Week 4 Lecture 2

Affine planes.

Affine Planes

1. Activity

We recall the axioms of incidence geometry:

- **(I1)** Given 2 distinct points, there is a unique line incident with both of them.
- **(I2)** Given a line, there exist two distinct points incident with it.
- **(I3)** There exists 3 non-collinear points.

(i) Can you find two non-isomorphic models with only 2 points satisfying **(I1)** and **(I2)**?

(ii) Assume that $(\mathcal{P}, \mathcal{L}, \mathcal{R})$ is an incidence geometry with 3 points only.
Show it is isomorphic to the model described in Example 4.

We say that there is a **unique incidence geometry with three points up to isomorphisms**.

Solutions

(i) Example of a model satisfying I1 and I2

Let

$$\mathcal{P} = \{1, 2\}, \qquad \mathcal{L} = \{\{1, 2\}\}.$$

$$P \, I \, \ell \iff P \in \ell \quad \text{(standard incidence)}.$$

I1 and **I2** are clearly satisfied.

Author(s):

Unique (up to isomorphism)?

Assume $(\mathcal{P}', \mathcal{L}', \mathcal{R}')$ is a model for axioms **I1** and **I2** and $\mathcal{P}'$ has only two elements.

Call x and y the two elements of $\mathcal{P}'$.

By **I1**, there exists $\ell \in \mathcal{L}'$ such that

$$x \, I \, \ell \quad \text{and} \quad y \, I \, \ell.$$

Are there any other lines?

Assume $\ell' \in \mathcal{L}'$ is a line.

By **I2**, it is incident with at least 2 points.

Since we only have 2 points, it is incident with x and y.

But ℓ is the only line incident with x and y.

Therefore $\ell' = \ell$.

Conclusion

$$\mathcal{P}' = \{x, y\}, \quad \mathcal{L}' = \{\ell\}, \quad \mathcal{R}' = \{(x, \ell), (y, \ell)\}.$$

This model is isomorphic to $(\mathcal{P}, \mathcal{L}, \mathcal{R})$.

Indeed define

$$f(1) = x, \quad f(2) = y, \qquad g(\{1, 2\}) = \ell.$$

(ii) The case of three points

Name our three points x, y, z.

We need to determine $\mathcal{L}$ and $\mathcal{R}$, i.e. the lines and incidences.

By **I1**, there exists a unique line

$$\ell_{xy} \text{ incident with } x \text{ and } y,$$

$$\ell_{yz} \text{ incident with } y \text{ and } z,$$

$$\ell_{xz} \text{ incident with } x \text{ and } z.$$

If two of these three lines were equal, then x, y, z are collinear, which contradicts **I3**.

For the same reason, none of these lines is incident to 3 points.

Therefore $\ell_{xy}, \ell_{yz}, \ell_{xz}$ are distinct and only incident to 2 points each.

Are these all the lines?

Assume $\ell \in \mathcal{L}$.

Then ℓ is incident with 2 points at least by **I2** and cannot be incident with 3.

So $\ell \in \{\ell_{xy}, \ell_{yz}, \ell_{xz}\}$.

Conclusion

$$\mathcal{L} = \{\ell_{xy}, \ell_{yz}, \ell_{xz}\}.$$

Now defining

$$f(x) = A, \quad f(y) = B, \quad f(z) = C,$$
$$g(\ell_{xy}) = \{A, B\}, \quad g(\ell_{yz}) = \{B, C\}, \quad g(\ell_{xz}) = \{A, C\},$$

gives an isomorphism. (Note that there are others!)

2. Affine planes

Definition 1. An affine plane is an incidence geometry $(\mathcal{P}, \mathcal{L}, \mathcal{R})$ satisfying the Euclidean parallel property.

Example questions

(a) Any affine planes with only 3 points?

(b) An affine plane with 4 points?

Example:

- 4 points
- 6 lines
- each line incident to only 2 points

Affine planes over a field

Definition

Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{Q}$ (or $\mathbb{C}$, or any field).

Points are elements of $\mathbb{F}^2$, i.e. ordered pairs (x, y) where $x, y \in \mathbb{F}$.

Lines are subsets of $\mathbb{F}^2$ of the form

$$\ell = \{(x, y) \in \mathbb{F}^2 : ax + by + c = 0\}$$

for some $(a, b, c) \in \mathbb{F}^3$ with $(a, b) \neq (0, 0)$.

$$P \, I \, \ell \iff P \in \ell.$$

Proposition 1. $(\mathbb{F}^2, \mathcal{L}, \mathcal{R})$ is an affine plane.

Proof **(I1)** Assume $(x_1, y_1) \neq (x_2, y_2)$.

Finding a line

$$\ell = \{ax + by + c = 0\}$$

such that $P_1, P_2 \in \ell$ amounts to solving:

$$\begin{cases} ax_1 + by_1 + c = 0 \\ ax_2 + by_2 + c = 0 \end{cases}$$

This system can be rewritten in matrix form as

$$\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The matrix

$$M = \begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{pmatrix}$$

has rank 2 since $(x_1, y_1) \neq (x_2, y_2)$.

Therefore the homogeneous system

$$M \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

has a nontrivial solution $(a, b, c) \neq (0, 0, 0)$.

We claim that necessarily $(a, b) \neq (0, 0)$. Indeed, if $a = b = 0$, then the equations reduce to

$$c = 0,$$

which contradicts $(a, b, c) \neq (0, 0, 0)$.

Thus we obtain a line

$$\ell = \{ax + by + c = 0\}$$

passing through P_1 and P_2.

Moreover, since $\operatorname{rank}(M) = 2$, the space of solutions is 1-dimensional.

Thus any two solutions (a, b, c) and $(a', b', c') \neq (0, 0, 0)$ are proportional:

$$(a', b', c') = \lambda(a, b, c)$$

for some $\lambda \neq 0$.

But proportional coefficients define the same line, so all solutions correspond to the same line.

Therefore there exists a unique line incident with P_1 and P_2.

(I2) Next HW or section.

(I3) Homework 4.

Euclidean parallel property

Let

$$\ell = \{ax + by + c = 0\}, \quad P = (x_0, y_0) \notin \ell.$$

Consider another line

$$m = \{dx + ey + f = 0\}.$$

We analyze the system:

$$\begin{cases} ax + by + c = 0 \\ dx + ey + f = 0 \end{cases}$$

Number of solutions:

- 0 solutions $\Rightarrow \ell \parallel m$
- 1 solution $\Rightarrow \ell \nparallel m$
- more than 1 solution $\Rightarrow \ell = m$

Compute the determinant:

$$\Delta = \begin{vmatrix} a & b \\ d & e \end{vmatrix} = ae - bd.$$

If $\Delta = 0$, then (a, b) and (d, e) are proportional.

Thus

$$m = \{ax + by + d = 0\}$$

for some d.

Hence lines parallel to ℓ take the form:

$$\{ax + by + d = 0\}, \quad d \neq c,$$

and one can show with similar arguments that such a line is indeed parallel to ℓ.

Now we want a line $m \parallel \ell$ passing through P.

Solve:

$$ax_0 + by_0 + d = 0.$$

Thus

$$d = -ax_0 - by_0.$$

Since $P \notin \ell$, we have $d \neq c$.

Thus $m \parallel \ell$ and it is unique.

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Remark (for algebra students)

For every $n \geq 1$ and prime p, there exists a field $\mathbb{F}_{p^n}$.

Thus there exists an affine plane with exactly

$$(p^n)^2 = p^{2n}$$

elements.