Week 1 Lecture 1: A first look at Euclid's

A first introduction to Euclid's geometry, the axiomatic method, primitive terms, postulates, models, and Euclid's first propositions.

Introduction

The first part of this course (Week 1 to Week 3) is centered around Euclid's *Elements*. Its goals include:

- learning classical compass and straightedge constructions;
- using dynamic geometry tools;
- understanding what an axiomatic theory is;
- understanding both the strengths and the limitations of Euclid's treatment.

A rough plan for the course is:

- Weeks 1–3: Euclid's *Elements*;
- Weeks 4–6: corrections and improvements via Hilbert's axioms;
- Weeks 7–10: the parallel postulate and non-Euclidean geometries.

Some history

The word *geometry* comes from the Greek *geometrein*, meaning "earth measurement."

Several early civilizations made important geometric contributions, including:

- the Egyptians,
- the Babylonians,
- the Hindus,
- the Chinese.

Author(s): Ximera Development Team

Their work was often motivated by practical needs such as measurement, surveying, construction, or astronomy. For example, approximate formulas for areas and volumes were known long before Greek mathematics developed a fully deductive style.

The Greeks introduced something fundamentally new: definitions, axioms, and proofs. Among the major figures are Thales, Pythagoras, and Plato. With the Greeks, geometry gradually became a deductive science rather than just a collection of useful rules.

Euclid gathered and organized much of the geometry known in his time into thirteen books called the *Elements*. He did not create all of this mathematics from scratch; rather, he collected, improved, and systematized the work of his predecessors.

A first look at Euclid's *Elements*

Reference: David E. Joyce's online edition of Euclid's *Elements*. This website provides free access to a beautifully presented online version of Euclid's *Elements*, together with diagrams and commentary.

Euclid's goal was to create a rigorous treatment of geometry.

For a theory to be rigorous, it must satisfy several requirements:

(a) There must be mutual understanding of the meaning of words: these are the *definitions*.

(b) Certain statements must be accepted without further justification: these are the *axioms* or *postulates*.

(c) There must be agreement on how and when one statement follows logically from another: these are the *rules of logic*.

A basic philosophical point is the following:

> It is impossible to define every word, since one would eventually end up in a loop.

Thus every axiomatic theory must begin with some primitive terms that are not defined in simpler language.

Primitive geometric forms

A convenient modern viewpoint is that there are a few primitive geometric forms from which the other notions are built. For Euclidean plane geometry, five such

primitive notions are essentially sufficient:

point, line, lie on, between, equal.

Here "equal" should often be understood in the geometric sense of *congruent*.

The purpose of the postulates is to describe how these primitive notions relate to one another.

Although we all have intuitive pictures in mind of what a point or a line is, full rigor requires us not to identify the mathematical object with the picture. In an axiomatic theory, points and lines are abstract objects whose behavior is governed only by the axioms.

The picture we have in mind is instead a *model* for the axiomatic theory.

Models of an axiomatic theory

A *model* of an axiomatic theory is a setting in which concrete objects are assigned to the primitive terms in such a way that the postulates become true.

Example 1: the usual plane

Take $\mathbb{R}^2$.

- Define points to be ordered pairs $(x, y) \in \mathbb{R}^2$.
- Define lines to be subsets given by linear equations

$$ax + by = c.$$

Then the usual incidence statements of Euclidean geometry hold in this model.

Example 2: a two-point model

Take the set

$$X = \{A, B\}.$$

- Define points to be the elements A and B.
- Define the unique line to be the subset

$$\{A, B\}.$$

Then through the two distinct points A and B there passes a line containing both.

So this is also a model of the statement "through any two distinct points there passes a line."

This example shows an important logical lesson: from a small number of primitive notions and postulates, one may *not* yet be able to deduce everything we expect from ordinary geometry. For instance, in this model, a line does *not* contain infinitely many points.

Thus Euclid must introduce enough postulates to recover the geometry he wants.

Book I: definitions and postulates

We begin with Book I of the *Elements*. Euclid starts with definitions, then postulates, then common notions, and then propositions.

He begins with many definitions. Some famous examples are:

Definition 1. A point is that which has no parts.

Definition 2. A line is breadthless length.

Definition 4. A straight line is a line which lies evenly with the points on itself.

These "definitions" are really attempts to define primitive terms such as points and lines by using other terms that are themselves left unclear. So they are not especially helpful as genuine mathematical definitions. From a modern point of view, it is better to keep point and line as primitive terms, whose properties and mutual relations are governed by the axioms.

Postulate 1. To draw a straight line from any point to any point.

This postulate relates the primitive terms "point", "line", and "lie on" by saying that given two points, there is a line on which both lie.

Postulate 2. To produce a finite straight line continuously in a straight line.

This wording is a little confusing at first. The meaning is that a given segment may be extended indefinitely in the same straight direction.

Definition 15–16. A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another. That distinguished point is called the center.

Here, I did not define what a plane figure is; for that, you may consult Definitions 7 (plane surface), 13 (boundary), and 14 (figure) in Book I. The term "equal" is also somewhat misleading here, since the radii are not literally equal

but rather congruent. In Euclid's framework, "equal" is best viewed as a primitive term whose use is governed by the common notions. For example:

Common Notion 1. Things which are equal to the same thing are also equal to one another.

Really, the term equal behaves like an equivalence relation. We may now state Postulate 3.

Postulate 3. To describe a circle with any center and radius.

This means that given a point A and a segment AB, one may construct a circle centered at A having AB as one of its radii.

Compass and straightedge

Euclid's first three postulates correspond closely to the basic constructions one can perform with:

- a straightedge, meaning a ruler without graduations;
- a compass.

Historically, Euclid's compass is thought of as a *collapsible compass*: once lifted from the page, it does not preserve the chosen opening. So one cannot simply transport a segment length directly without further argument.

Common Notion 1. Things which are equal to the same thing are also equal to one another.

In modern terms, this is a transitivity property for equality or congruence.

Euclid's first proposition

Proposition 1. To construct an equilateral triangle on a given finite straight line.

Proof Let AB be the given segment. Using Postulate 3, construct the circle centered at A with radius AB, and the circle centered at B with radius BA. Let C be a point of intersection of these two circles. Using Postulate 1, draw the segments AC and BC.

Since C lies on the circle centered at A with radius AB, we have

$$AC = AB.$$

Since C lies on the circle centered at B with radius BA, we have

$$BC = BA.$$

Therefore, by Common Notion 1,

$$AC = BC = AB.$$

Hence triangle ABC is equilateral. ■

Remark

From a modern perspective, there is an issue hidden in the construction: why must the two circles intersect? Euclid simply takes such an intersection point, but this is not justified by his postulates as stated. This is one of the places where the *Elements* is not fully rigorous by modern standards.