Week 3 Lecture 2: Tangents and angles in circles

A discussion of tangent lines, tangent circles, the relation between tangency and perpendicularity to the radius, and the main angle theorems in circles.

Goals for today

Today we will discuss:

(a) tangents to circles and tangent circles;

(b) the relation between tangency and perpendicularity to the radius;

(c) the alignment between the centers of tangent circles and the tangency point;

(d) the angle at the center and the angle at the circumference;

(e) angles in the same segment and the angle in a semicircle;

(f) the tangent-chord theorem (Proposition III.32).

Tangents

Definition

We say that a line *touches* or is *tangent to* a circle if it intersects the circle at a single point.

We also say that two circles *touch* or are *tangent to each other* if they intersect in a single point.

There are two possibilities:

- the circles are tangent *externally*;
- the circles are tangent *internally*.

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A basic fact about tangent circles

Proposition III.11–III.12. If two circles are tangent, then their centers and the point of tangency are aligned.

Proof. See Propositions III.11 and III.12.

Tangency and perpendicularity

Proposition. Consider a line ℓ intersecting a circle of center C at a point A. Then

$$\ell \text{ is tangent to the circle at } A \quad \Longleftrightarrow \quad \ell \perp CA.$$

Proof. We only prove the converse implication. For the direct implication, see Proposition III.16, and also Proposition III.18.

Assume that

$$\ell \perp CA.$$

We want to prove that A is the only intersection point of ℓ with the circle.

Proceed by contradiction and assume that ℓ intersects the circle at both A and B.

Since

$$CA = CB,$$

triangle CAB is isosceles. Hence the angles at A and B are equal. But $\angle A$ is a right angle, since $\ell \perp CA$. Therefore $\angle B$ is also a right angle.

So

$$\angle A + \angle B = 2 \text{ right angles.}$$

But then

$$\angle A + \angle B + \angle C > 2 \text{ right angles,}$$

which contradicts Proposition I.32.

This contradiction shows that A is the only point where ℓ meets the circle. Therefore ℓ is tangent to the circle at A.

Application

Consider a circle $\mathcal{C}$, and assume that AB and AC are tangent to $\mathcal{C}$ at B and C, respectively. Show that

$$AB = AC.$$

Angles in circles

Angle at the center and angle at the circumference

Proposition III.20. Consider a circle of center C, a chord AB, and a point D on the circle on the same side of the line (AB) as C. Then

$$2\angle ADB = \angle ACB.$$

Proof. We need to show that

$$\angle ACB = 2\angle ADB.$$

Draw the radius CD, and split the central angle:

$$\angle ACB = \angle ACD + \angle DCB.$$

Since

$$CA = CD,$$

triangle ACD is isosceles, so

$$\angle CDA = \angle DAC.$$

By Proposition I.32 in triangle ACD,

$$\angle CDA + \angle DAC + \angle ACD = 2 \text{ right angles.}$$

Since the first two angles are equal, this gives

$$2\angle CDA + \angle ACD = 2 \text{ right angles.}$$

Equivalently,

$$\angle ACD = 2\angle ADC.$$

Similarly, since

$$CD = CB,$$

triangle CDB is isosceles, so the same argument gives

$$\angle DCB = 2\angle DBC.$$

Therefore

$$\angle ACB = \angle ACD + \angle DCB = 2\angle ADC + 2\angle DBC = 2(\angle ADC + \angle DBC) = 2\angle ADB.$$

Application: equal angles in the same segment

Proposition III.21. Consider a chord AB of a circle of center C, and points D and E on the circle on the same side of C with respect to the line (AB). Show that

$$\angle ADB = \angle AEB.$$

Application: angle in a semicircle

Proposition III.31. Let AB be a diameter of a circle and C a point on the circle. Show that

$$\angle ACB$$

is a right angle.

The tangent-chord theorem

Proposition III.32. Let ℓ be tangent to a circle $\mathcal{C}$ at A, and let AB be a chord. Then the angle between the tangent ℓ and the chord AB is equal to the angle in the alternate segment.

In the notation of the figure:

$$\angle ADB = \angle EAB \qquad \text{and} \qquad \angle FAB = \angle BCA.$$

Proof. Thanks to Proposition III.21, we may assume without loss of generality that AD is a diameter of the circle.

We prove first that

$$\angle ADB = \angle EAB.$$

By Proposition I.32 applied to triangle ADB,

$$\angle ADB + \angle DBA + \angle BAD = 2 \text{ right angles.}$$

But since AD is a diameter, Proposition III.31 gives

$$\angle DBA = \text{ right angle.}$$

Hence

$$\angle ADB + \angle BAD = \text{ right angle.} \tag{1}$$

On the other hand, since ℓ is tangent at A, Proposition III.18 gives

$$\angle EAD = \text{ right angle.} \tag{2}$$

Also,

$$\angle EAD = \angle EAB + \angle BAD. \tag{3}$$

Combining (1), (2), and (3), we get

$$\angle ADB + \angle BAD = \angle EAB + \angle BAD.$$

Therefore

$$\angle ADB = \angle EAB.$$

We leave it to the reader, using Proposition III.32 in the same spirit, to show that

$$\angle FAB = \angle BCA.$$