# Week 10 Lecture 1

*Finishing the proof of Pappus' theorem in the normalized case, then concluding the course by discussing Hilbert planes and Euclidean planes.*

## 1. Finishing Pappus' theorem

Last time, we reduced Pappus' theorem to the following normalized projective situation. We work in $\mathbb{P}^2(\mathbb{F})$, and we assume

$$\ell = \{Y = 0\}, \qquad m = \{X = 0\}.$$

Then

$$O = \ell \cap m = [0 : 0 : 1].$$

Since $A, B, C \in \ell$ and none of them is equal to $O$, we may write

$$A = [1 : 0 : a], \qquad B = [1 : 0 : b], \qquad C = [1 : 0 : c],$$

where $a, b, c \in \mathbb{F}$ are distinct. Similarly, since $A', B', C' \in m$ and none of them is equal to $O$, we may write

$$A' = [0 : 1 : a'], \qquad B' = [0 : 1 : b'], \qquad C' = [0 : 1 : c'],$$

where $a', b', c' \in \mathbb{F}$ are distinct.

We want to show that

$$P = AB' \cap A'B, \qquad Q = AC' \cap A'C, \qquad R = BC' \cap B'C$$

are collinear.

## 2. Equation of the line $A'B$

The line $A'B$ is the unique projective line through

$$A' = [0 : 1 : a'] \qquad \text{and} \qquad B = [1 : 0 : b].$$

Recall that lines in $\mathbb{P}^2(\mathbb{F})$ correspond to planes in $\mathbb{F}^3$ passing through the origin. Thus $A'B$ corresponds to the plane through $(0, 0, 0)$ containing the vectors

$$(0, 1, a') \qquad \text{and} \qquad (1, 0, b).$$

Author(s):

A point $[X : Y : Z]$ lies on this line if and only if the vectors

$$(0, 1, a'), \qquad (1, 0, b), \qquad (X, Y, Z)$$

are linearly dependent. Equivalently,

$$\det \begin{pmatrix} 0 & 1 & X \\ 1 & 0 & Y \\ a' & b & Z \end{pmatrix} = 0.$$

This determinant is

$$a'Y + bX - Z.$$

Therefore

$$A'B = \{[X : Y : Z] \in \mathbb{P}^2(\mathbb{F}) : a'Y + bX - Z = 0\}.$$

Similarly,

$$AB' = \{[X : Y : Z] \in \mathbb{P}^2(\mathbb{F}) : b'Y + aX - Z = 0\}.$$

## 3. Expression for $P = AB' \cap A'B$

Let

$$P = [X : Y : Z] \in A'B \cap AB'.$$

Then

$$a'Y + bX - Z = 0$$

and

$$b'Y + aX - Z = 0.$$

Subtracting the two equations gives

$$(a' - b')Y + (b - a)X = 0.$$

Since $a' \neq b'$, we obtain

$$Y = \frac{b - a}{b' - a'}X.$$

Then

$$Z = a'Y + bX = a'\frac{b - a}{b' - a'}X + bX = \frac{bb' - aa'}{b' - a'}X.$$

Thus the solution space is one-dimensional:

$$\left\{ \left( X, \frac{b - a}{b' - a'}X, \frac{bb' - aa'}{b' - a'}X \right) : X \in \mathbb{F} \right\}.$$

Therefore

$$P = [b' - a' : b - a : bb' - aa'].$$

## 4. The other two intersection points

Proceeding in exactly the same way, we obtain

$$Q = AC' \cap A'C = [c' - a' : c - a : cc' - aa']$$

and

$$R = BC' \cap B'C = [c' - b' : c - b : cc' - bb'].$$

Now choose vector representatives

$$p = (b' - a', b - a, bb' - aa'),$$

$$q = (c' - a', c - a, cc' - aa'),$$

and

$$r = (c' - b', c - b, cc' - bb').$$

Then

$$q - p = (c' - b', c - b, cc' - bb') = r.$$

Hence

$$q - p - r = 0.$$

Therefore the three vectors $p, q, r$ are linearly dependent. Since collinearity in $\mathbb{P}^2(\mathbb{F})$ is equivalent to linear dependence of vector representatives, the points

$$P, Q, R$$

are collinear.

This proves Pappus' theorem in the normalized projective case. Since last time we showed that every projective Pappus configuration can be normalized by a projective transformation, this proves projective Pappus. Since affine Pappus follows from projective Pappus by the embedding

$$\mathbb{A}^2(\mathbb{F}) \longrightarrow \mathbb{P}^2(\mathbb{F}), \qquad (x, y) \longmapsto [x : y : 1],$$

we have also proved affine Pappus.

## 5. Course conclusion

We began this course with Euclid's *Elements*. This helped us:

- learn how to read mathematical proofs;
- learn compass-and-straightedge constructions;
- understand what an axiomatic theory is;

- notice gaps in some of Euclid's arguments;
- practice proving statements rigorously using results such as SAS, SSS, and ASA.

Then we moved toward a more rigorous treatment of geometry, especially through Hilbert's axioms. This led us to incidence geometry and to non-Euclidean geometries, including:

- finite geometries with different parallel properties;
- infinite geometries such as $\mathbb{A}^2(\mathbb{F})$, $\mathbb{P}^2(\mathbb{F})$, $\mathcal{M}_{\mathbb{Z}}$, and $\mathcal{M}_{\mathbb{Z}^2}$;
- transformations and symmetry groups;
- applications of transformations, such as the normalization proof of Pappus' theorem.

Along the way, we also practiced several proof strategies: direct proof, proof by contradiction, proof by induction, and proving that maps are injective, surjective, or bijective.

The last question is: what must we add to our incidence axioms if we want to recover Euclid's geometry more fully?

The incidence axioms **I1**–**I3** are not enough. They know about points and lines, but they do not know about segments, rays, angles, lengths, or angle measurements. To recover Euclid's geometry, Hilbert added two more families of axioms:

- betweenness axioms;
- congruence axioms.

# 6. Hilbert planes

A **Hilbert plane** is a model $\mathcal{M} = (\mathcal{P}, \mathcal{L}, \mathcal{R})$ satisfying the incidence axioms **I1-3** together with:

- a betweenness relation satisfying axioms **B1**–**B4**;
- congruence relations for segments and angles satisfying axioms **C1**–**C6**.

# 7. Betweenness

Betweenness is a relation on triples of points. We write

$$A * B * C$$

to mean that $B$ is between $A$ and $C$.

The betweenness axioms include the following ideas.

**B1.** If

$$A * B * C,$$

then $A, B, C$ are distinct and collinear.

**B2.** If $A \neq B$, then there exists a point $C$ such that

$$A * B * C.$$

This is a rigorous version of the idea that a segment can be extended beyond one of its endpoints. In particular, lines must contain infinitely many points.

**B3.** If $A, B, C$ are distinct and collinear, then exactly one of the three points lies between the other two.

**B4 (Pasch's axiom).** Let $A, B, C$ be three non-collinear points, and let $\ell$ be a line not incident with any of $A, B, C$. Suppose that $D \in \ell$ and

$$A * D * B.$$

Then either there exists $E \in \ell$ such that

$$B * E * C,$$

or there exists $E \in \ell$ such that

$$A * E * C,$$

but not both.

Geometrically, Pasch's axiom says that if a line enters a triangle through one side and does not pass through a vertex, then it must leave through exactly one of the other two sides.

## 8. Segments, rays, and angles

Once we have betweenness, we can define segments, rays, and angles.

If $A \neq B$, we write

$$(AB)$$

for the unique line through $A$ and $B$.

The segment from $A$ to $B$ is

$$[AB] = \{A, B\} \cup \{C \in \mathcal{P} : A * C * B\}.$$

The ray starting at $A$ and passing through $B$ is

$$[AB) = [AB] \cup \{C \in \mathcal{P} : A * B * C\}.$$

If $A, B, C$ are non-collinear, the angle $\angle BAC$ is the union of two rays:

$$\angle BAC = [AB) \cup [AC).$$

## 9. Congruence

Hilbert also introduces congruence relations:

- a congruence relation on line segments, written

$$[AB] \sim [CD];$$

- a congruence relation on angles, written

$$\angle BAC \sim \angle EDF.$$

The congruence axioms include the following ideas.

**C1.** Given a segment $[AB]$ and a ray $r$ starting at $C$, there exists a point $D \in r$ such that

$$[AB] \sim [CD].$$

This is the rigorous version of copying a segment onto a given ray.

**C2.** Segment congruence is an equivalence relation.

**C3.** If

$$A * B * C \qquad \text{and} \qquad D * E * F,$$

and

$$[AB] \sim [DE], \qquad [BC] \sim [EF],$$

then

$$[AC] \sim [DF].$$

This is an addition rule for segment lengths.

**C4.** Given an angle $\angle BAC$ and a ray $[DE)$, there exists a unique ray $[DF)$ such that

$$\angle BAC \sim \angle EDF.$$

This is the rigorous version of copying an angle onto a given ray. It corresponds to Euclid I.23.

**C5.** Angle congruence is an equivalence relation.

**C6 (SAS).** Let $ABC$ and $DEF$ be triangles. If

$$[AB] \sim [DE], \qquad [AC] \sim [DF], \qquad \angle BAC \sim \angle EDF,$$

then

$$[BC] \sim [EF],$$

and the remaining corresponding angles are congruent:

$$\angle ABC \sim \angle DEF, \qquad \angle ACB \sim \angle DFE.$$

This is the Hilbert-plane version of the SAS theorem.

## 10. Examples of Hilbert planes

The standard example is

$$\mathbb{A}^2(\mathbb{R}).$$

This is the usual Euclidean plane, with the usual notions of betweenness, segment congruence, and angle congruence.

Another example is

$$\mathbb{A}^2(\mathbb{K}),$$

where $\mathbb{K} \subset \mathbb{R}$ is the field of constructible numbers. A real number $k$ is constructible if, using straightedge and compass starting from a unit segment, we can construct a segment of length $|k|$. For example,

$$\sqrt{2} \in \mathbb{K},$$

because it is the diagonal length of a unit square.

Another example is

$$\mathbb{A}^2(\overline{\mathbb{Q}} \cap \mathbb{R}),$$

where $\overline{\mathbb{Q}} \cap \mathbb{R}$ is the field of real algebraic numbers. These are the real numbers which solve a polynomial equation

$$P(x) = 0$$

with rational coefficients. For example,

$$\sqrt[5]{2} \in \overline{\mathbb{Q}} \cap \mathbb{R},$$

because it is a solution of

$$x^5 - 2 = 0.$$

There are also hyperbolic Hilbert planes. These satisfy the Hilbert plane axioms, but they do not satisfy the Euclidean parallel property.

Another important example is the Moulton plane. Its point set is $\mathbb{R}^2$, but the lines are modified: usual lines are kept, except that lines of negative slope bend when they cross the $y$-axis. This gives a nonstandard geometry where many Euclidean-looking axioms still hold, but some familiar theorems such as Desargues' theorem fail.

## 11. Non-examples of Hilbert planes

The rational affine plane

$$\mathbb{A}^2(\mathbb{Q})$$

is not a Hilbert plane with the usual Euclidean congruence. The issue is segment-copying.

Consider the segment

$$[(0,0),(1,1)].$$

Its length is $\sqrt{2}$. If we try to copy this segment onto the ray from $(0,0)$ through $(1,0)$, the endpoint would have to be

$$(\sqrt{2},0).$$

But

$$(\sqrt{2},0) \notin \mathbb{Q}^2.$$

Thus axiom **C1** fails.

The complex affine plane

$$\mathbb{A}^2(\mathbb{C})$$

is also not a Hilbert plane in the usual sense, because there is no good order relation on $\mathbb{C}$ compatible with the geometry. In particular, complex lines do not behave like ordered real lines, so there is no suitable Hilbert-style betweenness relation.

## 12. Euclidean planes

A **Euclidean plane** is a Hilbert plane satisfying additional axioms. The most important additional axioms for us are:

- the Euclidean parallel property;
- the circle-circle intersection property;
- further axioms related to area and congruence of rectilinear figures.

The Euclidean parallel property says that, given a line $\ell$ and a point $p$ not incident with $\ell$, there is a unique line through $p$ parallel to $\ell$.

The circle-circle intersection property says, informally, that if one circle contains a point inside another circle and also a point outside that circle, then the two circles meet.

More precisely, fix two circles

$$C_1 = \{P \in \mathcal{P} : [A_1P] \sim [A_1B_1]\}$$

and

$$C_2 = \{P \in \mathcal{P} : [A_2P] \sim [A_2B_2]\}.$$

If $C_2$ contains a point $P$ inside $C_1$ and a point $Q$ outside $C_1$, then

$$C_1 \cap C_2 \neq \emptyset.$$

Here $P$ being inside $C_1$ means that

$$[A_1P] < [A_1B_1],$$

and $Q$ being outside $C_1$ means that

$$[A_1Q] > [A_1B_1].$$

This circle-circle intersection property is what guarantees constructions such as Euclid I.1, the construction of an equilateral triangle on a given segment.

## 13. Euclid's *Elements* inside Euclidean planes

A major theorem is:

**Theorem.** Euclid's *Elements*, Books I–IV, are valid in any Euclidean plane.

More precisely, different parts of the *Elements* require different amounts of structure.

For Book I:

| Propositions | Axioms needed |
|---|---|
| 1.1–1.28, except 1.1 and 1.22 | Hilbert plane |
| 1.1 and 1.22 | Hilbert plane + circle-circle intersection |
| 1.29–1.34 | Hilbert plane + Euclidean parallel property |
| 1.35–1.48 | Hilbert plane + parallel property + area theory |

For Book II:

| Propositions | Axioms needed |
|---|---|
| 2.1–2.13 | Hilbert plane + parallel property + area theory |
| 2.14 | Euclidean plane |

For Book III:

| Propositions | Axioms needed |
|---|---|
| 3.1–3.19, except 3.1 and 3.17 | Hilbert plane |
| 3.1 and 3.17 | Hilbert plane + circle-circle intersection |
| 3.20–3.34 | Hilbert plane + parallel property + circle-circle intersection |
| 3.35–3.37 | Euclidean plane + area theory |

For Book IV:

| Propositions | Axioms needed |
|---|---|
| 4.1–4.16 | Euclidean plane |

Thus, to recover the full Euclidean geometry of the *Elements*, incidence axioms are not enough. We need betweenness, congruence, a parallel axiom, circle intersection principles, and eventually area theory.

# 14. Closing

The purpose of this course was not only to learn a list of classical theorems. It was also to see how geometry can be built from axioms, how changing axioms changes the geometry, and how transformations can simplify difficult configurations.

I hope this class has sparked some interest in geometry. Hartshorne's book is a good reference for going further, and my door is always open if you would like to discuss more! Here is my personal email, for when I lose my UCSC email address:

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