, k), \qquad F(0, 1) = (\epsilon, k + \sigma).$$

Using the transformations from (c), precompose F with a suitable transformation G so that

$$(G \circ F)(0, 0) = (0, 0), \qquad (G \circ F)(0, 1) = (0, 1).$$

Hint: Use, as needed, S, a translation, and R.

(g) Prove the normalized case: if H is a transformation with $H(0, 0) = (0, 0)$ and $H(0, 1) = (0, 1)$, then H is the identity.

Hint: First show $H(1, 0) = (1, 0)$, $H(0, -1) = (0, -1)$, $H(1, 1) = (1, 1)$, and $H(0, 2) = (0, 2)$. Then use induction upward and downward on i, using vertical lines and rungs alternately.

(h) What can you conclude?

Challenge problems

You are welcome, and even encouraged, to invent your own infinite incidence geometry. For each model, try to draw a picture, guess some transformations, then guess all transformations. Below are two models you can think about.

C1. A wider ladder. Fix $r \geq 2$. Define a model with

$$\mathcal{P} = \{0, 1, \ldots, r\} \times \mathbb{Z},$$

where two points (a, i) and (b, j) form a line exactly when

$$|a - b| + |i - j| = 1.$$

What transformations do you expect?

Hint: You should at least find vertical translations, vertical reflections, and the horizontal reflection $a \mapsto r - a$. Are there any others? Try to distinguish boundary points from interior points by counting incident lines.

C2. An infinite tree. Consider the infinite rooted binary tree: one point is called the root; the root is joined to two new points; each of those is joined to two new points; and so on forever. What transformations can you imagine?

Hint: This model has many more transformations than the line or the ladder. For example, at each branching point, you may swap the two infinite subtrees growing from that point. Think about why this gives infinitely many independent choices.