Week 9 Lecture 2

Continuing Pappus' theorem: reducing the affine statement to the projective statement, introducing projective transformations, and normalizing the two lines.

1. Plan for the end of the course

Today, our main goal is to finish the proof of Pappus' theorem, one of the classical theorems of incidence geometry. This proof will use several ideas from the last few weeks:

- affine and projective planes;
- homogeneous coordinates;
- transformations;
- normalization.

The final exam will focus only on the material covered after the midterm: incidence geometry, affine and projective planes, finite and infinite models, counting arguments, homogeneous coordinates, projective completion, and transformations.

2. Reminder: affine Pappus

Let $\mathbb{F}$ be $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, or more generally any field. We want to prove Pappus' theorem in the affine plane

$$\mathbb{A}^2(\mathbb{F}).$$

Recall the statement. Let ℓ and m be two distinct lines in $\mathbb{A}^2(\mathbb{F})$. Let

$$A, B, C \in \ell$$

be three distinct points, and let

$$A', B', C' \in m$$

be three distinct points. Assume that none of these six points lies on $\ell \cap m$. Assume also that the three intersection points

$$P = AB' \cap A'B, \qquad Q = AC' \cap A'C, \qquad R = BC' \cap B'C$$

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exist in the affine plane. Then

$$P, Q, R$$

are collinear.

A direct affine proof is lengthy because there are cases to distinguish. The lines ℓ and m may be parallel, or they may intersect. The point of passing to the projective plane is to remove this distinction.

3. From the affine plane to the projective plane

Recall the map

$$\varphi : \mathbb{A}^2(\mathbb{F}) \longrightarrow \mathbb{P}^2(\mathbb{F}), \qquad \varphi(x, y) = [x : y : 1].$$

This map has two important properties:

- it sends affine lines to projective lines;
- points $P, Q, R \in \mathbb{A}^2(\mathbb{F})$ are collinear if and only if $\varphi(P), \varphi(Q), \varphi(R)$ are collinear in $\mathbb{P}^2(\mathbb{F})$.

Thus, starting with a Pappus configuration in the affine plane, we may apply φ to everything. The affine lines ℓ and m become projective lines

$$\widetilde{\ell} = \varphi(\ell), \qquad \widetilde{m} = \varphi(m),$$

and the points A, B, C, A', B', C' become points

$$\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{A}', \widetilde{B}', \widetilde{C}'$$

on $\widetilde{\ell}$ and $\widetilde{m}$.

If we can prove that

$$\varphi(P), \varphi(Q), \varphi(R)$$

are collinear in $\mathbb{P}^2(\mathbb{F})$, then P, Q, R are collinear in $\mathbb{A}^2(\mathbb{F})$.

Let us justify this last implication directly. Suppose

$$\varphi(P) = [x_P : y_P : 1], \qquad \varphi(Q) = [x_Q : y_Q : 1], \qquad \varphi(R) = [x_R : y_R : 1]$$

belong to a projective line

$$\widetilde{n} = \{[X : Y : Z] \in \mathbb{P}^2(\mathbb{F}) : aX + bY + cZ = 0\}.$$

Then

$$ax_P + by_P + c = 0, \qquad ax_Q + by_Q + c = 0, \qquad ax_R + by_R + c = 0.$$

Therefore the affine points

$$P = (x_P, y_P), \qquad Q = (x_Q, y_Q), \qquad R = (x_R, y_R)$$

all lie on the affine line

$$n = \{(x, y) \in \mathbb{A}^2(\mathbb{F}) : ax + by + c = 0\}.$$

Hence P, Q, R are collinear.

4. Projective Pappus

We now prove the projective version.

Theorem (Projective Pappus). Let ℓ and m be distinct lines in $\mathbb{P}^2(\mathbb{F})$. Let

$$A, B, C \in \ell$$

be three distinct points, and let

$$A', B', C' \in m$$

be three distinct points. Assume that none of these six points is equal to

$$O = \ell \cap m.$$

Define

$$P = AB' \cap A'B, \qquad Q = AC' \cap A'C, \qquad R = BC' \cap B'C.$$

Then P, Q, R are collinear.

Notice that in the projective plane, there is no need to assume that P, Q, R exist. Indeed, any two distinct projective lines meet.

There is also no need to distinguish the cases $\ell \parallel m$ and $\ell \cap m \neq \emptyset$, because in a projective plane any two distinct lines meet.

5. Why transformations help

The proof of Projective Pappus has two main steps.

(a) Use a projective transformation to send the two lines ℓ and m to the simple coordinate lines

$$\{X = 0\} \qquad \text{and} \qquad \{Y = 0\}.$$

(b) Prove Pappus' theorem in this normalized configuration.

This is a powerful use of transformations. Instead of proving the theorem for two completely arbitrary lines, we transform the whole configuration to a simpler one. Since transformations send lines to lines and preserve collinearity, proving the theorem after transformation proves it before transformation.

More precisely, suppose F is a projective transformation. If

$$F(P), F(Q), F(R)$$

are collinear, then P, Q, R are collinear, because applying F^{-1} sends the line through $F(P), F(Q), F(R)$ to a line through P, Q, R.

6. Projective transformations from matrices

Let $A \in GL_3(\mathbb{F})$ be an invertible 3×3 matrix. It defines a map

$$F_A : \mathbb{P}^2(\mathbb{F}) \longrightarrow \mathbb{P}^2(\mathbb{F})$$

by

$$F_A([X : Y : Z]) = [A(X, Y, Z)^T].$$

This is well-defined because if two nonzero vectors are proportional, then their images under A are still proportional.

Since A is invertible, F_A is bijective. Also, F_A sends projective lines to projective lines. Indeed, projective lines in $\mathbb{P}^2(\mathbb{F})$ correspond to planes in $\mathbb{F}^3$ passing through the origin, and an invertible linear map sends planes through the origin to planes through the origin.

Thus F_A is a transformation of $\mathbb{P}^2(\mathbb{F})$.

Example. Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$A \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} X + Y + Z \\ Y + Z \\ Z \end{pmatrix}.$$

So

$$F_A([X : Y : Z]) = [X + Y + Z : Y + Z : Z].$$

7. Why translations do not work directly

In the affine plane, translations are natural transformations. In the projective plane, the situation is different.

Recall that points of $\mathbb{P}^2(\mathbb{F})$ correspond to lines in $\mathbb{F}^3$ passing through the origin. If we translate all vectors in $\mathbb{F}^3$ by a fixed nonzero vector, a line through the origin need not be sent to a line through the origin.

For example, let

$$v = (1, 0, 0).$$

Consider the translation

$$T_v(X, Y, Z) = (X, Y, Z) + v.$$

The line

$$\{(0, \lambda, 0) : \lambda \in \mathbb{F}\}$$

passes through the origin. Its image under T_v is

$$\{(1, \lambda, 0) : \lambda \in \mathbb{F}\},$$

which does not pass through the origin. Therefore this translation does not define a transformation of $\mathbb{P}^2(\mathbb{F})$.

8. The normalization step

We now prove the normalization step. Let ℓ and m be distinct lines in $\mathbb{P}^2(\mathbb{F})$. Write

$$\ell = \{[X : Y : Z] \in \mathbb{P}^2(\mathbb{F}) : aX + bY + cZ = 0\},$$

and

$$m = \{[X : Y : Z] \in \mathbb{P}^2(\mathbb{F}) : dX + eY + fZ = 0\}.$$

Let

$$L = (a, b, c), \qquad M = (d, e, f).$$

Since ℓ and m are distinct lines, the vectors L and M are linearly independent.

Choose a vector

$$N = (g, h, i)$$

such that L, M, N form a basis of $\mathbb{F}^3$. Define

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$

Then A is invertible.

We claim that the projective transformation F_A sends ℓ to the line $\{X = 0\}$ and m to the line $\{Y = 0\}$.

Indeed, if $[X : Y : Z] \in \ell$, then

$$aX + bY + cZ = 0.$$

The first coordinate of

$$A \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

is $aX + bY + cZ$, so the first coordinate of $F_A([X : Y : Z])$ is zero. Hence

$$F_A(\ell) \subseteq \{X = 0\}.$$

Conversely, since A is invertible, every point on $\{X = 0\}$ has a preimage, and the same computation applied to A^{-1} shows that this preimage lies on ℓ. Therefore

$$F_A(\ell) = \{X = 0\}.$$

Similarly,

$$F_A(m) = \{Y = 0\}.$$

Thus it is enough to prove Projective Pappus in the normalized case

$$\ell = \{X = 0\}, \qquad m = \{Y = 0\}.$$

9. Beginning the normalized coordinate proof

Assume from now on that

$$\ell = \{X = 0\}, \qquad m = \{Y = 0\}.$$

First, we find the intersection point

$$O = \ell \cap m.$$

Let

$$O = [X : Y : Z].$$

Since $O \in \ell$, we have $X = 0$. Since $O \in m$, we have $Y = 0$. Since $[X : Y : Z]$ is a projective point, the vector (X, Y, Z) is not zero, so $Z \neq 0$. Therefore

$$O = [0 : 0 : Z] = [0 : 0 : 1].$$

Now let $A' \in m$. Write

$$A' = [X : Y : Z].$$

Since $A' \in m = \{Y = 0\}$, we have $Y = 0$. We claim that $X \neq 0$. If $X = 0$, then since $(X, Y, Z) \neq (0, 0, 0)$, we would have $Z \neq 0$, and hence

$$A' = [0 : 0 : Z] = [0 : 0 : 1] = O.$$

But by hypothesis $A' \neq O$. Therefore $X \neq 0$. Dividing by X, we can write

$$A' = [1 : 0 : a']$$

for some $a' \in \mathbb{F}$.

The same argument gives

$$B' = [1 : 0 : b'], \qquad C' = [1 : 0 : c']$$

for some $b', c' \in \mathbb{F}$. Since A', B', C' are distinct, the scalars

$$a', b', c'$$

are distinct.

Similarly, because $A, B, C \in \ell = \{X = 0\}$ and none of them is equal to O, we can write

$$A = [0 : 1 : a], \qquad B = [0 : 1 : b], \qquad C = [0 : 1 : c],$$

where $a, b, c \in \mathbb{F}$ are distinct.

This is the normalized coordinate setup for the rest of the proof.