Week 3 Lecture 1: Angles in triangles, RASS, and the center of a circle

A discussion of the sum of the angles in a triangle, polygons and convexity, the RASS congruence theorem, an application to chords in a circle, and the construction of the center of a circle.

Goals for today

Today we will discuss:

(a) the sum of the angles in a triangle (Proposition I.32);

(b) polygons, convexity, and sums of interior angles;

(c) the RASS congruence theorem;

(d) an application to chords in a circle;

(e) locating the center of a circle.

Sum of the angles in a triangle

Proposition I.32. In any triangle, the sum of the three interior angles equals two right angles. Moreover, an exterior angle equals the sum of the two remote interior angles.

Let the three interior angles of a triangle be denoted by $\angle 1$, $\angle 2$, and $\angle 3$, and let $\angle 4$ be an exterior angle. Then:

$$\angle 1 + \angle 2 + \angle 3 = 2 \text{ right angles}$$

and

$$\angle 4 = \angle 1 + \angle 3.$$

Proof. Let ABC be a triangle. Construct a line ℓ through C parallel to (AB) (Proposition I.31). Next, fix a point D on the ray $[BC)$ so that C lies between B and D. Finally, fix a point E on ℓ on the same side of (BC) as A.

Since $(AB) \parallel \ell$, Proposition I.29 gives

$$\angle A = \angle ACE \qquad \text{and} \qquad \angle B = \angle ECD.$$

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Since B, C, D are aligned, Proposition I.13 gives

$$\angle ACB + \angle ACE + \angle ECD = 2 \text{ right angles.}$$

Substituting the two equalities above, we obtain

$$\angle A + \angle B + \angle C = 2 \text{ right angles.}$$

Moreover,

$$\angle ACD = \angle ACE + \angle ECD = \angle A + \angle B.$$

Thus the exterior angle equals the sum of the two remote interior angles.

Polygons and convexity

Definition

A polygon is a finite broken line made of segments, where consecutive sides meet at their common endpoint, and non-consecutive sides do not intersect.

Examples

Examples of polygons include:

- triangles;
- quadrilaterals;
- pentagons.

Convex polygons

A polygon is said to be *convex* if every segment joining two vertices of the polygon stays inside the polygon.

In a convex polygon, diagonals stay inside the polygon. In a non-convex polygon, some diagonals may go outside.

Exercise

Compute the sum of the interior angles for:

(a) a convex quadrilateral;

(b) a convex pentagon;

(c) a convex polygon with n sides, where $n \geq 5$.

Hint. Cut the polygon into triangles.

For a quadrilateral, this gives

4 right angles.

Question. Is the same result still true for non-convex polygons?

Remark. Yes, but the proof is much more difficult.

RASS congruence theorem

Theorem (RASS). If two right triangles have equal hypotenuse and one corresponding side equal, then the triangles are congruent.

Remark. This theorem is not in Euclid's *Elements*. It follows from the Pythagorean theorem (Proposition I.47), and we will see it again in the homework.

Application to a chord

Exercise (also Proposition III.3). Consider a chord AB on a circle of center C. Assume that D is a point on AB. Show that

$$D \text{ is the midpoint of } AB \iff \angle ADC \text{ is a right angle.}$$

Hint.

($\Rightarrow$) Use the SSS theorem.

($\Leftarrow$) Use the RASS theorem.

Circle and center

Definition

A chord is a segment joining two points on a circle.

Proposition III.1

The perpendicular bisector of a chord contains the center of the circle.

Proof. Let AB be a chord of a circle of center C, and let ℓ be the perpendicular bisector of AB. We want to show that C lies on ℓ.

Assume for contradiction that C does not lie on ℓ. Let D be the midpoint of AB.

Since $CA = CB$, triangle ABC is isosceles, so the angles at A and B are equal.

Now compare triangles ADC and CDB. We have

$$AD = DB, \qquad AC = CB, \qquad CD = CD.$$

Hence by SSS,

$$\triangle ADC = \triangle CDB.$$

In particular,

$$\angle ADC = \angle CDB.$$

But A, D, B are aligned, so the two adjacent angles $\angle ADC$ and $\angle CDB$ sum to two right angles. Since they are equal, each must be a right angle. Therefore CD is perpendicular to AB.

Since D is the midpoint of AB, it follows that the line through D perpendicular to AB is precisely the perpendicular bisector ℓ. Hence C lies on ℓ, contradiction.

Therefore the perpendicular bisector of a chord contains the center of the circle.

Activity

Using only a compass and straightedge, make a circle and then "forget" where the center is. Construct the center in as few moves as possible, without lifting your compass.