Week 4 Lecture 1: Hilbert's axioms, models, and parallel properties

An introduction to Hilbert's axiomatic approach to Euclidean geometry, the axioms of incidence, models of incidence geometry, the Euclidean/elliptic/hyperbolic parallel properties, and isomorphisms of models.

Hilbert's axioms

Hilbert (1899) proposed additional axioms in order to establish Euclidean geometry as a rigorous mathematical theory.

Recall Euclid's proof of Proposition I.1:

- take distinct points A and B;
- the circles centered at A and B of radius AB intersect at a point C;
- then ABC is an equilateral triangle.

Why do the circles intersect? For example, in the model where the points are the integers and there is only one line, namely the whole set itself, circles may fail to intersect.

So Euclid's proof relies on several hidden assumptions. Hilbert's goal was to introduce a new set of axioms making Euclid's *Elements* fully rigorous.

Goals for today

Today we will:

(a) re-introduce axiomatic theories and models;

(b) introduce the axioms of incidence;

(c) introduce the Euclidean, elliptic, and hyperbolic parallel properties;

(d) study various models of incidence geometry;

(e) define isomorphisms of models.

Author(s): Ximera Development Team

1. Axioms of incidence and models

To introduce the axioms of incidence, we use three primitive terms:

- points,
- lines,
- incident.

(I1) For any two distinct points A and B, there exists a unique line ℓ incident with both A and B.

(I2) For every line ℓ, there exist at least two distinct points incident with ℓ.

(I3) There exist three distinct points with the property that no line is incident with all three of them. Such points are called *non-collinear*.

What is a model?

A model for an axiomatic theory containing the primitive words *points*, *lines*, and *incidence* can be rigorously defined as:

- a set P of points,
- a set L of lines,
- an incidence subset $R \subseteq P \times L$.

Remark. The role of the incidence subset is to tell us which points and lines are incident. In set theory, R is called a *relation*.

Notation. If $p \in P$ and $\ell \in L$ are a point and a line, we write

$$p \, I \, \ell$$

(read "p is incident with ℓ") when

$$(p, \ell) \in R.$$

An *incidence geometry* is any model for which axioms **I1**, **I2**, and **I3** are satisfied.

Remark. Points, lines, and incidence can be anything.

Example 1

Let

$$P = \{\text{straightedge}, \text{chalk}\}, \qquad L = \{\text{table}\},$$

and let

$$R = \{(\text{straightedge}, \text{table}), (\text{chalk}, \text{table})\}.$$

Equivalently, we may say:

$$\text{straightedge } I \text{ table} \qquad \text{and} \qquad \text{chalk } I \text{ table}.$$

Then **I1** and **I2** are satisfied, but **I3** is not.

Example 2

Let

$P = \{\text{all the words in the English language}\}, \qquad L = \{\text{all the books written in English}\}.$

Define incidence by declaring that

$$p \, I \, \ell \quad \text{if the word } p \text{ appears at least once in the book } \ell.$$

Question: are **I1**, **I2**, and **I3** satisfied?

Example 3

Let

$$P = \{\varnothing, \{1, 2\}, \star\}, \qquad L = \{\varnothing\}, \qquad R = \varnothing.$$

So there is one line and no incidences.

Example 4

Let

$$P = \{A, B, C\}, \qquad L = \big\{\{A, B\}, \{B, C\}, \{A, C\}\big\}.$$

Define incidence by

$$p \, I \, \ell \iff p \in \ell.$$

In this example, incidence is given by the usual set-theoretic notion of belonging.
For example,

$$A \, I \, \{A, B\} \qquad \text{and} \qquad B \, I \, \{A, B\}.$$

This example may be represented by a triangle diagram: bold dots for points and strokes for lines.

Question: is (P, L, R) an incidence geometry? In other words, does it satisfy **I1**, **I2**, and **I3**?

Activity 1

Show that, assuming only axioms **I1** and **I3**, the following properties hold.

(i) There exist three distinct and non-concurrent lines. In other words, there exist distinct lines ℓ, ℓ', ℓ'' such that there is no point P satisfying

$$P \, I \, \ell, \qquad P \, I \, \ell', \qquad P \, I \, \ell''.$$

(ii) If ℓ and ℓ' are lines and $P \neq Q$ are distinct points such that

$$P \, I \, \ell, \qquad P \, I \, \ell', \qquad Q \, I \, \ell, \qquad Q \, I \, \ell',$$

then

$$\ell = \ell'.$$

(iii) For every line ℓ, there exists at least one point not lying on it (that is, not incident with it).

2. Parallel properties

Definition. We say that an incidence geometry satisfies:

(i) the *Euclidean parallel property* if, for every line $\ell \in L$ and every point $P \in P$ such that $P \not I \ \ell$, there exists a unique line ℓ' such that

$$P \, I \, \ell' \qquad \text{and} \qquad \ell \parallel \ell';$$

(ii) the *elliptic parallel property* if there exist no parallel lines;

(iii) the *hyperbolic parallel property* if, for every line ℓ and every point P such that $P \not I \, \ell$, there exist at least two distinct lines parallel to ℓ and incident with P.

Activity 2

(i) In Example 4, determine whether the Euclidean, elliptic, or hyperbolic parallel property is satisfied. Or maybe none of these are satisfied.

(ii) Consider the following model:

- $P = \{\text{points on a sheet of paper}\}$,
- $L = \{\text{circles of positive radius}\}$,
- $p \, I \, \ell$ if $p \in \ell$ (standard incidence).

Is this an incidence geometry? What parallel property is satisfied?

3. Isomorphisms between models

Example 5

Let

$$L = \{a, b, c\},$$

a set with three elements, and let

$$P = \big\{\{a, b\}, \{b, c\}, \{a, c\}\big\},$$

the set of all subsets of L with two elements.

Define incidence by

$$p \, I \, \ell \iff \ell \in p.$$

Here lines can be subsets of points. For example,

$$\{a, b\} \, I \, a \qquad \text{and} \qquad \{a, b\} \, I \, b,$$

since $a \in \{a, b\}$ and $b \in \{a, b\}$. But

$$\{a, b\} \not I \, c$$

since $c \notin \{a, b\}$.

This example also has a triangle diagram. Examples 4 and 5 look very similar. We say that they are *isomorphic*.

Definition. Two models (P, L, R) and (P', L', R') are *isomorphic* if we can find:

- a bijective map $f : P \to P'$,
- a bijective map $g : L \to L'$,

such that for every $p \in P$ and $\ell \in L$, we have

$$(p, \ell) \in R \quad \Longleftrightarrow \quad \big(f(p), g(\ell)\big) \in R'.$$

In other words, two models are isomorphic if we can map their points and lines while preserving incidence.

Example

Consider Examples 4 and 5 and define

$$f(A) = \{a, b\}, \qquad f(B) = \{b, c\}, \qquad f(C) = \{a, c\},$$

and

$$g(\{A, B\}) = b, \qquad g(\{B, C\}) = c, \qquad g(\{A, C\}) = a.$$

Then f and g are bijective and preserve incidence. Therefore Examples 4 and 5 are isomorphic.