Week 1 Lecture 2: Euclid's rigor, constructions, and angles

A discussion of the six parts of a proposition, Euclid's rigor in Propositions I.1, I.2, and I.4, constructions in GeoGebra Classic, angle definitions. Click on activity W1L2 for the GeoGebra constructions we did in class.

Goals for today

The goals for today are:

(a) understand the structure of a rigorous proof;

(b) evaluate Euclid's rigor in a given proof;

(c) perform the constructions of Propositions I.1 and I.2 in GeoGebra Classic;

(d) discuss angle definitions and Proposition I.4.

The six parts of a proposition

Euclid's propositions may be analyzed into six parts.

Enunciation: what is given and what is sought.

Exposition: what is given in specific form.

Specification: makes clear what is sought.

Construction: adds what is needed from the postulates.

Proof: infers deductively what is sought from definitions and previously demonstrated propositions.

Conclusion: confirms what has been proven from the definition.

Let us identify these parts for Proposition I.1.

Enunciation. To construct an equilateral triangle on a given finite straight line.

Exposition. Let AB be a given finite straight line.

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Specification. We must construct a point C such that AB, AC, and BC are equal to one another.

Construction. Construct the circle with center A and radius AB using Postulate 3. Construct the circle with center B and radius AB using Postulate 3. Then get a point C that lies on each circle.

Proof. Since C lies on the circle centered at A with radius AB, we have $AB = AC$ by Definitions 15–16. Since C lies on the circle centered at B with radius AB, we have $BC = AB$ by Definitions 15–16. Hence

$$AB = AC \qquad \text{and} \qquad AB = BC,$$

so

$$AC = BC$$

by Common Notion 1.

Conclusion. The triangle ABC is equilateral since all sides are equal.

A first issue

There is already a gap in the construction: why should the two circles intersect? This is not justified from Euclid's postulates as stated.

Constructing Proposition I.1 in GeoGebra Classic

Remark. Why GeoGebra Classic rather than the GeoGebra app? While the GeoGebra app is great, it does not allow you to create new tools, and this is something we will want in order to perform more complex constructions. Think of it this way: once you went through the trouble of creating a geometric figure using some given data, you have now unlocked a new tool which you can use in the next construction.

To construct Proposition I.1 in GeoGebra Classic:

(a) Open GeoGebra Classic.

(b) Click on the symbol just below the top right to remove axes and grid.

(c) Click on the third tool on the top left and choose "Segment" to create the segment AB.

(d) Click on the sixth tool on the top left panel and choose "Circle with Center through Point" to create the circle centered at A through B, and repeat to create the circle centered at B through A.

(e) Click on the second tool and choose "Intersect," then click at one of the intersections of the two circles to create C.

(f) Create AC and BC using the segment tool.

You now have your equilateral triangle ABC.

Creating a new tool

Now let us make a new tool.

(a) Click on the top right panel with three bars and choose "Tools," then "Create new tool."

(b) We first have to choose our output. Here there are various options. We could choose:

- the point C and the segments AC and BC;
- just the point C.

I prefer the second option, since the point C is sufficient to create the triangle, and often we will just need this point, not the entire triangle.

(c) The input is now just A and B.

(d) Give a name to the tool such as "I.1 equilateral" and click on finish.

Testing the new tool

To test the new tool:

(a) erase your construction with the eraser tool;

(b) create a segment AB;

(c) click on the new tool, which is the last symbol in the tool panel;

(d) click on A and then on B, and you get C.

What happens if you click on B and then on A?

Now it is your turn: Proposition I.2

Now it is your turn.

(a) Open Proposition I.2 from Book I on Joyce's website:

Proposition I.2.

(b) Carry on the construction in GeoGebra Classic.

(c) The construction involves making an equilateral triangle, so use your new tool.

(d) Evaluate Euclid's proof by stating the six parts of the proof:

Enunciation, Exposition, Specification, Construction, Proof, Conclusion.

Can you see any gap?

(e) Save your new tool.

Proposition I.2

Enunciation. To place a straight line equal to a given line with one end at a given point.

Exposition. Let A be the point and BC the given straight line.

Specification. We must create a point P such that $AP = BC$.

Construction.

- By Proposition I.1, construct an equilateral triangle ABD.
- By Postulate 2, extend DA and DB.
- By Postulate 3, construct the circle centered at B with radius BC; call it c_1.
- Take E at the intersection of the ray DB and c_1.
- By Postulate 3, construct the circle centered at D with radius DE; call it c_2.
- Take P at the intersection of c_2 and the ray DA.

Proof. We have $DP = DE$ and $DA = DB$. But

$$DP - DA = AP \qquad \text{and} \qquad DE - DB = BE.$$

By Common Notion 3, if equals are subtracted from equals, then the remainders are equal, so

$$AP = BE.$$

By Definitions 15–16,

$$BE = BC.$$

Hence, by Common Notion 1,

$$AP = BC.$$

Conclusion. We have built a point P such that $AP = BC$, which was our goal.

Gaps in Proposition I.2

There are several issues.

(a) What is a ray? What is a segment?

(b) Why does the ray intersect the circle? This seems impossible to justify with just Postulates 1, 2, and 3.

(c) What does it mean to add or subtract segments? Here one should use the primitive form *between*.

Angles

Definition 8. A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.

Remark. It is not extremely precise what is meant.

Definition 9. And when the lines containing the angle are straight, the angle is called rectilinear.

Remark. This still lacks precision.

Remark. Euclid does not consider zero angles and straight angles to be angles themselves.

Proposition I.4

Proposition 4. If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equal to the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides.

This is also called the SAS theorem, for *side-angle-side*.

Exposition. Let ABC and DEF be two triangles such that

$$AB = DE, \qquad AC = DF, \qquad \angle BAC = \angle EDF.$$

Specification. We need to show

$$BC = EF, \qquad \angle ABC = \angle DEF, \qquad \angle ACB = \angle DFE.$$

Construction. Superpose ABC on DEF by placing A on D and AB on DE.

Proof.

- B coincides with E since $AB = DE$.
- Since $\angle BAC = \angle EDF$, the side AC coincides with DF.
- Since $AC = DF$, the point C coincides with F.
- But B also coincides with E, so BC coincides with EF, hence

  $$BC = EF$$

  by Common Notion 4: two things which coincide with one another are equal to one another.
- Therefore the triangles ABC and DEF are equal as triangles.
- Hence the remaining angles coincide and therefore are equal.

Conclusion. We showed

$$ABC = DEF, \qquad BC = EF, \qquad \angle ABC = \angle DEF, \qquad \angle ACB = \angle DFE.$$

Many issues

There are many issues here.

(a) Superposing is not a construction coming from the postulates. Hilbert will actually add the SAS theorem as an axiom.

(b) It is not clear that if B coincides with E and C with F, then BC coincides with EF. One could imagine different possible segments joining B and C.