Week 9 Lecture 1

Finishing the transformations of the integer line, then introducing Pappus' theorem and the reduction from the affine plane to the projective plane.

1. Finishing the integer line

Recall the model $\mathcal{M}_{\mathbb{Z}}$. Its points are the integers,

$$\mathcal{P} = \mathbb{Z},$$

and its lines are the subsets

$$\{k, k+1\}, \qquad k \in \mathbb{Z}.$$

Last time, we introduced the translations

$$T_\ell(k) = k + \ell, \qquad \ell \in \mathbb{Z},$$

and the reflection

$$s(k) = -k.$$

Each T_ℓ is a transformation, and s is also a transformation. Since transformations form a group under composition, every map of the form

$$T_\ell \circ S, \qquad \ell \in \mathbb{Z}, \quad S \in \{\mathrm{id}, s\},$$

is a transformation.

The goal is to prove that these are all the transformations of $\mathcal{M}_{\mathbb{Z}}$.

2. Normalization

Let

$$f \in \mathrm{Aut}(\mathcal{M}_{\mathbb{Z}}).$$

Set

$$L = f(0).$$

We want to simplify f by composing it with transformations that we already understand. Define

$$g := T_{-L} \circ f.$$

Author(s):

Then

$$g(0) = T_{-L}(f(0)) = T_{-L}(L) = 0.$$

Since $\{0, 1\}$ is a line and g is a transformation,

$$g(\{0, 1\}) = \{g(0), g(1)\} = \{0, g(1)\}$$

is a line. The only lines containing 0 are $\{-1, 0\}$ and $\{0, 1\}$. Hence

$$g(1) = 1 \qquad \text{or} \qquad g(1) = -1.$$

Choose $S \in \{\mathrm{id}, s\}$ as follows:

$$S = \begin{cases} \mathrm{id}, & \text{if } g(1) = 1, \\ s, & \text{if } g(1) = -1. \end{cases}$$

Define

$$h := S \circ g = S \circ T_{-L} \circ f.$$

Then

$$h(0) = 0 \qquad \text{and} \qquad h(1) = 1.$$

This is the normalization step: starting from an arbitrary transformation f, we constructed a simpler transformation h fixing both 0 and 1.

3. The normalized transformation is the identity

We now prove that if $h \in \mathrm{Aut}(\mathcal{M}_{\mathbb{Z}})$, $h(0) = 0$, and $h(1) = 1$, then

$$h = \mathrm{id}\,.$$

From last time, we already proved by strong induction that

$$h(k) = k$$

for every $k \geq 0$. It remains to prove the same statement for negative integers.

First, since $\{-1, 0\}$ is a line,

$$h(\{-1, 0\}) = \{h(-1), h(0)\} = \{h(-1), 0\}$$

is a line. Thus $h(-1) = 1$ or $h(-1) = -1$. But $h(1) = 1$, and h is injective. Since $-1 \neq 1$, we cannot have $h(-1) = 1$. Therefore

$$h(-1) = -1.$$

We now prove by strong induction that

$$h(-k) = -k$$

for every $k \geq 0$. We already know this for $k = 0$ and $k = 1$.

Assume that $n \geq 1$, and suppose that

$$h(0) = 0, \quad h(-1) = -1, \quad \ldots, \quad h(-n) = -n.$$

We prove that

$$h(-(n+1)) = -(n+1).$$

Since $\{-(n+1), -n\}$ is a line, its image is a line:

$$h(\{-(n+1), -n\}) = \{h(-(n+1)), h(-n)\} = \{h(-(n+1)), -n\}.$$

Thus $h(-(n+1))$ is adjacent to $-n$. Hence

$$h(-(n+1)) = -(n+1) \qquad \text{or} \qquad h(-(n+1)) = -(n-1).$$

But $h(-(n-1)) = -(n-1)$ by the induction hypothesis, and h is injective. Since $-(n+1) \neq -(n-1)$, we cannot have

$$h(-(n+1)) = -(n-1).$$

Therefore

$$h(-(n+1)) = -(n+1).$$

By strong induction,

$$h(-k) = -k$$

for every $k \geq 0$. Combining this with the result for nonnegative integers, we get

$$h(k) = k$$

for every $k \in \mathbb{Z}$. Hence

$$h = \mathrm{id}\,.$$

4. Classification of the transformations

Recall that

$$h = S \circ T_{-L} \circ f.$$

Since $h = \mathrm{id}$, we have

$$S \circ T_{-L} \circ f = \mathrm{id}\,.$$

We now solve this equation for f. Composing on the left by S^{-1}, we get

$$T_{-L} \circ f = S^{-1}.$$

Composing on the left by T_L, we get

$$f = T_L \circ S^{-1}.$$

Now S is either id or s. In both cases,

$$S^{-1} = S.$$

Indeed, $\mathrm{id}^{-1} = \mathrm{id}$, and

$$s(s(k)) = s(-k) = k,$$

so $s^{-1} = s$. Therefore

$$f = T_L \circ S, \qquad S \in \{\mathrm{id}, s\}.$$

Thus the transformations of $\mathcal{M}_{\mathbb{Z}}$ are exactly the maps

$$k \longmapsto k + L \qquad \text{and} \qquad k \longmapsto -k + L, \qquad L \in \mathbb{Z}.$$

5. Pappus' theorem in the affine plane

We now turn to a classical theorem of incidence geometry.
Let $\mathbb{F}$ be $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, or more generally any field. We work in the affine plane

$$\mathbb{A}^2(\mathbb{F}).$$

Theorem (Affine Pappus). Let ℓ and m be two distinct lines in $\mathbb{A}^2(\mathbb{F})$. Let

$$A, B, C \in \ell$$

be three distinct points, and let

$$A', B', C' \in m$$

be three distinct points. Assume the following non-degeneracy conditions:

- none of the six points A, B, C, A', B', C' lies on $\ell \cap m$;
- the three intersection points

  $$P = AB' \cap A'B, \qquad Q = AC' \cap A'C, \qquad R = BC' \cap B'C$$

  exist in the affine plane.

Then P, Q, R are collinear.

Question 1 *Construct this configuration in GeoGebra or Cinderella. Move the points A, B, C, A', B', C'. What seems to happen to P, Q, R?*

The condition that P, Q, R exist is important in the affine plane. For example, it may happen that the lines AB' and $A'B$ are parallel. In that case, the point

$$P = AB' \cap A'B$$

does not exist in the affine plane.

However, if AB' and $A'B$ are parallel, then they meet at a point at infinity in the projective completion. This suggests that the theorem should become cleaner in the projective plane.

6. Why pass to the projective plane?

There are two main reasons to prove Pappus' theorem projectively.

First, in the affine plane, there are different cases. The two lines ℓ and m may be parallel or they may meet. A direct affine proof has to treat these cases separately.

Second, some of the intersection points P, Q, R may fail to exist in the affine plane because the corresponding lines are parallel. In the projective plane, distinct lines always meet.

Thus the projective plane removes both difficulties: there is no parallel case, and the missing affine intersections become points at infinity.

7. From affine to projective

Recall the standard embedding

$$\varphi : \mathbb{A}^2(\mathbb{F}) \longrightarrow \mathbb{P}^2(\mathbb{F}), \qquad \varphi(x, y) = [x : y : 1].$$

This map lets us view the affine plane as the part of the projective plane where the last homogeneous coordinate is nonzero.

A key property is the following.

Collinearity is preserved. For points $P, Q, R \in \mathbb{A}^2(\mathbb{F})$,

$$P, Q, R \text{ are collinear in } \mathbb{A}^2(\mathbb{F})$$

if and only if

$$\varphi(P), \varphi(Q), \varphi(R) \text{ are collinear in } \mathbb{P}^2(\mathbb{F}).$$

Indeed, an affine line has an equation

$$ax + by + c = 0,$$

and its projective completion has equation

$$aX + bY + cZ = 0.$$

If (x, y) lies on the affine line, then $[x : y : 1]$ lies on the corresponding projective line. Conversely, a projective line containing points with $Z = 1$ gives an affine line after setting $Z = 1$.

Therefore, to prove affine Pappus, it is enough to prove a projective version of Pappus.

8. Pappus' theorem in the projective plane

Theorem (Projective Pappus). Let ℓ and m be two distinct lines in $\mathbb{P}^2(\mathbb{F})$. Let

$$A, B, C \in \ell$$

be three distinct points, and let

$$A', B', C' \in m$$

be three distinct points. Assume that none of these six points is equal to the point

$$\ell \cap m.$$

Define

$$P = AB' \cap A'B, \qquad Q = AC' \cap A'C, \qquad R = BC' \cap B'C.$$

Then P, Q, R are collinear.

In the projective plane, the points P, Q, R always exist, because any two distinct projective lines meet. Thus the theorem has a cleaner statement than in the affine plane.