Week 6 Lecture 1

Counting points in affine planes, projective planes, and the first idea of projective completion.

1. Counting points in affine planes

In this entire section, let

$$\alpha = (\mathcal{P}, \mathcal{L}, \mathcal{R})$$

be an affine plane.

Given a line $\ell \in \mathcal{L}$, we recall the notation

$$R(\ell) := \{p \in \mathcal{P} : p \, I \, \ell\}.$$

This is the set of points incident with ℓ.

During last class and this week's homework, we proved the following proposition.

Proposition 1. Let $\ell, m \in \mathcal{L}$ be two lines. Then there exists a bijective map

$$f : R(\ell) \longrightarrow R(m).$$

In particular, all lines in an affine plane are incident with the same number of points.

Using the proposition above, we will prove the following theorem.

Theorem 1. Let $n \geq 2$ be an integer, and assume that for some line $\ell \in \mathcal{L}$, we have

$$\#R(\ell) = n.$$

In other words, assume that one line is incident with exactly n points.

Then

$$\#\mathcal{P} = n^2.$$

That is, there are exactly n^2 points in our affine plane. The integer n is called the **order** of the affine plane.

Proof By axiom **I3**, there exists a point

$$p \in \mathcal{P}$$

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such that

$$p \not I \ell.$$

Choose a point $q \in R(\ell)$. Let m be the unique line incident with p and q. This line exists and is unique by axiom **I1**.

Since $p \not I \ell$, we have

$$m \neq \ell.$$

Moreover, the point q is the unique intersection point of ℓ and m. Indeed, if there were another point $q' \neq q$ incident with both ℓ and m, then ℓ and m would be two distinct lines incident with both q and q', contradicting axiom **I1**.

Since $\#R(\ell) = n$, write the points of ℓ as

$$R(\ell) = \{q, q_1, \ldots, q_{n-1}\}.$$

For each $1 \leq i \leq n-1$, define m_i to be the unique line parallel to m and passing through q_i.

This is possible by the Euclidean parallel property, because

$$q_i \not I m.$$

Indeed, $m \cap \ell = \{q\}$, and $q_i \neq q$.

We now observe that

$$R(m_i) \cap R(\ell) = \{q_i\}.$$

Indeed, $q_i \in R(m_i) \cap R(\ell)$. Moreover, since $m_i \parallel m$, we have $m_i \neq \ell$, and m_i cannot meet ℓ in any point other than q_i.

Let $i \neq j$. Suppose, for contradiction, that there exists a point $x \in \mathcal{P}$ such that

$$x \, I \, m_i \qquad \text{and} \qquad x \, I \, m_j.$$

Since $m_i \parallel m$, we have $x \not I m$. Indeed, if $x \, I \, m$, then m_i and m would meet at x, contradicting $m_i \parallel m$.

Therefore m_i and m_j are both lines through x parallel to m. By the Euclidean parallel property, there is a unique line through x parallel to m. Hence

$$m_i = m_j.$$

But m_i and m_j meet ℓ at the unique points q_i and q_j, respectively. Hence

$$q_i = q_j.$$

This contradicts the fact that q_i and q_j are distinct. Therefore the lines m_i and m_j do not meet.

Therefore the lines

$$m, m_1, \ldots, m_{n-1}$$

are pairwise parallel.

By the proposition from last class, each of these lines has exactly n points. Therefore the union

$$R(m) \cup R(m_1) \cup \cdots \cup R(m_{n-1})$$

contains exactly n^2 points.

We now prove that this union is all of $\mathcal{P}$.

Suppose, for contradiction, that there exists a point $x \in \mathcal{P}$ such that

$$x \not I\, m$$

and, for every $1 \leq i \leq n-1$,

$$x \not I\, m_i.$$

Let t be the unique line incident with q and x. This line exists and is unique by axiom **I1**.

We claim first that

$$t \neq m.$$

Indeed, if $t = m$, then $x\, I\, m$, contradicting the choice of x.

We also claim that t is not parallel to m. Indeed, both t and m are incident with q, so they meet at q.

Now fix $1 \leq i \leq n-1$. We claim that

$$t \neq m_i \qquad \text{and} \qquad t \text{ is not parallel to } m_i.$$

First, $t \neq m_i$, because $x \not I\, m_i$, whereas $x\, I\, t$.

Second, suppose for contradiction that

$$t \parallel m_i.$$

Since $m_i \parallel m$, both t and m would be lines through q parallel to m_i. This contradicts the uniqueness part of the Euclidean parallel property, because $t \neq m$.

Therefore t and m_i are distinct and not parallel. Hence they meet in a unique point. Call this point x_i:

$$x_i\, I\, t \qquad \text{and} \qquad x_i\, I\, m_i.$$

Thus the line t is incident with the points

$$q, x_1, \ldots, x_{n-1}, x.$$

These $n+1$ points are all distinct. Indeed, the points x_i lie on the distinct parallel lines m_i, the point q lies on m, and x was chosen to lie on none of the lines

$$m, m_1, \ldots, m_{n-1}.$$

Therefore t is incident with at least $n+1$ points. But by the proposition from last class, every line is incident with exactly n points. This is a contradiction.

Hence no such point x exists, and therefore

$$\mathcal{P} = R(m) \cup R(m_1) \cup \cdots \cup R(m_{n-1}).$$

Since the lines

$$m, m_1, \ldots, m_{n-1}$$

are pairwise parallel and each contains exactly n points, this union contains exactly

$$n \cdot n = n^2$$

points.

Therefore

$$\#\mathcal{P} = n^2.$$

■

2. Projective planes

Definition 1. A **projective plane** is a model of incidence geometry

$$(\mathcal{P}, \mathcal{L}, \mathcal{R})$$

with the elliptic parallel property, and such that every line has at least three distinct points incident with it.

Recall that the **elliptic parallel property** says that any two distinct lines meet. In other words, if

$$\ell, m \in \mathcal{L} \qquad \text{and} \qquad \ell \neq m,$$

then there exists a point $p \in \mathcal{P}$ such that

$$p \, I \, \ell \qquad \text{and} \qquad p \, I \, m.$$

Example. The Fano plane is a projective plane with 7 points and 7 lines.

3. Activity: A point avoiding two lines

Problem 1 *Show that the following statement holds in any projective plane:*
For every pair of distinct lines $\ell, m \in \mathcal{L}$, there exists a point $P \in \mathcal{P}$ such that

$$P \not I\, \ell \qquad \text{and} \qquad P \not I\, m.$$

In other words, show that in a projective plane, given two distinct lines, there exists a point incident with neither of these two lines.

Proof Let $\ell, m \in \mathcal{L}$ be two distinct lines.

By the elliptic parallel property, there exists a point $x \in \mathcal{P}$ such that

$$x\, I\, \ell \qquad \text{and} \qquad x\, I\, m.$$

Since we are in a projective plane, every line is incident with at least three distinct points. Therefore, choose distinct points

$$x, a, b \in R(\ell)$$

and distinct points

$$x, c, d \in R(m).$$

Observe that a, c, d are distinct. Indeed, if one of c or d were equal to a, then the two distinct lines ℓ and m would have two distinct points in common, namely x and a. This would contradict axiom **I1**. Similarly, b, c, d are distinct.

By axiom **I1**, there exists a unique line n incident with a and c.

We note that

$$n \neq \ell.$$

Indeed, if $n = \ell$, then $c\, I\, \ell$. Since $c\, I\, m$ and $c \neq x$, the lines ℓ and m would have two distinct points in common, contradicting axiom **I1**.

Similarly,

$$n \neq m.$$

Indeed, if $n = m$, then $a\, I\, m$, again contradicting the fact that ℓ and m are distinct and already meet at x.

Since $n \neq \ell$, the point b is not incident with n. Otherwise n and ℓ would both contain a and b, so $n = \ell$.

Similarly, since $n \neq m$, the point d is not incident with n.

Also,

$$x \not I\, n.$$

Indeed, if $x \, I \, n$, then n and ℓ would both contain x and a, so $n = \ell$, a contradiction.

The line n must be incident with at least three points. It is already incident with a and c, so there exists a point

$$P \in R(n)$$

such that

$$P \neq a \qquad \text{and} \qquad P \neq c.$$

We claim that

$$P \not I \, \ell \qquad \text{and} \qquad P \not I \, m.$$

Indeed, if $P \, I \, \ell$, then the lines n and ℓ would both contain a and P. Since $P \neq a$, axiom **I1** would imply

$$n = \ell,$$

a contradiction.

Similarly, if $P \, I \, m$, then the lines n and m would both contain c and P. Since $P \neq c$, axiom **I1** would imply

$$n = m,$$

also a contradiction.

Thus

$$P \not I \, \ell \qquad \text{and} \qquad P \not I \, m.$$

■

4. Activity: All projective lines have the same number of points

Let

$$\pi = (\mathcal{P}, \mathcal{L}, \mathcal{R})$$

be a projective plane.

Given a line $\ell \in \mathcal{L}$, define

$$R(\ell) := \{P \in \mathcal{P} : P \, I \, \ell\}.$$

This is the set of points incident with ℓ.

Problem 2 *Show that if ℓ and m are distinct lines, then there is a bijective map*

$$f : R(\ell) \longrightarrow R(m).$$

In other words, show that all lines in a projective plane have the same number of points on them.

Proof If $\ell = m$, then the identity map

$$\mathrm{id} : R(\ell) \longrightarrow R(m)$$

is a bijection.

So assume that

$$\ell \neq m.$$

By the previous activity, there exists a point $P \in \mathcal{P}$ such that

$$P \not I\, \ell \qquad \text{and} \qquad P \not I\, m.$$

Let $x \in R(\ell)$. By axiom **I1**, there exists a unique line ℓ_{xP} incident with both x and P:

$$x\, I\, \ell_{xP} \qquad \text{and} \qquad P\, I\, \ell_{xP}.$$

Since $P \not I\, m$, the line ℓ_{xP} is not equal to m.

Thus, by the elliptic parallel property, ℓ_{xP} and m meet in a unique point. Define

$$f(x) := \text{the unique point in } R(\ell_{xP}) \cap R(m).$$

Equivalently,

$$f(x) = \ell_{xP} \cap m.$$

This defines a map

$$f : R(\ell) \longrightarrow R(m).$$

You will prove in section that this map is bijective. ■

5. Projective completion

Goal

The goal is to construct a projective plane from an affine plane.

Let

$$\alpha = (\mathcal{P}, \mathcal{L}, \mathcal{R})$$

be an affine plane.

The idea is the following: think of train rails disappearing toward the horizon. Parallel lines do not meet inside the affine plane, but they appear to meet at a common point at infinity.

For example, every line parallel to a fixed line ℓ should meet at one common point at infinity. Every line parallel to another line m should also meet at a common point at infinity, but a different one if

$$m \nparallel \ell.$$

Thus, in the projective completion, each parallel class of affine lines will give one new point at infinity.