Week 2 Lecture 2: ASA, parallel lines, and angle sums in triangles

A discussion of the ASA and AAS congruence theorems, parallel lines and Postulate 5, basic properties of parallel lines, the construction of a parallel line through a point, and the angle sum in a triangle.

Learnt in section

In section, we discussed the following points.

- one more Euclid-style proof using a rhombus;
- angles made by intersecting straight lines, especially Propositions I.13–I.15;
- angle and side comparisons in triangles, especially Propositions I.16–I.21;
- constructing a triangle with three given sides, Proposition I.22;
- constructing an angle equal to a given angle, Proposition I.23.

In particular:

- Proposition I.13 says that when a straight line stands on a straight line, the adjacent angles sum to two right angles.
- Proposition I.14 says that if two adjacent angles sum to two right angles, then the outer sides lie on a straight line.
- Proposition I.15 says that vertically opposite angles are equal.
- Propositions I.16–I.21 compare sides and angles in triangles.
- Proposition I.20 is the triangle inequality.

Goals for today

Today we will discuss:

(a) the ASA and AAS theorems (Proposition I.26);

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(b) parallel lines: definition and postulate;

(c) basic properties of parallel lines (Propositions I.27–I.30);

(d) constructing a parallel to a line through a point;

(e) the sum of the angles in a triangle (Proposition I.32).

This finishes the part of Book I on triangles and parallelism.

The ASA and AAS theorems

A comparison result: Propositions I.24 and I.25

Propositions I.24 and I.25 compare angles and sides in triangles.

Proposition I.24. If two triangles have two sides equal respectively, but the angle contained by the equal straight lines in one triangle is greater than the angle contained by the equal straight lines in the other, then the base opposite the greater angle is greater.

Proposition I.25. Conversely, if two triangles have two sides equal respectively, but the base of one is greater than the base of the other, then the angle contained by the equal straight lines in the triangle with greater base is greater.

ASA theorem (Proposition I.26)

The ASA theorem states the following.

If two triangles have two angles equal respectively, and the side included between those angles equal, then the triangles coincide in all corresponding sides and angles.

In symbols, if triangles ABC and DEF satisfy

$$\angle B = \angle E, \qquad BC = EF, \qquad \angle C = \angle F,$$

then the corresponding remaining parts also agree:

$$AB = DE, \qquad AC = DF, \qquad \angle A = \angle D.$$

Remark. As Euclid does, we will sometimes write

$$ABC = DEF$$

to mean that the two triangles coincide in all corresponding sides and angles.

Proof idea

We prove this by contradiction.

Assume, for instance, that

$$AB > DE.$$

Then by Proposition I.3, we can construct a point G on the segment BA such that

$$BG = DE.$$

Now compare triangles GBC and DEF. We have

$$BG = DE, \qquad BC = EF, \qquad \angle GBC = \angle DEF.$$

Hence by Proposition I.4,

$$\angle GCB = \angle DFE.$$

But by assumption,

$$\angle DFE = \angle ACB.$$

So

$$\angle GCB = \angle ACB.$$

This is impossible, since G lies on the segment BA, and so triangle GBC is properly contained in triangle ABC, forcing

$$\angle ACB > \angle GCB.$$

This contradiction shows that AB cannot be greater than DE.

Proceeding in the same way, one shows that DE cannot be greater than AB. Hence

$$AB = DE.$$

Then Proposition I.4 gives the remaining equalities:

$$AC = DF, \qquad \angle A = \angle D.$$

AAS theorem

A closely related version is the AAS theorem: if two angles and a non-included side agree, then the two triangles are equal.

For example, if

$$\angle B = \angle E, \qquad \angle C = \angle F, \qquad AC = DF,$$

then

$$ABC = DEF.$$

Remark. Instead of assuming $AC = DF$, one could also assume

$$AB = DE.$$

So there are two symmetric versions of AAS.

Application

Consider an isosceles triangle ABC with

$$AB = AC.$$

Assume the bisector of $\angle B$ meets AC at E, and the bisector of $\angle C$ meets AB at D.

Show that

$$CD = BE \qquad \text{and} \qquad CE = BD.$$

Step 1: diagram. Draw the triangle together with the two bisectors.

Step 2: identify triangles where ASA applies.

Since $AB = AC$, Proposition I.5 gives

$$\angle ABC = \angle ACB.$$

But

$$\angle ABC = \angle EBD + \angle DBC = 2\angle DBC$$

because BD bisects $\angle ABC$, and similarly

$$\angle ACB = 2\angle ECB$$

because CE bisects $\angle ACB$.

So from $\angle ABC = \angle ACB$, we conclude

$$\angle DBC = \angle ECB.$$

Now compare triangles EBC and DCB. We have

$$\angle EBC = \angle DCB, \qquad BC = BC, \qquad \angle ECB = \angle DBC.$$

Hence by ASA,

$$\triangle EBC = \triangle DCB.$$

Therefore

$$DB = EC \qquad \text{and} \qquad BE = DC.$$

Parallel lines

Definition

Definition 23. Two straight lines are said to be parallel if they do not intersect.

Postulate 5

Postulate 5. If a straight line falling on two straight lines makes the interior angles on the same side sum to less than two right angles, then the two straight lines, if extended indefinitely, meet on that side on which the angles are less than two right angles.

Remark. This is an axiom of plane geometry, so one should include in the statement that the lines lie in a plane.

Historical remark. For about two thousand years, mathematicians tried either to prove Postulate 5 from Postulates 1–4 or to replace it by seemingly weaker assumptions. One famous reformulation is Playfair's axiom.

Playfair's axiom. Given a line ℓ and a point P not on ℓ, there exists a unique line ℓ' through P parallel to ℓ.

We will see that Euclid's postulates imply Playfair's axiom. The converse, is more subtle and was established later.

Basic propositions about parallel lines

We now summarize Propositions I.27–I.30.

Suppose a line falls on two lines (AB) and (CD).

- If $(AB) \parallel (CD)$, then alternate angles are equal.
- If $(AB) \parallel (CD)$, then the exterior angle equals the interior opposite angle.
- If $(AB) \parallel (CD)$, then the interior angles on the same side sum to two right angles.
- Conversely, each of the properties above implies that the two lines are parallel.

Proof of one implication

Let us prove that if two lines are parallel, then alternate angles are equal.

Assume

$$(AB) \parallel (CD).$$

We want to show that the relevant alternate angles, say $\angle 1$ and $\angle 2$, are equal.

Assume by contradiction that

$$\angle 1 > \angle 2.$$

Now observe that

$$\angle 1 + \angle 3 = 2 \text{ right angles.}$$

Hence

$$\angle 2 + \angle 3 < \angle 1 + \angle 3 = 2 \text{ right angles.}$$

Therefore, by Postulate 5, the two lines would meet on that side. This contradicts the assumption that they are parallel.

By the same argument, $\angle 2 > \angle 1$ is impossible. Hence

$$\angle 1 = \angle 2.$$

So the alternate angles are equal.

Proposition I.30

Proposition I.30. Straight lines parallel to the same straight line are parallel to one another.

Proof sketch. Assume that (AB) and (CD) are both parallel to (EF). Euclid then considers a line intersecting all three lines and applies Proposition I.29 twice to conclude that the same alternate angles are equal, and then uses Proposition I.27 to deduce that

$$(AB) \parallel (CD).$$

Remark. There is a gap here: Euclid simply fixes a line intersecting the three given lines, but this existence is not justified from the earlier postulates alone. This will later be repaired by strengthening the axiomatic framework.

Another remark. If ℓ is a line and P is a point not on ℓ, there is at most one line through P parallel to ℓ.

Indeed, if there were two such lines ℓ' and ℓ'', both through P and both parallel to ℓ, then Proposition I.30 would imply

$$\ell' \parallel \ell''.$$

But ℓ' and ℓ'' meet at P, contradiction.

Constructing a parallel line through a given point

Proposition I.31. Given a line ℓ and a point P not on ℓ, we can construct a line ℓ' through P parallel to ℓ.

Activity. Follow the construction and create the line using GeoGebra Classic. You should use the tools created so far, such as Proposition I.23.

Sum of the angles in a triangle

Proposition I.32. In any triangle, the sum of the three interior angles equals two right angles.

Moreover, the exterior angle equals the sum of the two remote interior angles.

So if the angles of a triangle are labeled $\angle 1$, $\angle 2$, and $\angle 3$, and $\angle 4$ is an exterior angle, then

$$\angle 1 + \angle 2 + \angle 3 = 2 \text{ right angles}$$

and

$$\angle 4 = \angle 1 + \angle 3.$$

Proof. Read Proposition I.32 at home.