Week 6 Lecture 2

Projective completion, points at infinity, and models of the real projective plane.

1. Projective completion

Goal

Our goal is to construct a projective plane from an affine plane.

Let

$$\alpha = (\mathcal{P}, \mathcal{L}, \mathcal{R})$$

be an affine plane.

The motivating idea is perspective drawing. Think of train tracks disappearing toward the horizon. Parallel lines do not meet inside the affine plane, but they appear to meet at a point at infinity.

For example, every line parallel to a fixed line ℓ should meet at a common point at infinity. Similarly, every line parallel to another line m should meet at another common point at infinity. If

$$m \not\parallel \ell,$$

then these should be different points at infinity.

How do we formalize this? We begin by defining an equivalence relation on the set of lines.

2. Parallel classes

Definition 1. Let S be a set. A relation $\sim$ on S is called an **equivalence relation** if it satisfies the following three properties:

(a) **Reflexivity.** For every $x \in S$,

$$x \sim x.$$

Author(s):

(b) **Symmetry.** For every $x, y \in S$, if

$$x \sim y,$$

then

$$y \sim x.$$

(c) **Transitivity.** For every $x, y, z \in S$, if

$$x \sim y \qquad \text{and} \qquad y \sim z,$$

then

$$x \sim z.$$

Given two lines $\ell, \ell' \in \mathcal{L}$, we say that ℓ is equivalent to ℓ', and we write

$$\ell \sim \ell',$$

if

$$\ell = \ell' \qquad \text{or} \qquad \ell \parallel \ell'.$$

In other words, $\ell \sim \ell'$ if the two lines are equal or parallel.

Proposition 1. The relation $\sim$ is an equivalence relation on $\mathcal{L}$.

Proof We must check reflexivity, symmetry, and transitivity.

Reflexivity. For every line $\ell \in \mathcal{L}$, we have

$$\ell = \ell.$$

Therefore

$$\ell \sim \ell.$$

Symmetry. Suppose that

$$\ell \sim \ell'.$$

Then either $\ell = \ell'$, or $\ell \parallel \ell'$.

If $\ell = \ell'$, then $\ell' = \ell$, so $\ell' \sim \ell$.

If $\ell \parallel \ell'$, then by symmetry of the notion of parallelism,

$$\ell' \parallel \ell,$$

and therefore $\ell' \sim \ell$.

Transitivity.

See HW5.

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3. Equivalence classes

Something very useful about equivalence relations is that they allow us to define equivalence classes.

Definition 2. Given a line $\ell \in \mathcal{L}$, we define

$$[\ell] := \{\ell' \in \mathcal{L} : \ell' \sim \ell\}.$$

This is the **equivalence class** of ℓ.

In this context, it is also called the **parallel class of** ℓ.

Thus $[\ell]$ consists of all lines parallel to ℓ, together with ℓ itself.

By transitivity, if

$$\ell \sim \ell',$$

then

$$[\ell] = [\ell'].$$

We denote by

$$\boxed{\mathcal{L}_{/\sim} := \{[\ell] : \ell \in \mathcal{L}\}}$$

the set of all equivalence classes. This is called the **quotient set** of $\mathcal{L}$ by the equivalence relation $\sim$.

Proposition 2. The set of lines $\mathcal{L}$ is partitioned into equivalence classes:

$$\mathcal{L} = \bigsqcup_{A \in \mathcal{L}_{/\sim}} A.$$

Proof First, every line belongs to some equivalence class. Indeed, if $\ell \in \mathcal{L}$, then

$$\ell \in [\ell].$$

Therefore

$$\mathcal{L} = \bigcup_{A \in \mathcal{L}_{/\sim}} A.$$

It remains to prove that this union is disjoint. Suppose that

$$A, B \in \mathcal{L}_{/\sim} \qquad \text{and} \qquad A \cap B \neq \emptyset.$$

Since A and B are equivalence classes, there exist lines $\ell, m \in \mathcal{L}$ such that

$$A = [\ell], \qquad B = [m].$$

Choose $r \in A \cap B$. Then

$$r \sim \ell \qquad \text{and} \qquad r \sim m.$$

By symmetry and transitivity,

$$\ell \sim m.$$

Therefore

$$[\ell] = [m],$$

and hence

$$A = B.$$

Thus two equivalence classes are either equal or disjoint, so

$$\mathcal{L} = \bigsqcup_{A \in \mathcal{L}_{/\sim}} A.$$

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4. Example: the affine plane with four points

Consider the affine plane with the smallest possible number of points: the affine plane of order 2. It has 4 points and 6 lines.

Label the four points by

$$0, 1, 2, 3.$$

We write ℓ_{ij} for the line incident with i and j.

Problem 1 *Find the equivalence classes of parallel lines and describe the quotient set*

$$\mathcal{L}_{/\sim}.$$

Solution: There are three pairs of parallel lines:

$$\ell_{12} \parallel \ell_{03},$$

$$\ell_{01} \parallel \ell_{23},$$

and

$$\ell_{02} \parallel \ell_{13}.$$

Therefore

$$[\ell_{12}] = [\ell_{03}] = \{\ell_{12}, \ell_{03}\},$$
$$[\ell_{01}] = [\ell_{23}] = \{\ell_{01}, \ell_{23}\},$$

and

$$[\ell_{02}] = [\ell_{13}] = \{\ell_{02}, \ell_{13}\}.$$

Thus there are three equivalence classes, each containing two lines. Hence

$$\mathcal{L}_{/\sim} = \big\{[\ell_{03}], [\ell_{01}], [\ell_{02}]\big\}.$$

Equivalently, we could also write

$$\mathcal{L}_{/\sim} = \big\{[\ell_{12}], [\ell_{23}], [\ell_{13}]\big\}.$$

This is because

$$[\ell_{12}] = [\ell_{03}], \qquad [\ell_{23}] = [\ell_{01}], \qquad [\ell_{13}] = [\ell_{02}].$$

We say that ℓ is a **representative** of the equivalence class $[\ell]$. For example, in

$$[\ell_{03}] = \{\ell_{12}, \ell_{03}\},$$

both ℓ_{12} and ℓ_{03} are representatives of the same equivalence class.

Observe that

$$\mathcal{L} = \{\ell_{01}, \ell_{23}\} \sqcup \{\ell_{02}, \ell_{13}\} \sqcup \{\ell_{03}, \ell_{12}\}.$$

This is precisely the partition of $\mathcal{L}$ into parallel classes.

5. Points at infinity

We now return to our affine plane

$$\alpha = (\mathcal{P}, \mathcal{L}, \mathcal{R}).$$

We have decomposed the set of affine lines into disjoint equivalence classes:

$$\mathcal{L}_{/\sim} = \{[\ell] : \ell \in \mathcal{L}\}.$$

We will say that ℓ and all lines parallel to ℓ meet at infinity in a common point, which we call

$$[\ell].$$

Thus each equivalence class of parallel lines becomes one new point at infinity.

Lines in different equivalence classes meet in different points at infinity. The horizon should connect all of these points at infinity. This gives the idea of the **line at infinity**.

6. The projective completion

We now define the projective completion of an affine plane.

Definition 3. Let

$$\alpha = (\mathcal{P}, \mathcal{L}, \mathcal{R})$$

be an affine plane. Its **projective completion** is the incidence geometry

$$\alpha^* = (\mathcal{P}^*, \mathcal{L}^*, \mathcal{R}^*)$$

defined as follows.

The new set of points

We define

$$\mathcal{P}^* := \mathcal{P} \sqcup \mathcal{L}_{/\sim}.$$

The points of $\mathcal{P}$ are called **ordinary points** (aka old points), and the points of

$$\mathcal{L}_{/\sim}$$

are called **points at infinity** (aka new points).

Thus we are adding one new point for each parallel class of affine lines.

The new set of lines

We define

$$\mathcal{L}^* := \mathcal{L} \sqcup \{\ell_\infty\}.$$

Here ℓ_∞ is a new line, called the **line at infinity** (aka new line). It represents the horizon. Lines $\ell \in \mathcal{L}$ will be referred to as **ordinary lines** (aka old lines).

The new incidence relation

What matters now is the new incidence relation $\mathcal{R}^*$.

(i) Old incidences are kept.

If

$$p \in \mathcal{P} \qquad \text{and} \qquad \ell \in \mathcal{L},$$

then

$$p \, I^* \, \ell$$

if and only if

$$p \, I \, \ell.$$

(ii) Incidences involving points at infinity.

Let $[\ell] \in \mathcal{L}_{/\sim}$. Then $[\ell]$ is incident with every affine line in its equivalence class. That is, if $m \in \mathcal{L}$, then

$$[\ell] \, I^* \, m$$

if and only if

$$m \in [\ell].$$

Equivalently,

$$[\ell] \, I^* \, m \qquad \Longleftrightarrow \qquad m = \ell \text{ or } m \parallel \ell.$$

An old line $\ell \in \mathcal{L}$ is incident to precisely one point at infinity, namely $[\ell]$.

(iii) The line at infinity.

Every point at infinity is incident with the line at infinity:

$$[\ell] \, I^* \, \ell_\infty$$

for every

$$[\ell] \in \mathcal{L}_{/\sim}.$$

No ordinary point is incident with ℓ_∞.

This completes the definition of the projective completion.

7. Example: completing the order-2 affine plane

Return to the affine plane with four points.

We found that

$$\mathcal{L}_{/\sim} = \big\{[\ell_{01}], [\ell_{02}], [\ell_{03}]\big\}.$$

Thus the projective completion has three new points:

$$[\ell_{01}], \qquad [\ell_{02}], \qquad [\ell_{03}].$$

The point $[\ell_{01}]$ is incident with both

$$\ell_{01} \qquad \text{and} \qquad \ell_{23},$$

since

$$\ell_{01} \parallel \ell_{23}.$$

Similarly,

$$[\ell_{02}]\, I^*\, \ell_{02} \qquad \text{and} \qquad [\ell_{02}]\, I^*\, \ell_{13},$$

and

$$[\ell_{03}]\, I^*\, \ell_{03} \qquad \text{and} \qquad [\ell_{03}]\, I^*\, \ell_{12}.$$

Finally, we add the line at infinity

$$\ell_\infty,$$

which is incident with all three points at infinity:

$$[\ell_{01}], \qquad [\ell_{02}], \qquad [\ell_{03}].$$

Corollary 1. The Fano plane is the projective completion of the affine plane with four points.

8. Why is the projective completion projective?

Question 2 *Why is the projective completion of an affine plane a projective plane?*

This will be a HW6 problem, but we prove Axiom 1 for inspiration.

Let

$$\alpha^* = (\mathcal{P}^*, \mathcal{L}^*, \mathcal{R}^*)$$

be the projective completion of an affine plane

$$\alpha = (\mathcal{P}, \mathcal{L}, \mathcal{R}).$$

We want to show that α^* is a projective plane. We must verify the incidence axioms and the elliptic parallel property.

Axiom I1

Let

$$p, q \in \mathcal{P}^* \qquad \text{with} \qquad p \neq q.$$

We must show that there exists a unique line

$$\ell \in \mathcal{L}^*$$

incident with both p and q.

Case 1:$(p, q) \in \mathcal{P}$, i.e., p and q are ordinary points (aka old points)

By axiom **I1** in the affine plane, there exists a unique affine line

$$\ell \in \mathcal{L}$$

such that

$$p \, I \, \ell \qquad \text{and} \qquad q \, I \, \ell.$$

Therefore

$$p \, I^* \, \ell \qquad \text{and} \qquad q \, I^* \, \ell.$$

Moreover, neither p nor q is incident with ℓ_∞, because ordinary points are not incident with the line at infinity.

If another affine line $\ell' \in \mathcal{L}$ were incident with both p and q, i.e.:

$$p \; I^* \; \ell' \quad \& \quad q \; I^* \; \ell',$$

then we would have:

$$p \; I \; \ell' \quad \& \quad q \; I \; \ell'.$$

By axiom **I1** in the affine plane, this implies $\ell' = \ell$.

Thus the unique line through p and q is ℓ.

Case 2: $p, q \in \mathcal{L}_{/\sim}$

Suppose that p and q are points at infinity. Write

$$p = [\ell], \qquad q = [m],$$

with

$$[\ell] \neq [m].$$

By definition, both p and q are incident with

$$\ell_\infty.$$

We claim that no affine line is incident with both p and q. Indeed, each affine line belongs to exactly one equivalence class, so it has exactly one point at infinity.

Therefore the unique line incident with both p and q is

$$\ell_\infty.$$

Case 3: $p \in \mathcal{P}$ **and** $q \in \mathcal{L}_{/\sim}$

Suppose that p is an ordinary point and

$$q = [\ell]$$

is a point at infinity.

If

$$p \, I \, \ell,$$

then ℓ itself is incident with both p and $[\ell]$ for the new incident relation, i.e., $[\ell] \; I^* \; \ell$ and $p \; I^* \; \ell$.

If

$$p \not I \; \ell,$$

then by the Euclidean parallel property, there exists a unique affine line ℓ' through p parallel to ℓ. Then

$$[\ell] = [\ell']$$

and hence

$$[\ell] \, I^* \, \ell'.$$

So ℓ' is incident with both p and $[\ell]$ for the new incident relation.

In either case, there exists a line incident with both p and q.

For uniqueness, suppose that ℓ' and ℓ'' are affine lines incident with both p and $[\ell]$. Then both ℓ' and ℓ'' pass through p, and both belong to the parallel class $[\ell]$. Hence they are both parallel to ℓ. By the Euclidean parallel property, there is a unique line through p parallel to ℓ. Therefore

$$\ell' = \ell''.$$

Thus **I1** holds in the projective completion.