Week 5 Lecture 1

Affine planes and the number of points on lines.

Affine Planes

In the previous lecture, we defined affine planes as incidence geometries satisfying the Euclidean parallel property.

Definition 1. An **affine plane** is an incidence geometry

$$\alpha = (\mathcal{P}, \mathcal{L}, \mathcal{R})$$

satisfying the Euclidean parallel property.

Recall that, if $P \in \mathcal{P}$ and $\ell \in \mathcal{L}$, we write

$$P \, I \, \ell$$

to mean that the point P is incident with the line ℓ.

1. Activity

(1) Show that in an affine plane, the following property holds:

For every pair of distinct and non-parallel lines ℓ and m, there exists a unique point P incident with both lines.

In other words, if $\ell \neq m$ and ℓ is not parallel to m, then there exists a unique point $P \in \mathcal{P}$ such that

$$P \, I \, \ell \qquad \text{and} \qquad P \, I \, m.$$

(2) Let

$$\alpha = (\mathcal{P}, \mathcal{L}, \mathcal{R})$$

be an affine plane. Given a line $\ell \in \mathcal{L}$, we denote

$$R(\ell) := \{P \in \mathcal{P} : P \, I \, \ell\}.$$

The goal is to show that, given any two lines ℓ and m, there exists a bijection

$$f : R(\ell) \longrightarrow R(m).$$

Author(s):

2. Intersections of non-parallel lines

We first prove the property from Activity 1.

Proposition 1. Let $\alpha = (\mathcal{P}, \mathcal{L}, \mathcal{R})$ be an affine plane. Let $\ell, m \in \mathcal{L}$ be two distinct lines.

If ℓ and m are not parallel, then there exists a unique point $P \in \mathcal{P}$ such that

$$P \, I \, \ell \qquad \text{and} \qquad P \, I \, m.$$

Proof Since ℓ and m are not parallel, by definition there exists at least one point $P \in \mathcal{P}$ such that

$$P \, I \, \ell \qquad \text{and} \qquad P \, I \, m.$$

We now prove uniqueness. Suppose that P and Q are two points incident with both ℓ and m. If $P \neq Q$, then by axiom **I1**, there is a unique line incident with both P and Q.

But both ℓ and m are incident with P and Q. Therefore $\ell = m$, contradicting the assumption that $\ell \neq m$.

Hence $P = Q$. Therefore the point of intersection is unique. ■

3. All lines have the same number of points

Let

$$\alpha = (\mathcal{P}, \mathcal{L}, \mathcal{R})$$

be an affine plane. For a line ℓ, recall that

$$R(\ell) = \{P \in \mathcal{P} : P \, I \, \ell\}.$$

Proposition 2. Given any two lines $\ell, m \in \mathcal{L}$, there exists a bijection

$$f : R(\ell) \longrightarrow R(m).$$

In particular, all lines in an affine plane have the same number of points.

Proof If $\ell = m$, then the identity map

$$\mathrm{id} : R(\ell) \longrightarrow R(m)$$

is a bijection. So assume from now on that

$$\ell \neq m.$$

Step 1: Choose two special points.

We first show that there exist two distinct points $a, b \in \mathcal{P}$ such that

$$a \, I \, \ell, \qquad a \not I m,$$

and

$$b \not I \ell, \qquad b \, I \, m.$$

By axiom **I2**, there exist two distinct points on ℓ. If both of these points were also on m, then by axiom **I1** the lines ℓ and m would be equal, contradicting the assumption that $\ell \neq m$. Therefore at least one of these two points lies on ℓ but not on m. Call this point a.

Similarly, by axiom **I2**, there exist two distinct points on m. If both of these points were also on ℓ, then by axiom **I1** the lines m and ℓ would be equal, again contradicting $\ell \neq m$. Therefore at least one of these two points lies on m but not on ℓ. Call this point b.

In particular, $a \neq b$.

Step 2: Define an auxiliary line.

Let n be the unique line incident with both a and b. This line exists and is unique by axiom **I1**.

We claim that

$$n \neq \ell \qquad \text{and} \qquad n \neq m.$$

Indeed, if $n = \ell$, then $b \, I \, \ell$, contradicting the choice of b. Similarly, if $n = m$, then $a \, I \, m$, contradicting the choice of a.

Step 3: Intersections of n with ℓ and m.

Since n is incident with a, and $a \, I \, \ell$, the lines n and ℓ meet at a. Since $n \neq \ell$, they meet in exactly one point by the first problem. Hence

$$R(n) \cap R(\ell) = \{a\}.$$

Similarly,

$$R(n) \cap R(m) = \{b\}.$$

Step 4: Construct the map.

Let $x \in R(\ell)$.

If $x = a$, we will define

$$f(a) = b.$$

Now assume that

$$x \in R(\ell) \qquad \text{and} \qquad x \neq a.$$

Since $x \in R(\ell)$, $x \neq a$, and

$$R(n) \cap R(\ell) = \{a\},$$

we have

$$x \not I n.$$

Therefore, by the Euclidean parallel property, there exists a unique line n_x such that

$$x \, I \, n_x \qquad \text{and} \qquad n_x \parallel n.$$

We now show that m and n_x meet in a unique point. First, note that

$$n_x \neq m.$$

Indeed, if $n_x = m$, then $m \parallel n$, because $n_x \parallel n$. But this is impossible, since m and n meet at the point b.

Next, we show that m is not parallel to n_x. Suppose, for contradiction, that

$$m \parallel n_x.$$

Since $n_x \parallel n$, both m and n would be lines through the point b parallel to n_x. Also, $m \neq n$, by step 2. This contradicts the uniqueness part of the Euclidean parallel property.

Therefore m and n_x are distinct and not parallel. By the previous proposition, they meet in a unique point. Let $y \in \mathcal{P}$ be this unique point:

$$y \, I \, m \qquad \text{and} \qquad y \, I \, n_x.$$

We define

$$f(x) = y.$$

Thus

$$f : R(\ell) \longrightarrow R(m)$$

is given by

$$f(x) = \begin{cases} b, & \text{if } x = a, \\ \text{the unique point in } R(m) \cap R(n_x), & \text{if } x \neq a. \end{cases}$$

Step 5: The constructed map $f : R(\ell) \longrightarrow R(m)$ is bijective.

Next HW. ■

Corollary 1. In an affine plane, if one line is incident with exactly n points (where $n \geq 2$ is an integer), then all lines are incident with exactly n points.