Week 7 Lecture 1: Visualizing Projective Completions

Visualizing the real projective plane and introducing homogeneous coordinates.

1. Visualizing the real projective plane

Let

$$\mathcal{A} = (\mathbb{R}^2, \mathcal{L}_{\mathbb{R}^2}, I_{\mathbb{R}^2})$$

be the affine plane on $\mathbb{R}^2$, also called the **real affine plane**.

The goal of this lecture is to understand how we can visualize its projective completion

$$\mathcal{A}^*,$$

also called the **real projective plane**.

To help visualize this construction, you can click here for the Desmos 3D animation introduced in class.

2. Step 1: An isomorphic model for the real affine plane

Let

$$\mathbb{S}^2_+ := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1,\ z > 0\}.$$

This is the open northern hemisphere.

We identify a point $(x, y) \in \mathbb{R}^2$ with the point $(x, y, 1) \in \mathbb{R}^3$. In other words, we identify the real affine plane with the plane $z = 1$ in $\mathbb{R}^3$. This plane is tangent to the unit sphere at the north pole

$$N = (0, 0, 1).$$

Now consider the segment from the origin $(0, 0, 0)$ to the point $(x, y, 1)$. This segment intersects the open northern hemisphere in a unique point. We define

$$f(x, y) := \frac{(x, y, 1)}{\sqrt{x^2 + y^2 + 1}}.$$

Thus we have a map

$$f : \mathbb{R}^2 \longrightarrow \mathbb{S}^2_+.$$

Author(s):

Claim 1. The map $f : \mathbb{R}^2 \to \mathbb{S}^2_+$ is bijective.

Proof Left as a calculus exercise. ■

Remark 1. We have found a bijection from $\mathbb{R}^2$ to $\mathbb{S}^2_+$. We can therefore define lines in $\mathbb{S}^2_+$ to be the images under f of lines in $\mathbb{R}^2$.

More precisely, define

$$\mathcal{L}_{\mathbb{S}^2_+} := \{f(\ell) : \ell \in \mathcal{L}_{\mathbb{R}^2}\}.$$

Since lines in $\mathbb{S}^2_+$ are subsets of $\mathbb{S}^2_+$, we use the standard incidence relation of belonging:

$$p \, I \, \ell \qquad \Longleftrightarrow \qquad p \in \ell.$$

Claim 2. The map

$$\mathcal{L}_{\mathbb{R}^2} \longrightarrow \mathcal{L}_{\mathbb{S}^2_+}, \qquad \ell \longmapsto f(\ell),$$

is bijective.

Proof This follows from the fact that $f : \mathbb{R}^2 \to \mathbb{S}^2_+$ is bijective. ■

Corollary 1. The incidence geometries

$$(\mathbb{R}^2, \mathcal{L}_{\mathbb{R}^2}, I_{\mathbb{R}^2}) \qquad \text{and} \qquad (\mathbb{S}^2_+, \mathcal{L}_{\mathbb{S}^2_+}, I_{\mathbb{S}^2_+})$$

are isomorphic.

Remark 2. Lines in $\mathbb{S}^2_+$ are exactly the semi-great circles, that is, intersections of $\mathbb{S}^2_+$ with planes through the origin.

Corollary 2. The projective completion of the real affine plane is isomorphic to the projective completion of the open northern hemisphere model.

3. Step 2: Projective completion of the open northern hemisphere model

Let

$$\overline{\mathbb{S}^2_+} := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1, \ z \geq 0\}.$$

This is the closed northern hemisphere. It is the open northern hemisphere together with the equator

$$\mathbb{S}^1_{\text{eq}} := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1, \ z = 0\}.$$

Observation 1. Lines in $\mathcal{L}_{\mathbb{S}^2_+}$ are parallel if and only if, when continued to the closed northern hemisphere, they intersect the equator at the same pair of antipodal points p and $-p$.

Therefore, an equivalence class of parallel lines in $\mathcal{L}_{\mathbb{S}^2_+}$ is determined by a pair

$$\{p, -p\},$$

where p is a point on the equator.

Thus, as a model of the projective completion of the real affine plane, we may consider the closed northern hemisphere with antipodal points on the equator identified.

4. Identifying antipodal points on the equator

We now make this rigorous. We define an equivalence relation on $\overline{\mathbb{S}^2_+}$ as follows.

If $x \in \mathbb{S}^2_+$, so x is in the open northern hemisphere, define

$$x \sim y \qquad \Longleftrightarrow \qquad y = x.$$

Thus points in the open hemisphere are only equivalent to themselves.

If $x \in \overline{\mathbb{S}^2_+} \setminus \mathbb{S}^2_+$, so x is on the equator, define

$$x \sim y \qquad \Longleftrightarrow \qquad y = x \quad \text{or} \quad y = -x.$$

Thus points on the equator are identified with their antipodal points.

Claim 3. This is an equivalence relation.

We denote the quotient set by

$$\overline{\mathbb{S}^2_+}/\sim .$$

Remark 3. Points in the open hemisphere are alone in their equivalence classes. Points on the equator have equivalence classes of the form

$$\{p, -p\}.$$

5. The bijection from the projective completion to the quotient hemisphere

Let ${\mathbb{S}^2_+}^*$ be the projective completion of the open northern hemisphere model. Recall that its points are

$${\mathbb{S}^2_+}^* = \mathbb{S}^2_+ \sqcup \mathcal{L}_{\mathbb{S}^2_+}/\!\parallel .$$

Here $\mathcal{L}_{\mathbb{S}^2_+}/\!\parallel$ denotes the set of parallel classes of lines in $\mathbb{S}^2_+$.

We define a map

$$F : \mathbb{S}_+^{2\,*} \longrightarrow \overline{\mathbb{S}_+^2}/\sim$$

as follows.

If $x \in \mathbb{S}_+^2$, define

$$F(x) = [x].$$

If $[\ell]$ is a parallel class of lines in $\mathbb{S}_+^2$, then every line in this class, when continued to the closed northern hemisphere, meets the equator at the same antipodal pair $\{p, -p\}$. Define

$$F([\ell]) = [p].$$

Claim 4. The map

$$F : \mathbb{S}_+^{2\,*} \longrightarrow \overline{\mathbb{S}_+^2}/\sim$$

is bijective.

Proof This is left as an exercise. ■

6. Mapping lines to lines

We now describe the corresponding lines.

Recall that the lines in $\mathbb{S}_+^{2\,*}$ are

$$\mathcal{L}^*_{\mathbb{S}_+^2} = \mathcal{L}_{\mathbb{S}_+^2} \sqcup \{\ell_\infty\}.$$

Given a line $\ell \in \mathcal{L}_{\mathbb{S}_+^2}$, let $\overline{\ell} \subseteq \overline{\mathbb{S}_+^2}$ be the continuation of ℓ to the closed northern hemisphere. This is a closed half-great circle, and it meets the equator at two antipodal points p and $-p$.

Let

$$\pi : \overline{\mathbb{S}_+^2} \longrightarrow \overline{\mathbb{S}_+^2}/\sim, \qquad x \longmapsto [x],$$

be the quotient map. Define

$$G(\ell) := \pi(\overline{\ell}).$$

Equivalently,

$$G(\ell) = \{[x] : x \in \overline{\ell}\}.$$

Thus $G(\ell)$ is the closed half-great circle corresponding to ℓ, with antipodal points identified on the equator.

For the line at infinity, define

$$G(\ell_\infty) := \{[x] : x \in \mathbb{S}^1_{\text{eq}}\}.$$

This is the equator with antipodal points identified.

Claim 5. The map G is bijective, and together with F, it gives an isomorphism of incidence geometries.

Proof This is left as an exercise. ■

Corollary 3. The real projective plane can be modeled as follows:

- points are elements of the closed northern hemisphere, with antipodal points on the equator identified;
- lines are closed half-great circles on the closed northern hemisphere, with antipodal points on the equator identified.

7. The sphere model

We can also use the whole sphere. Equip

$$\mathbb{S}^2$$

with the antipodal equivalence relation:

$$p \sim q \qquad \Longleftrightarrow \qquad q = p \quad \text{or} \quad q = -p.$$

We denote the quotient by

$$\mathbb{S}^2/\{\pm 1\}.$$

There is a bijection

$$\overline{\mathbb{S}^2_+}/\sim \longrightarrow \mathbb{S}^2/\{\pm 1\}, \qquad [p] \longmapsto [p].$$

As a result, we may define lines in $\mathbb{S}^2/\{\pm 1\}$ as the images of lines in $\overline{\mathbb{S}^2_+}/\sim$, and we get an isomorphic model of the real projective plane.

Corollary 4. In the sphere model of the real projective plane:

- points are points on the sphere, with antipodal points identified;
- lines are great circles, with antipodal points identified.

Remark 4. This is the model used by Cinderella for elliptic geometry.

Remark 5. Cinderella is very easy to install and use. You can download the current beta version here. We will use it to visualize the sphere model of the real projective plane.

8. Projective completion of F^2

We now describe an algebraic model of the projective completion of the affine plane over a field.

Let F be a field, for example

$$F = \mathbb{R}, \quad \mathbb{Q}, \quad \mathbb{C},$$

or any finite field.

9. Definition of $\mathbb{P}^2(F)$

Consider

$$F^3 \setminus \{(0,0,0)\}.$$

We define an equivalence relation on this set as follows.

Given two nonzero triples

$$(x,y,z), (x',y',z') \in F^3 \setminus \{(0,0,0)\},$$

we say that

$$(x,y,z) \sim (x',y',z')$$

if there exists $\lambda \in F^\times = F \setminus \{0\}$ such that

$$(x',y',z') = \lambda(x,y,z).$$

The equivalence class of (x,y,z) is denoted by

$$[x:y:z].$$

Thus

$$[x:y:z] = \{(\lambda x, \lambda y, \lambda z) : \lambda \in F^\times\}.$$

Definition 1. The **projective plane over** F is

$$\mathbb{P}^2(F) := \big(F^3 \setminus \{(0,0,0)\}\big)/\sim .$$

Remark 6. The equivalence classes in $\mathbb{P}^2(F)$ correspond to lines in F^3 through the origin.

10. Lines in $\mathbb{P}^2(F)$

The points of $\mathbb{P}^2(F)$ are the equivalence classes

$$[x:y:z].$$

The lines of $\mathbb{P}^2(F)$ are subsets of the form

$$\ell_{a,b,c} := \{[x : y : z] \in \mathbb{P}^2(F) : ax + by + cz = 0\},$$

where

$$(a, b, c) \in F^3 \setminus \{(0, 0, 0)\}.$$

Incidence is the standard incidence relation of belonging:

$$p \, I \, \ell \qquad \Longleftrightarrow \qquad p \in \ell.$$

This incidence geometry is called the **projective plane over** F.

11. $\mathbb{P}^2(F)$ as a projective completion

To help visualize homogeneous coordinates and the projective completion of F^2, you can click here for the Desmos 3D animation introduced in class.

Proposition 1. The projective plane $\mathbb{P}^2(F)$ is isomorphic to the projective completion of the affine plane F^2.

Proof Let

$$\mathcal{A} = (F^2, \mathcal{L}_{F^2}, I_{F^2})$$

be the affine plane over F, and let

$$\mathcal{A}^* = (\mathcal{P}^*, \mathcal{L}^*, I^*)$$

be its projective completion.

We define a map

$$f : \mathcal{P}^* \longrightarrow \mathbb{P}^2(F)$$

as follows.

For an ordinary affine point $(x, y) \in F^2$, define

$$f(x, y) := [x : y : 1].$$

Now let $[\ell]$ be a point at infinity, where ℓ is an affine line of the form

$$ax + by + c = 0.$$

Define

$$f([\ell]) := [-b : a : 0].$$

This is well-defined because every line parallel to ℓ has the form

$$ax + by + d = 0$$

for some $d \in F$, and all such lines determine the same point

$$[-b : a : 0].$$

Claim 6. The map f is well-defined and bijective.

Proof This is left as an exercise. ■

We now define a map on lines

$$g : \mathcal{L}^* \longrightarrow \mathcal{L}_{\mathbb{P}^2(F)}.$$

If $\ell \in \mathcal{L}_{F^2}$ is the affine line

$$ax + by + c = 0,$$

define

$$g(\ell) := \{[x : y : z] \in \mathbb{P}^2(F) : ax + by + cz = 0\}.$$

For the line at infinity, define

$$g(\ell_\infty) := \{[x : y : z] \in \mathbb{P}^2(F) : z = 0\}.$$

Equivalently,

$$g(\ell_\infty) = \{[x : y : 0] : (x, y) \neq (0, 0)\}.$$

Claim 7. The map g is well-defined and bijective.

Proof This is left as an exercise. ■

Claim 8. The maps

$$f : \mathcal{P}^* \longrightarrow \mathbb{P}^2(F)$$

and

$$g : \mathcal{L}^* \longrightarrow \mathcal{L}_{\mathbb{P}^2(F)}$$

give an isomorphism of incidence geometries.

Proof This is left as an exercise. ■

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12. Another isomorphic model

There is another useful model of $\mathbb{P}^2(F)$.

Define $\widehat{\mathcal{P}}$ to be the set of lines in F^3 through the origin. In other words, its elements are subsets of the form

$$P = \{\lambda(x, y, z) : \lambda \in F\},$$

for some

$$(x, y, z) \neq (0, 0, 0).$$

Define $\widehat{\mathcal{L}}$ to be the set of planes in F^3 through the origin. In other words, its elements are subsets of the form

$$L = \{(x, y, z) \in F^3 : ax + by + cz = 0\},$$

for some

$$(a, b, c) \neq (0, 0, 0).$$

Incidence is inclusion:

$$P \; I \; L \qquad \Longleftrightarrow \qquad P \subseteq L.$$

The map

$$\mathbb{P}^2(F) \longrightarrow \widehat{\mathcal{P}}, \qquad [x : y : z] \longmapsto \{\lambda(x, y, z) : \lambda \in F\},$$

is bijective.

Together with the corresponding map sending projective lines to planes through the origin, this gives an isomorphic model of $\mathbb{P}^2(F)$.