Week 2 Lecture 1: "summary diagrams", "Euclid-style" proofs, and more on postulates

A discussion of proposition "summary diagrams", "Euclid-style" proofs, stronger forms of Postulates 1 and 2, and the definition of right angles and Postulate 4.

Goals for this week

(a) Create a proposition "summary diagram" for a given proposition and use them to remember Propositions I.1 to I.34.

(b) Remember the definition of parallel lines and right angles; the stronger forms of Postulates 1 and 2, Postulates 4 and 5.

(c) Apply the known propositions to create new "Euclid-style" proofs.

Roughly speaking:

- the diagram shows the situation;
- the construction adds the needed objects;
- the proof justifies the claim.

Roughly, Book I may be viewed in three parts:

- triangles: Propositions I.1–I.26;
- parallel lines: Propositions I.27–I.34;
- area: Propositions I.35–I.48.

"Summary diagrams" of propositions

We begin with Proposition I.4.

Proposition I.4 (SAS). If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal,

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then they also have the base equal to the base, the triangle equal to the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides.

Question. How can we make a "summary diagram" for this proposition?

The idea is the following:

- mark identically the sides and angles that are given to be equal;
- use three colors: one for the initial data, one for the construction, and one for the conclusion.

For Proposition I.4, the initial data may be summarized as

$$AB = DE, \qquad AC = DF, \qquad \angle BAC = \angle EDF,$$

and the conclusion as

$$BC = EF, \qquad \angle CBA = \angle FED, \qquad \angle ACB = \angle DFE.$$

Activity 1

Read the enunciation part of Propositions I.5, I.6, and I.8, and create "summary diagrams".

Here are the proposition summaries in words.

Proposition I.5. If $AB = AC$ in triangle ABC, then

$$\angle CBA = \angle ACB.$$

Moreover, if the equal sides are extended, then the exterior angles are also equal.

Proposition I.6. If in triangle ABC we have

$$\angle CBA = \angle ACB,$$

then

$$AB = AC.$$

Proposition I.8 (SSS). If two triangles satisfy

$$AB = DE, \qquad AC = DF, \qquad BC = EF,$$

then the corresponding angles are equal:

$$\angle CBA = \angle FED, \qquad \angle BAC = \angle EDF, \qquad \angle ACB = \angle DFE.$$

This is also called the SSS theorem, for *side-side-side*.

"Euclid-style" proofs

We now practice writing short proofs in the style of Euclid.

Activity 2

Using the postulates and propositions we have seen so far, prove the following statements using the style of Euclid.

Problem 1

Let ABC be an isosceles triangle with $AB = AC$. Suppose D lies on the segment BC and that $BD = DC$, that is, D is the midpoint of BC. Show that

$$\angle BAD = \angle DAC.$$

Idea. The diagram displays the initial information and helps us see what proposition to apply.

Proof. Consider triangles BAD and DAC. We have

$$AB = AC, \qquad BD = DC, \qquad AD = AD.$$

Hence the three corresponding sides are equal, so by Proposition I.8 (SSS theorem),

$$\angle BAD = \angle DAC.$$

This is what we wanted to prove.

Problem 2

Show the converse of Problem 1.

Let ABC be an isosceles triangle with $AB = AC$, and let D be a point on the segment BC. Assume

$$\angle BAD = \angle DAC.$$

Show that

$$BD = DC.$$

Proof. Consider triangles BAD and DAC. We have

$$AB = AC, \qquad \angle BAD = \angle DAC, \qquad AD = AD.$$

Thus two sides and the included angle are equal, so by Proposition I.4 (SAS theorem),

$$BD = DC.$$

Therefore D is the midpoint of BC.

Revisiting Proposition I.4

We now review again the proof of the SAS theorem.

Suppose

$$AB = DE, \qquad AC = DF, \qquad \angle BAC = \angle EDF.$$

Euclid argues that one may superpose triangle ABC onto triangle DEF by placing A on D and B on E. Then the equality of the included angles and sides forces C to coincide with F, and so the side BC coincides with EF.

Important issues

There are at least two important problems with this argument.

(a) Superposition is not introduced by the postulates or by previous propositions.

(b) Even if B coincides with E and C coincides with F, it is not automatic from the weak form of Postulate 1 that the line joining B to C is the same as the line joining E to F.

So Euclid's proof of Proposition I.4 implicitly uses a stronger form of Postulate 1.

Stronger forms of Postulates 1 and 2

A stronger form of Postulate 1

We will use the following stronger version of Postulate 1.

Postulate 1 (strong form). There exists a unique line passing through any two distinct points.

This stronger form justifies notation such as:

- $\overleftrightarrow{AB}$ or (AB) for the unique line passing through A and B;
- AB or $[AB]$ for the segment between A and B, that is, the points D on (AB) such that D is between A and B;
- $\overrightarrow{AB}$ for the ray emanating from A and passing through B, that is, the set of points D on (AB) such that B is between A and D.

Bonus question

Can you think of a model which has points and lines and satisfies Postulate 1 in its weak form,

between two points there is a line,

but not in its strong form?

A stronger form of Postulate 2

We will also use the following stronger form of Postulate 2.

Postulate 2 (strong form). If A and B are distinct points on a line, then we can find a point D on the line, distinct from A and B, such that B is between A and D.

In other words, a segment may be extended beyond one of its endpoints.

Bonus question

Can you think of a model that has points and lines and satisfies Postulates 1 and 2 in their strong forms, but such that certain lines contain only one point?

Right angles and Postulate 4

We now discuss Euclid's definition of a right angle.

Definition 10. When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is called a perpendicular to that on which it stands.

Interpretation. Consider an angle $\angle BAC$, and choose a point D on the line (AB) such that A is between D and B. Then $\angle BAC$ is a right angle if

$$\angle BAC = \angle CAD.$$

This gives meaning to the picture of a line standing on another line and creating two equal adjacent angles.

Postulate 4. All right angles are equal to one another.

Final activity

In GeoGebra Classic, create the constructions for Propositions I.9 to I.12 and save the corresponding new tools. You may see working tools using this link GeoGebra activity.

If you have time, try to answer the following two questions:

(a) Can you think of a model which has points and lines and satisfies Postulate 1 in its weak form but not in its strong form?

(b) Can you think of a model which satisfies the strong forms of Postulates 1 and 2 but has certain lines with only one point on them?