# MATH 128A

**Dimitri Navarro**

**October 8, 2026**

## Contents

# Part I

# Syllabus and and other important documents

## 1 Syllabus

*This syllabus contains important information about the course, including its goals, assignments, policies, resources, and expectations.*

### Communication and Office Hours

**Important:** While I appreciate your enthusiasm for discussing mathematics, please save mathematical questions for classes, sections, and office hours. For matters not directly related to course content, I am happy to communicate by email.

**Canvas:** All announcements will be managed through Canvas.

**Instructor:** Dimitri Navarro (he/him). You may call me Professor or Dimitri.

**Email:** dnavar17@ucsc.edu

**Office:** McHenry Library, Office 4192

**Office Hours:** Mondays from 2:00 to 3:30 pm and Wednesdays from 10:30 am to 12:00 pm, McHenry 4192.

Purpose → I am happy to answer questions about the course or assignments, review prerequisite material, or discuss more advanced topics. There are no stupid questions, and you do not need to arrive with anything specific in mind. You are welcome to come listen, study, or work through ideas with your peers.

**TA:** Mike Williams

**Email:** mijawill@ucsc.edu

**Office:** McHenry Library 1266/1261

**Office Hours:** Tuesdays and Thursdays from 2:00-3:00pm, McHenry 1261.

If these times don't work for you, feel free to schedule a meeting with Mike.

## Course Goals

The main goal of this course is to introduce you to classical geometry from both a historical and a modern point of view. We will begin with Euclid's *Elements*, studying both the power and the limitations of Euclid's treatment of geometry. From there, we will move toward a more rigorous axiomatic framework through Hilbert's axioms, and we will see how questions surrounding the parallel postulate eventually led to the development of non-Euclidean geometries.

By the end of the course, you should be able to read and write geometric proofs, carry out and justify straightedge-and-compass constructions, distinguish clearly between intuition and rigorous argument, and understand several concrete models of Euclidean and non-Euclidean geometry. More broadly, the course aims to develop both your geometric intuition and your ability to express mathematical ideas clearly and rigorously.

I look forward to a wonderful quarter together!

**Prerequisites:** Either MATH 100 or CSE 101.

## Classes and Sections

**Important:** If any part of the course is inaccessible to you, please contact me so we can discuss accommodations and ensure your full participation.

**Classes:** TuTh 3:20–4:55 pm, Cowell Clrm 131.

Format → Class meetings will combine the introduction of new material with problem-solving activities designed to reinforce and deepen your understanding.

**Sections:** Wed 2:40–3:45 pm or 4:00–5:05 pm, McHenry Clrm 1267.

Format → Each section will be based on a worksheet. The class will be divided into groups, each assigned one special problem. Groups will work collaboratively for the first 40 minutes; in the final 20 minutes, each group will give a 5-minute presentation on its special problem.

Purpose → These sessions are designed to help you strengthen key mathematical skills: (1) problem-solving, (2) collaboration, and (3) oral presentation.

Remark → Presenting your work can feel daunting at first, but it is an essential skill developed only through practice. These sessions are low-stakes and ungraded—they exist to let you experiment, make mistakes, and learn together. Everyone (including me!) will make mistakes, so let us keep these sessions open, supportive, and fun.

## Lecture Notes, References, and Tools

**Important:** You will access the course materials via our Ximera course page. I created this site to make the course materials as accessible as possible. My LaTeX files are automatically converted into HTML and posted there, so they can be read by screen readers. On the site, you will find the lecture notes and worksheets for the course.

**Notes:** I aim to post the lecture notes within one day of each class.

**Recordings:** You may access class recordings on YuJa (left panel in Canvas).

**References:** This course draws on several books and online resources. In particular, I will use:

(i) David E. Joyce's online edition of Euclid's *Elements*. This website provides free access to a beautifully presented online version of Euclid's *Elements*, together with diagrams and commentary.

(ii) Marvin J. Greenberg, *Euclidean and Non-Euclidean Geometries: Development and History*

(iii) Robin Hartshorne, *Geometry: Euclid and Beyond*

(iv) David W. Henderson and Daina Taimiņa, *Experiencing Geometry: Euclidean and Non-Euclidean with History*

(v) Ana Irene Ramírez Galarza and José Seade, *Introduction to Classical Geometries*

Most of the course will focus on (i), (ii), and (iii). During the first part of the course, we will spend time reading portions of Euclid's *Elements* through (i), where we encounter the first great attempt at a rigorous treatment of geometry. Euclid's constructions and ideas, but also his oversights and lack of rigor, will serve as a guiding thread toward a more precise treatment of geometry through Hilbert's axioms. They will also help us understand the two-thousand-year effort to prove the parallel postulate, which eventually led to the development of non-Euclidean geometries. I will rely on (ii) for its historical perspective and philosophical approach, and on (iii) for its rigor and its rich collection of exercises.

If time permits, I would also like to spend some time on (iv), which offers a very hands-on approach to a variety of geometric concepts on a broader class of spaces than Euclidean or hyperbolic geometry alone. I would also like to present a few aspects of Klein's point of view on geometry through group actions, which provides a beautiful unifying framework for thinking about what geometry is.

**Dynamic Geometry Tools:** One of the main goals of this course is to become acquainted with straightedge-and-compass constructions. While I strongly encourage you to purchase a ruler and a compass and to carry out constructions

by hand on paper, the tools described below add a dimension that paper-and-pencil constructions cannot provide: they allow you to move points and lines dynamically and observe how a construction changes under such variations.

(ia) GeoGebra. GeoGebra is freely accessible online and is especially well suited to two- and three-dimensional Euclidean geometry. You may create an account and save your constructions, and there is also an iPad app if you prefer to work that way.

(ib) GeoGebra Classic. A more powerful version of GeoGebra which allows you to also create new tools.

(ii) Spherical Easel. Spherical Easel is, in spirit, a kind of GeoGebra for spherical geometry. It allows you to create and save interactive spherical constructions and is particularly well suited to experimenting with great circles, spherical triangles, and other objects of spherical geometry.

(iii) Cinderella. Unlike the previous two tools, Cinderella is software that you install and use locally rather than a web-based platform. It is a beautiful and powerful tool for two-dimensional Euclidean, hyperbolic, and elliptic geometry, though it is somewhat less intuitive at first than GeoGebra or Spherical Easel. We will use it in particular when we begin our study of hyperbolic geometry.

The purpose of these tools is to help us visualize geometric phenomena, explore examples, and test conjectures, but they are not a substitute for careful reasoning and proof.

**Tentative class schedule:** I will do my best to follow the schedule below.

| **Week** | **Content** |
|---|---|
| 1 | Introduction to the class and Euclid's Elements Book 1 |
| 2 | Euclid's Elements Book 1 and 2 |
| 3 | Euclid's Elements Book 2 and 3 |
| 4 | Euclid's Elements Book 3 and 4 |
| 5 | Hilbert's axioms of Incidence I1, I2, and I3 and Midterm Exam |
| 6 | Finite geometry |
| 7 | Affine planes |
| 8 | Projective planes |
| 9 | Transformations |
| 10 | Pappus Theorem and back to Euclid's Elements |

Table 1: Tentative Class Schedule

## Assignments & Assessment

**Homework (40%):** Homework will be released weekly on Thursday and due the next Thursday. During exam weeks, different dates may apply. You will access your homework and submit your work via Canvas. Homework may be submitted either handwritten (clearly scanned as a single PDF) or typeset in LaTeX. While you should always answer a problem in your own words, I strongly encourage you to discuss the problems with your classmates.

**Midterm Exam (25%):** The midterm will take place on **Thursday, April 30th (week 5), at the usual class time and in the usual classroom.**

**Final Exam (35%):** The final exam will be on **Thursday, June 11th from 12:00 to 3:00 pm in the usual classroom.**

| **Excellent** | **Good** | **Satisfactory** | **Poor** | **Failing** |
|---|---|---|---|---|
| A+ (95–100) | B+ (80–84) | C+ (65–69) | D+ (50–54) | F ($< 40$) |
| A (90–94) | B (75–79) | C (60–64) | D (45–49) | |
| A- (85–89) | B- (70–74) | C- (55–59) | D- (40–44) | |

Table 2: Grading Scale

## Feedback

**Instructor and TA Feedback:** Mike and I will provide direct comments and feedback on your assignments.

**Weekly Student Feedback:** Most homework assignments will include a bonus question inviting you to comment on the course, describe concepts you find difficult, and reflect on your own learning. Your responses will be anonymous and will help me adjust my teaching throughout the quarter. Please share what is working well for you and what could be improved.

**End-of-quarter Feedback:** At the end of the quarter you will be asked to complete a Student Experience of Teaching survey for this course. SETs provide an opportunity for you to give valuable feedback on your learning that is honest and constructive. This anonymous feedback will help me consider modifications to the course that will help future students learn more effectively.

## Policies: Course, Grading, and AI

**Grading Homework:** You are welcome to use LaTeX to write your answers, although handwritten submissions are equally acceptable. Emphasis should be

placed on the presentation and clarity of your reasoning. In grading, I value the process over the outcome: partial credit will be awarded for sound reasoning, even if the final answer is incorrect.

**Homework Corrections:** You may revisit graded homework and submit corrections to recover up to full credit. Read the feedback and posted solution, then explain in your own words what was incorrect or incomplete in your original answer and how to correct it. Include a corrected argument; simply copying the posted solution is not sufficient. Email your corrections to Mike and me and submit a regrade request for the corresponding question. The purpose of this policy is to encourage you to use feedback, learn from your mistakes, and revisit earlier material throughout the quarter.

**Late & Skipped Homework:** I understand that life can be unpredictable and sometimes unexpected events might interfere with your ability to submit assignments on time. If you find yourself in such a situation, please reach out to me *at least one day before the assignment's due date.*

**Make-Up Exams:** I am aware that unforeseen events may prevent you from attending the exams. If you find yourself in such a situation, please reach out to me as early as you can. We will work together to find a suitable solution such as a make-up exam.

**Generative Artificial Intelligence:** You are allowed to use AI tools (such as ChatGPT) provided you clearly disclose how you used them. If you choose to use AI for an assignment, please include both the prompts you entered and a brief explanation of how the AI assisted you. AI should be used to support your learning rather than replace it. Appropriate uses include asking for clarification of concepts, requesting additional practice questions, checking your understanding, or receiving feedback on your own written attempts. In contrast, it is not appropriate to paste homework problems into an AI tool and copy its solution, or to rely on AI to write up homework that you then submit as your own.

The goal of this course is to develop your own mathematical thinking. AI tools make mistakes, especially in mathematics, and over-reliance on them can prevent you from mastering the material. Moreover, a majority of your grade—coming from the midterm and final exam—will be based on assessments in which AI cannot be used. Genuine understanding is therefore essential. I encourage you to use AI in ways that make you smarter, not in ways that make you dependent.

## Student Services and Accommodation

**Small Group Tutoring:** Small Group Tutoring provides a small interactive setting for students to be comfortable engaging in conversations about course material with other students. Each session is led by a current UCSC undergraduate student who has taken the course and done well in it. Tutoring sessions are

1-hour long and generally have a capacity of 6 students. When students sign up for tutoring, they make a commitment to attend tutoring weekly for the entire quarter. All students can sign up for up to 2 sessions of tutoring a week per course they are enrolled in.

**Academic Excellence (ACE) Program:** For over thirty years, the Academic Excellence (ACE) program has been increasing the diversity of students earning their bachelor's degrees in science, technology, engineering, and mathematics (STEM) at UC Santa Cruz. ACE advances educational equity by engaging students in active learning, peer mentoring, skill-building, and community development.

**DRC accommodation:** UC Santa Cruz is committed to creating an academic environment that supports its diverse student body. If you are a student with a disability who requires accommodations to achieve equal access in this course, please affiliate with the DRC (by phone at (831) 459-2089 or by email at drc@ucsc.edu). For students already affiliated, make sure that you have requested Academic Access Letters, where you intend to use accommodations. You can also request to meet privately with me during my office hours or by appointment as soon as possible. I would like us to discuss how we can implement your accommodations in this course to ensure your access and full engagement in this course.

**Religious Accommodation:** UC Santa Cruz welcomes diversity of religious beliefs and practices, recognizing the contributions differing experiences and viewpoints can bring to the community. There may be times when an academic requirement conflicts with religious observances and practices. If that happens, you may request reasonable accommodation for religious practices. I will review the situation in an effort to provide a reasonable accommodation without penalty. You should first discuss the conflict and your requested accommodation with me early in the term. We may also seek assistance from the Dean of Students Office.

# 2 How to make custom tools on GeoGebra Classic?

*A tutorial to create custom tools on GeoGebra Classic.*

In GeoGebra Classic, it is possible to create your own tools (for example, a tool that constructs the midpoint of a segment or the perpendicular bisector of a segment). This allows you to reuse constructions without repeating all the steps.

## Example: creating a midpoint tool

We explain how to create a tool that takes two points $A, B$ as input and returns their midpoint.

**Step 1. Construct the object.**

- Create two points $A$ and $B$.
- Construct two circles: one centered at $A$ passing through $B$, and one centered at $B$ passing through $A$.
- Let $C$ and $D$ be their two intersection points.
- Draw the line $(CD)$.
- Let $M$ be the intersection of $(CD)$ with the segment $[AB]$.

**Step 2. Create the tool.**

- Click on **Tools** in the menu bar.
- Select **Create New Tool**.

A window opens.

**Step 3. Select the output.**

- In the list of objects, select the point $M$.
- Click **Next**.

**Step 4. Select the input.**

- Select the points $A$ and $B$ as input objects.
- Click **Next**.

**Step 5. Name the tool.**

- Give a name to your tool (for example: *Midpoint*).
- Click **Finish**.

## Using the tool

Your new tool now appears in the toolbar.

To use it:

- Select the tool,
- Click on two points $A$ and $B$,
- GeoGebra automatically constructs their midpoint.

## Remarks

- You can create tools for more complex constructions (perpendicular bisector, angle bisector, tangent to a circle, etc.).
- Tools are especially useful when performing many constructions in the same activity.
- You can save an activity and reuse the tools later by copying the file.

# Part II

# Euclid's Elements

## 3 Week 1 Lecture 1: A first look at Euclid's

*A first introduction to Euclid's geometry, the axiomatic method, primitive terms, postulates, models, and Euclid's first propositions.*

### Introduction

The first part of this course (Week 1 to Week 3) is centered around Euclid's *Elements*. Its goals include:

- learning classical compass and straightedge constructions;
- using dynamic geometry tools;
- understanding what an axiomatic theory is;
- understanding both the strengths and the limitations of Euclid's treatment.

A rough plan for the course is:

- Weeks 1–3: Euclid's *Elements*;
- Weeks 4–6: corrections and improvements via Hilbert's axioms;
- Weeks 7–10: the parallel postulate and non-Euclidean geometries.

### Some history

The word *geometry* comes from the Greek *geometrein*, meaning "earth measurement."

Several early civilizations made important geometric contributions, including:

- the Egyptians,
- the Babylonians,

- the Hindus,
- the Chinese.

Their work was often motivated by practical needs such as measurement, surveying, construction, or astronomy. For example, approximate formulas for areas and volumes were known long before Greek mathematics developed a fully deductive style.

The Greeks introduced something fundamentally new: definitions, axioms, and proofs. Among the major figures are Thales, Pythagoras, and Plato. With the Greeks, geometry gradually became a deductive science rather than just a collection of useful rules.

Euclid gathered and organized much of the geometry known in his time into thirteen books called the *Elements*. He did not create all of this mathematics from scratch; rather, he collected, improved, and systematized the work of his predecessors.

## A first look at Euclid's *Elements*

**Reference:** David E. Joyce's online edition of Euclid's *Elements*. This website provides free access to a beautifully presented online version of Euclid's *Elements*, together with diagrams and commentary.

Euclid's goal was to create a rigorous treatment of geometry.

For a theory to be rigorous, it must satisfy several requirements:

(a) There must be mutual understanding of the meaning of words: these are the *definitions*.

(b) Certain statements must be accepted without further justification: these are the *axioms* or *postulates*.

(c) There must be agreement on how and when one statement follows logically from another: these are the *rules of logic*.

A basic philosophical point is the following:

> It is impossible to define every word, since one would eventually end up in a loop.

Thus every axiomatic theory must begin with some primitive terms that are not defined in simpler language.

# Primitive geometric forms

A convenient modern viewpoint is that there are a few primitive geometric forms from which the other notions are built. For Euclidean plane geometry, five such primitive notions are essentially sufficient:

point, line, lie on, between, equal.

Here "equal" should often be understood in the geometric sense of *congruent*.

The purpose of the postulates is to describe how these primitive notions relate to one another.

Although we all have intuitive pictures in mind of what a point or a line is, full rigor requires us not to identify the mathematical object with the picture. In an axiomatic theory, points and lines are abstract objects whose behavior is governed only by the axioms.

The picture we have in mind is instead a *model* for the axiomatic theory.

# Models of an axiomatic theory

A *model* of an axiomatic theory is a setting in which concrete objects are assigned to the primitive terms in such a way that the postulates become true.

## Example 1: the usual plane

Take $\mathbb{R}^2$.

- Define points to be ordered pairs $(x, y) \in \mathbb{R}^2$.
- Define lines to be subsets given by linear equations

$$ax + by = c.$$

Then the usual incidence statements of Euclidean geometry hold in this model.

## Example 2: a two-point model

Take the set

$$X = \{A, B\}.$$

- Define points to be the elements $A$ and $B$.

- Define the unique line to be the subset

$$\{A, B\}.$$

Then through the two distinct points $A$ and $B$ there passes a line containing both.

So this is also a model of the statement "through any two distinct points there passes a line."

This example shows an important logical lesson: from a small number of primitive notions and postulates, one may *not* yet be able to deduce everything we expect from ordinary geometry. For instance, in this model, a line does *not* contain infinitely many points.

Thus Euclid must introduce enough postulates to recover the geometry he wants.

## Book I: definitions and postulates

We begin with Book I of the *Elements*. Euclid starts with definitions, then postulates, then common notions, and then propositions.

He begins with many definitions. Some famous examples are:

**Definition 1.** A point is that which has no parts.

**Definition 2.** A line is breadthless length.

**Definition 4.** A straight line is a line which lies evenly with the points on itself.

These "definitions" are really attempts to define primitive terms such as points and lines by using other terms that are themselves left unclear. So they are not especially helpful as genuine mathematical definitions. From a modern point of view, it is better to keep point and line as primitive terms, whose properties and mutual relations are governed by the axioms.

**Postulate 1.** To draw a straight line from any point to any point.

This postulate relates the primitive terms "point", "line", and "lie on" by saying that given two points, there is a line on which both lie.

**Postulate 2.** To produce a finite straight line continuously in a straight line.

This wording is a little confusing at first. The meaning is that a given segment may be extended indefinitely in the same straight direction.

**Definition 15–16.** A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another. That distinguished point is called the center.

Here, I did not define what a plane figure is; for that, you may consult Definitions 7 (plane surface), 13 (boundary), and 14 (figure) in Book I. The term "equal" is also somewhat misleading here, since the radii are not literally equal but rather congruent. In Euclid's framework, "equal" is best viewed as a primitive term whose use is governed by the common notions. For example:

**Common Notion 1.** Things which are equal to the same thing are also equal to one another.

Really, the term equal behaves like an equivalence relation. We may now state Postulate 3.

**Postulate 3.** To describe a circle with any center and radius.

This means that given a point $A$ and a segment $AB$, one may construct a circle centered at $A$ having $AB$ as one of its radii.

### Compass and straightedge

Euclid's first three postulates correspond closely to the basic constructions one can perform with:

- a straightedge, meaning a ruler without graduations;
- a compass.

Historically, Euclid's compass is thought of as a *collapsible compass*: once lifted from the page, it does not preserve the chosen opening. So one cannot simply transport a segment length directly without further argument.

**Common Notion 1.** Things which are equal to the same thing are also equal to one another.

In modern terms, this is a transitivity property for equality or congruence.

## Euclid's first proposition

**Proposition 1.** To construct an equilateral triangle on a given finite straight line.

***Proof*** Let $AB$ be the given segment. Using Postulate 3, construct the circle centered at $A$ with radius $AB$, and the circle centered at $B$ with radius $BA$. Let $C$ be a point of intersection of these two circles. Using Postulate 1, draw the segments $AC$ and $BC$.

Since $C$ lies on the circle centered at $A$ with radius $AB$, we have

$$AC = AB.$$

Since $C$ lies on the circle centered at $B$ with radius $BA$, we have

$$BC = BA.$$

Therefore, by Common Notion 1,

$$AC = BC = AB.$$

Hence triangle $ABC$ is equilateral. ■

## Remark

From a modern perspective, there is an issue hidden in the construction: why must the two circles intersect? Euclid simply takes such an intersection point, but this is not justified by his postulates as stated. This is one of the places where the *Elements* is not fully rigorous by modern standards.

# 4 Week 1 Lecture 2: Euclid's rigor, constructions, and angles

*A discussion of the six parts of a proposition, Euclid's rigor in Propositions I.1, I.2, and I.4, constructions in GeoGebra Classic, angle definitions. Click on activity W1L2 for the GeoGebra constructions we did in class.*

## Goals for today

The goals for today are:

(a) understand the structure of a rigorous proof;

(b) evaluate Euclid's rigor in a given proof;

(c) perform the constructions of Propositions I.1 and I.2 in GeoGebra Classic;

(d) discuss angle definitions and Proposition I.4.

## The six parts of a proposition

Euclid's propositions may be analyzed into six parts.

**Enunciation:** what is given and what is sought.

**Exposition:** what is given in specific form.

**Specification:** makes clear what is sought.

**Construction:** adds what is needed from the postulates.

**Proof:** infers deductively what is sought from definitions and previously demonstrated propositions.

**Conclusion:** confirms what has been proven from the definition.

Let us identify these parts for Proposition I.1.

**Enunciation.** To construct an equilateral triangle on a given finite straight line.

**Exposition.** Let $AB$ be a given finite straight line.

**Specification.** We must construct a point $C$ such that $AB$, $AC$, and $BC$ are equal to one another.

**Construction.** Construct the circle with center $A$ and radius $AB$ using Postulate 3. Construct the circle with center $B$ and radius $AB$ using Postulate 3. Then get a point $C$ that lies on each circle.

**Proof.** Since $C$ lies on the circle centered at $A$ with radius $AB$, we have $AB = AC$ by Definitions 15–16. Since $C$ lies on the circle centered at $B$ with radius $AB$, we have $BC = AB$ by Definitions 15–16. Hence

$$AB = AC \qquad \text{and} \qquad AB = BC,$$

so

$$AC = BC$$

by Common Notion 1.

**Conclusion.** The triangle $ABC$ is equilateral since all sides are equal.

### A first issue

There is already a gap in the construction: why should the two circles intersect? This is not justified from Euclid's postulates as stated.

## Constructing Proposition I.1 in GeoGebra Classic

**Remark.** Why GeoGebra Classic rather than the GeoGebra app? While the GeoGebra app is great, it does not allow you to create new tools, and this is something we will want in order to perform more complex constructions. Think of it this way: once you went through the trouble of creating a geometric figure using some given data, you have now unlocked a new tool which you can use in the next construction.

To construct Proposition I.1 in GeoGebra Classic:

(a) Open GeoGebra Classic.

(b) Click on the symbol just below the top right to remove axes and grid.

(c) Click on the third tool on the top left and choose "Segment" to create the segment $AB$.

(d) Click on the sixth tool on the top left panel and choose "Circle with Center through Point" to create the circle centered at $A$ through $B$, and repeat to create the circle centered at $B$ through $A$.

(e) Click on the second tool and choose "Intersect," then click at one of the intersections of the two circles to create $C$.

(f) Create $AC$ and $BC$ using the segment tool.

You now have your equilateral triangle $ABC$.

### Creating a new tool

Now let us make a new tool.

(a) Click on the top right panel with three bars and choose "Tools," then "Create new tool."

(b) We first have to choose our output. Here there are various options. We could choose:

- the point $C$ and the segments $AC$ and $BC$;
- just the point $C$.

I prefer the second option, since the point $C$ is sufficient to create the triangle, and often we will just need this point, not the entire triangle.

(c) The input is now just $A$ and $B$.

(d) Give a name to the tool such as "I.1 equilateral" and click on finish.

### Testing the new tool

To test the new tool:

(a) erase your construction with the eraser tool;

(b) create a segment $AB$;

(c) click on the new tool, which is the last symbol in the tool panel;

(d) click on $A$ and then on $B$, and you get $C$.

What happens if you click on $B$ and then on $A$?

## Now it is your turn: Proposition I.2

Now it is your turn.

(a) Open Proposition I.2 from Book I on Joyce's website:

*Proposition I.2*.

(b) Carry on the construction in GeoGebra Classic.

(c) The construction involves making an equilateral triangle, so use your new tool.

(d) Evaluate Euclid's proof by stating the six parts of the proof:

Enunciation, Exposition, Specification, Construction, Proof, Conclusion.

Can you see any gap?

(e) Save your new tool.

## Proposition I.2

**Enunciation.** To place a straight line equal to a given line with one end at a given point.

**Exposition.** Let $A$ be the point and $BC$ the given straight line.

**Specification.** We must create a point $P$ such that $AP = BC$.

**Construction.**

- By Proposition I.1, construct an equilateral triangle $ABD$.
- By Postulate 2, extend $DA$ and $DB$.
- By Postulate 3, construct the circle centered at $B$ with radius $BC$; call it $c_1$.
- Take $E$ at the intersection of the ray $DB$ and $c_1$.
- By Postulate 3, construct the circle centered at $D$ with radius $DE$; call it $c_2$.
- Take $P$ at the intersection of $c_2$ and the ray $DA$.

**Proof.** We have $DP = DE$ and $DA = DB$. But

$$DP - DA = AP \qquad \text{and} \qquad DE - DB = BE.$$

By Common Notion 3, if equals are subtracted from equals, then the remainders are equal, so

$$AP = BE.$$

By Definitions 15–16,

$$BE = BC.$$

Hence, by Common Notion 1,

$$AP = BC.$$

**Conclusion.** We have built a point $P$ such that $AP = BC$, which was our goal.

### Gaps in Proposition I.2

There are several issues.

(a) What is a ray? What is a segment?

(b) Why does the ray intersect the circle? This seems impossible to justify with just Postulates 1, 2, and 3.

(c) What does it mean to add or subtract segments? Here one should use the primitive form *between*.

## Angles

**Definition 8.** A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.

**Remark.** It is not extremely precise what is meant.

**Definition 9.** And when the lines containing the angle are straight, the angle is called rectilinear.

**Remark.** This still lacks precision.

**Remark.** Euclid does not consider zero angles and straight angles to be angles themselves.

## Proposition I.4

**Proposition 4.** If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equal to the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides.

This is also called the SAS theorem, for *side-angle-side*.

**Exposition.** Let $ABC$ and $DEF$ be two triangles such that

$$AB = DE, \qquad AC = DF, \qquad \angle BAC = \angle EDF.$$

**Specification.** We need to show

$$BC = EF, \qquad \angle ABC = \angle DEF, \qquad \angle ACB = \angle DFE.$$

**Construction.** Superpose $ABC$ on $DEF$ by placing $A$ on $D$ and $AB$ on $DE$.

**Proof.**

- $B$ coincides with $E$ since $AB = DE$.
- Since $\angle BAC = \angle EDF$, the side $AC$ coincides with $DF$.
- Since $AC = DF$, the point $C$ coincides with $F$.
- But $B$ also coincides with $E$, so $BC$ coincides with $EF$, hence

  $$BC = EF$$

  by Common Notion 4: two things which coincide with one another are equal to one another.
- Therefore the triangles $ABC$ and $DEF$ are equal as triangles.
- Hence the remaining angles coincide and therefore are equal.

**Conclusion.** We showed

$$ABC = DEF, \qquad BC = EF, \qquad \angle ABC = \angle DEF, \qquad \angle ACB = \angle DFE.$$

## Many issues

There are many issues here.

(a) Superposing is not a construction coming from the postulates. Hilbert will actually add the SAS theorem as an axiom.

(b) It is not clear that if $B$ coincides with $E$ and $C$ with $F$, then $BC$ coincides with $EF$. One could imagine different possible segments joining $B$ and $C$.

# 5 Homework 1

*Homework 1 for MATH 128A.*

**MATH 128A: Homework 1** **Due date: Thursday, April 9th**

**Instructions.** You will submit your solutions on Gradescope, which you can access through our Canvas course page by clicking on the assignment.

You are welcome to submit your homework either handwritten or in LaTeX. If you choose to submit handwritten work, please use pen rather than pencil, and make sure that both your handwriting and your scans are as readable as possible in order to facilitate grading.

Please also feel free to attend office hours if you would like hints on the homework or if you would like to discuss more advanced material.

You will see that our Ximera course page proposes you to see the answers, but this will only become available once I publish the solution.

**Exercise 1** ***Multiple Choice (10 points).*** *Which of the following is* not *one of the six parts of a proposition discussed in class?*

***Multiple Choice:***

(a) *Enunciation*

(b) *Exposition*

(c) *Construction*

(d) *Corollary* ✓

***Answer.*** *The correct answer is* Corollary. *The six parts discussed in class were:*

*Enunciation, Exposition, Specification, Construction, Proof, Conclusion.*

**Exercise 2** ***Multiple Choice (10 points).*** *According to the discussion in class, what is the main gap in Euclid's proof of Proposition I.1?*

***Multiple Choice:***

(a) *Euclid never uses Common Notion 1.*

(b) *Euclid does not justify why the two circles intersect.* ✓

(c) *Euclid assumes that all triangles are equilateral.*

(d) *Euclid forgets to draw the segment $AB$.*

***Answer.*** *The correct answer is that* Euclid does not justify why the two circles intersect. *His construction of an equilateral triangle on a given segment $AB$ requires the two circles centered at $A$ and $B$ with radius $AB$ to meet, but this is not guaranteed by the postulates stated so far.*

**Exercise 3** ***Multiple Choice (10 points).*** *Which of the following was presented in class as a primitive geometric form?*

***Multiple Choice:***

(a) *Circle*

(b) *Triangle*

(c) *Between* ✓

(d) *Angle*

***Answer.*** *The correct answer is* Between. *In class, primitive notions included objects such as points and lines, together with primitive relations such as betweenness.*

**Problem 4** ***Construction of Proposition I.3 (20 points).***

*Using GeoGebra, GeoGebra Classic, or pen and paper, perform the construction of Proposition I.3.*

*If you use pen and paper, include a drawing of your construction with your homework.*

*If you submit your homework in LaTeX, you may either include an image of your construction (for example, as a* `.png` *file), or share your construction using the sharing option provided by GeoGebra. In that case, your answer may simply be a link to your construction.*

***Answer.*** *https://www.geogebra.org/classic/mdyp9mnc*

***Remark.*** *Observe how, when you move the initial points $A$, $B$, $C$, and $D$, the output (named $f$ on my solution) moves with it. In general, when using GeoGebra, your inputs have the color blue. Everything else is something constructed from this initial data, and takes the color grey. If you construction is correct, your final construction point should move when you move your initial points. If it does not, please reach out to me as there is likely a gap in your understanding of GeoGebra.*

**Answer.** We choose Proposition I.5.

**Enunciation.** In isosceles triangles the angles at the base equal one another, and, if the equal straight lines are produced further, then the angles under the base equal one another.

**Exposition.** Let $ABC$ be an isosceles triangle having the side $AB$ equal to the side $AC$, and let the straight lines $BD$ and $CE$ be produced further in a straight line with $AB$ and $AC$.

**Specification.** I say that the angle $ABC$ equals the angle $ACB$, and the angle $CBD$ equals the angle $BCE$.

**Construction.** Take an arbitrary point $F$ on $BD$. Cut off $AG$ from $AE$, the greater, equal to $AF$, the less, by Proposition I.3, and join the straight lines $FC$ and $GB$.

**Proof.** Since $AF$ equals $AG$, and $AB$ equals $AC$, therefore the two sides $FA$ and $AC$ equal the two sides $GA$ and $AB$, respectively, and they contain a common angle, the angle $FAG$.

Therefore the base $FC$ equals the base $GB$, the triangle $AFC$ equals the triangle $AGB$, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides, that is, the angle $ACF$ equals the angle $ABG$, and the angle $AFC$ equals the angle $AGB$, by Proposition I.4.

Since the whole $AF$ equals the whole $AG$, and in these $AB$ equals $AC$, therefore the remainder $BF$ equals the remainder $CG$, by Common Notion 3.

But $FC$ was also proved equal to $GB$, therefore the two sides $BF$ and $FC$ equal the two sides $CG$ and $GB$ respectively, and the angle $BFC$ equals the angle $CGB$, while the base $BC$ is common to them. Therefore the triangle $BFC$ also equals the triangle $CGB$, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides. Therefore the angle $FBC$ equals the angle $GCB$, and the angle $BCF$ equals the angle $CBG$, by Proposition I.4.

Accordingly, since the whole angle $ABG$ was proved equal to the angle $ACF$, and in these the angle $CBG$ equals the angle $BCF$, the remaining angle $ABC$ equals the remaining angle $ACB$, by Common Notion 3, and they are at the base of the triangle $ABC$. But the angle $FBC$ was also proved equal to the angle $GCB$, and they are under the base.

**Conclusion.** Therefore in isosceles triangles the angles at the base equal one another, and, if the equal straight lines are produced further, then the angles under the base equal one another.

**Comment on rigor.** I do not see an actual logical gap in Euclid's proof of Proposition I.5. Rather, the interesting point is that a modern reader may

be tempted to derive the equality of the exterior angles from the equality of the interior base angles by subtracting from straight angles. However, Euclid cannot argue this way here, because the relevant result about adjacent angles on a straight line only appears later in Proposition I.13, and using it here would be circular. So the proof is not missing a step; instead, it is carefully written to avoid relying on facts that have not yet been established.

**Problem 5** ***Axiomatic theories and models (30 points total).***

(a) ***(15 points)*** *In your own words, what is the difference between an* axiomatic theory *and a* model *for an axiomatic theory?*

***Answer.*** *An axiomatic theory is a collection of primitive terms together with axioms describing how those terms are supposed to behave. A model for an axiomatic theory is a concrete interpretation of those primitive terms in which all the axioms are true.*

*For example, a theory might use the primitive words* point *and* line *and say that through any two distinct points there passes a line. A model would then specify what the points and lines actually are, and verify that this statement holds in that setting.*

(b) ***(15 points)*** *Can you imagine a model with three elements for the axiomatic theory consisting only of the primitive terms* point, line, *and Postulate 1?*

*If so, describe your model clearly by explaining:*

(i) *what the points are,*
(ii) *what the lines are,*
(iii) *why Postulate 1 is satisfied.*

***Answer.*** *Yes.*

*Let the points be three distinct objects:*

$$P, \quad Q, \quad R.$$

*Let the lines be the sets*

$$\{P, Q\}, \qquad \{P, R\}, \qquad \{Q, R\}.$$

*In other words, each line consists of exactly two of the three points.*

*Then Postulate 1 is satisfied, because given any two distinct points, there is a line containing them:*

- *the points $P$ and $Q$ lie on the line $\{P, Q\}$,*
- *the points $P$ and $R$ lie on the line $\{P, R\}$,*
- *the points $Q$ and $R$ lie on the line $\{Q, R\}$.*

*So this gives a model with three points and three lines satisfying Postulate 1.*

# 6 Week 2 Lecture 1: "summary diagrams", "Euclid-style" proofs, and more on postulates

*A discussion of proposition "summary diagrams", "Euclid-style" proofs, stronger forms of Postulates 1 and 2, and the definition of right angles and Postulate 4.*

## Goals for this week

(a) Create a proposition "summary diagram" for a given proposition and use them to remember Propositions I.1 to I.34.

(b) Remember the definition of parallel lines and right angles; the stronger forms of Postulates 1 and 2, Postulates 4 and 5.

(c) Apply the known propositions to create new "Euclid-style" proofs.

Roughly speaking:

- the diagram shows the situation;
- the construction adds the needed objects;
- the proof justifies the claim.

Roughly, Book I may be viewed in three parts:

- triangles: Propositions I.1–I.26;
- parallel lines: Propositions I.27–I.34;
- area: Propositions I.35–I.48.

## "Summary diagrams" of propositions

We begin with Proposition I.4.

**Proposition I.4 (SAS).** If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equal to the triangle,

and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides.

**Question.** How can we make a "summary diagram" for this proposition?

The idea is the following:

- mark identically the sides and angles that are given to be equal;
- use three colors: one for the initial data, one for the construction, and one for the conclusion.

For Proposition I.4, the initial data may be summarized as

$$AB = DE, \qquad AC = DF, \qquad \angle BAC = \angle EDF,$$

and the conclusion as

$$BC = EF, \qquad \angle CBA = \angle FED, \qquad \angle ACB = \angle DFE.$$

## Activity 1

Read the enunciation part of Propositions I.5, I.6, and I.8, and create "summary diagrams".

Here are the proposition summaries in words.

**Proposition I.5.** If $AB = AC$ in triangle $ABC$, then

$$\angle CBA = \angle ACB.$$

Moreover, if the equal sides are extended, then the exterior angles are also equal.

**Proposition I.6.** If in triangle $ABC$ we have

$$\angle CBA = \angle ACB,$$

then

$$AB = AC.$$

**Proposition I.8 (SSS).** If two triangles satisfy

$$AB = DE, \qquad AC = DF, \qquad BC = EF,$$

then the corresponding angles are equal:

$$\angle CBA = \angle FED, \qquad \angle BAC = \angle EDF, \qquad \angle ACB = \angle DFE.$$

This is also called the SSS theorem, for *side-side-side*.

# "Euclid-style" proofs

We now practice writing short proofs in the style of Euclid.

## Activity 2

Using the postulates and propositions we have seen so far, prove the following statements using the style of Euclid.

### Problem 1

Let $ABC$ be an isosceles triangle with $AB = AC$. Suppose $D$ lies on the segment $BC$ and that $BD = DC$, that is, $D$ is the midpoint of $BC$. Show that

$$\angle BAD = \angle DAC.$$

**Idea.** The diagram displays the initial information and helps us see what proposition to apply.

**Proof.** Consider triangles $BAD$ and $DAC$. We have

$$AB = AC, \qquad BD = DC, \qquad AD = AD.$$

Hence the three corresponding sides are equal, so by Proposition I.8 (SSS theorem),

$$\angle BAD = \angle DAC.$$

This is what we wanted to prove.

### Problem 2

Show the converse of Problem 1.

Let $ABC$ be an isosceles triangle with $AB = AC$, and let $D$ be a point on the segment $BC$. Assume

$$\angle BAD = \angle DAC.$$

Show that

$$BD = DC.$$

**Proof.** Consider triangles $BAD$ and $DAC$. We have

$$AB = AC, \qquad \angle BAD = \angle DAC, \qquad AD = AD.$$

Thus two sides and the included angle are equal, so by Proposition I.4 (SAS theorem),

$$BD = DC.$$

Therefore $D$ is the midpoint of $BC$.

# Revisiting Proposition I.4

We now review again the proof of the SAS theorem.

Suppose

$$AB = DE, \qquad AC = DF, \qquad \angle BAC = \angle EDF.$$

Euclid argues that one may superpose triangle $ABC$ onto triangle $DEF$ by placing $A$ on $D$ and $B$ on $E$. Then the equality of the included angles and sides forces $C$ to coincide with $F$, and so the side $BC$ coincides with $EF$.

## Important issues

There are at least two important problems with this argument.

(a) Superposition is not introduced by the postulates or by previous propositions.

(b) Even if $B$ coincides with $E$ and $C$ coincides with $F$, it is not automatic from the weak form of Postulate 1 that the line joining $B$ to $C$ is the same as the line joining $E$ to $F$.

So Euclid's proof of Proposition I.4 implicitly uses a stronger form of Postulate 1.

# Stronger forms of Postulates 1 and 2

## A stronger form of Postulate 1

We will use the following stronger version of Postulate 1.

**Postulate 1 (strong form).** There exists a unique line passing through any two distinct points.

This stronger form justifies notation such as:

- $\overleftrightarrow{AB}$ or $(AB)$ for the unique line passing through $A$ and $B$;
- $AB$ or $[AB]$ for the segment between $A$ and $B$, that is, the points $D$ on $(AB)$ such that $D$ is between $A$ and $B$;
- $\overrightarrow{AB}$ for the ray emanating from $A$ and passing through $B$, that is, the set of points $D$ on $(AB)$ such that $B$ is between $A$ and $D$.

### Bonus question

Can you think of a model which has points and lines and satisfies Postulate 1 in its weak form,

between two points there is a line,

but not in its strong form?

### A stronger form of Postulate 2

We will also use the following stronger form of Postulate 2.

**Postulate 2 (strong form).** If $A$ and $B$ are distinct points on a line, then we can find a point $D$ on the line, distinct from $A$ and $B$, such that $B$ is between $A$ and $D$.

In other words, a segment may be extended beyond one of its endpoints.

### Bonus question

Can you think of a model that has points and lines and satisfies Postulates 1 and 2 in their strong forms, but such that certain lines contain only one point?

## Right angles and Postulate 4

We now discuss Euclid's definition of a right angle.

**Definition 10.** When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is called a perpendicular to that on which it stands.

**Interpretation.** Consider an angle $\angle BAC$, and choose a point $D$ on the line $(AB)$ such that $A$ is between $D$ and $B$. Then $\angle BAC$ is a right angle if

$$\angle BAC = \angle CAD.$$

This gives meaning to the picture of a line standing on another line and creating two equal adjacent angles.

**Postulate 4.** All right angles are equal to one another.

## Final activity

In GeoGebra Classic, create the constructions for Propositions I.9 to I.12 and save the corresponding new tools. You may see working tools using this link GeoGebra activity.

If you have time, try to answer the following two questions:

(a) Can you think of a model which has points and lines and satisfies Postulate 1 in its weak form but not in its strong form?

(b) Can you think of a model which satisfies the strong forms of Postulates 1 and 2 but has certain lines with only one point on them?

# 7 Week 2 Section Worksheet

*A worksheet on Euclid-style proofs, summary diagrams, and constructions from Book I.*

**Problem 6** ***Euclid-style proof: diagonals of a rhombus.***

*Definition.* *A rhombus is a quadrilateral with four equal sides.*

*Exercise.* *Show that the diagonal of a rhombus meet orthogonal.*

*Remark. At this stage, you should assume that the diagonals meet.*

*Suggestion. Begin by drawing an annotated diagram containing the information you know from the definition of a rhombus.*

*Hint. Consider a rhombus $ABCD$ and assume the diagonals $AC$ and $BD$ intersect in a point $E$. By the definition of right angles, it is enough to show that*

$$\angle BEA = \angle CEB.$$

*Indeed, since $A$, $E$, and $C$ lie on the same straight line, equal adjacent angles imply that the diagonals are orthogonal.*

**Problem 7** ***"Summary diagrams".*** *Read the enunciation of Propositions I.13 through I.21. For each proposition, draw a summary diagram that helps you remember what the proposition says.*

**Problem 8** ***Recognizing Euclid's propositions.***

*For each of the following statements, identify which proposition of Euclid's* Book I *states this result (or can be directly applied).*

(a) *Let $ABC$ be a triangle. Suppose that*

$$AB > AC.$$

*Then*

$$\angle ACB > \angle ABC.$$

(b) *Let $ABC$ be a triangle. Suppose that*

$$\angle ACB > \angle ABC.$$

*Then*

$$AB > AC.$$

(c) *Let $ABC$ be a triangle, and let $D$ be a point on $BC$. Then*

$$AB + AC > BD + DC.$$

(d) *Let $ABC$ be a triangle, and let $D$ be a point on $BC$ extended beyond $C$. Then*

$$\angle ACD > \angle ABC.$$

**Problem 9** ***Further activities.***

(a) *In GeoGebra, GeoGebra Classic, or with pen and paper, carry out the constructions of Propositions I.22 and I.23. You may use the following GeoGebra activity where the necessary preliminary tools have already been created.*

*See page 4 once you have tried making Proposition I.22.*

*Remark. Proposition I.23 uses an alternative version of Proposition I.22 which, given a triangle $ABC$ and a ray $[DE)$, allows you to "move" the triangle $ABC$ so that $A$ is sent to $D$ and $B$ lies on the ray $[DE)$.*

*More precisely, the construction produces points $F$ and $G$ such that $F$ lies on the ray $[DE)$ and*

$$DF = AB, \qquad DG = AC, \qquad FG = BC.$$

*Thus the triangle $DFG$ is congruent to the triangle $ABC$, with the side corresponding to $AB$ placed along the given ray.*

*I created this tool, and you can play around with it using the corresponding link GeoGebra activity.*

*As a last exercise, you can try to design this tool yourself. In my construction, I only used Proposition I.2.*

(b) *Choose one proposition among Propositions I.13 through I.23. Identify its six parts as clearly as you can:*

*Enunciation, Exposition, Specification, Construction, Proof, Conclusion.*

*If you find any gap, unclear step, or point where Euclid's proof is not fully rigorous from a modern point of view, explain it briefly.*

(c) *Suppose again that $ABCD$ is a rhombus and that the diagonals $AC$ and $BD$ meet at a point $E$. Can you show that $E$ is the midpoint of both diagonals?*

*See the next page for a possible approach to the rhombus exercise, after you have spent time thinking about it.*

## A possible approach to the rhombus exercise

Since $AB = AD$, Proposition I.5 gives

$$\angle ABD = \angle BDA.$$

Since $CB = CD$, Proposition I.5 gives

$$\angle CBD = \angle BDC.$$

Now compare triangles $ABD$ and $CBD$. We have

$$AB = BC, \qquad AD = DC, \qquad BD = BD.$$

Hence, by Proposition I.8,

$$\angle ABD = \angle CBD.$$

Since $E$ lies on the line $BD$, this gives

$$\angle ABD = \angle ABE \qquad \text{and} \qquad \angle CBD = \angle CBE.$$

Therefore

$$\angle ABE = \angle CBE.$$

Now compare triangles $ABE$ and $CBE$. We have

$$BA = BC, \qquad BE = BE, \qquad \angle ABE = \angle CBE.$$

Hence, by Proposition I.4,

$$\angle BEA = \angle CEB.$$

Since $A$, $E$, and $C$ lie on the same straight line, it follows that the equal adjacent angles $\angle BEA$ and $\angle CEB$ are right angles. Therefore,

$$BD \perp AC \qquad \text{at } E.$$

# GeoGebra note on Proposition I.22

One difficulty with GeoGebra is that Euclid's original constructions are not always well suited for building a custom tool.

Recall the goal of Proposition I.22:

> Given three straight lines equal to three given straight lines, to construct a triangle.

In Euclid's notation, the three given segments are usually called $A$, $B$, and $C$. The goal is to construct a triangle whose three sides are equal to $A$, $B$, and $C$.

Ideally, we would like to create a GeoGebra tool whose only inputs are the three given segments, and which works in any configuration where the triangle inequality is satisfied. In other words, no one segment should be longer than the sum of the other two.

As it stands, Euclid's construction has a few problems from the point of view of GeoGebra.

First, it asks you to create a new point $D$, which has nothing to do with the initial data you started with, and then to create new objects based on this point. As a result, if you want to turn the construction into a custom tool, you would have to add this extra point to your initial input, even though it is not really part of the given data. This is unnecessary.

Second, the construction does not work well in all configurations. For example, in order to carry out Euclid's construction, you need to choose a point $E$ very far along a ray emanating from $D$. The reason is that later you must remove from the longer segment $DE$ a segment equal to $A$ using Proposition I.3. But this is already problematic in GeoGebra, because the point $E$ is chosen arbitrarily and is not tied to the initial data $A$, $B$, and $C$. As soon as your given sides become too long relative to your original choice of $E$, the construction will no longer work properly, and your tool will break.

For GeoGebra, it is therefore better to use a modified construction that depends only on the three given segments and works uniformly whenever the triangle inequality holds.

Suppose the three given segments are $AB$, $CD$, and $EF$.

(a) Create the three segments $AB$, $CD$, and $EF$.

(b) Construct the full line $(AB)$.

(c) Use your Proposition I.2 tool to copy the segment $CD$ starting at the point $B$.

In GeoGebra, click in the order

$$B,\ C,\ D.$$

This produces a point $G$ such that

$$BG = CD.$$

(d) Draw the circle centered at $B$ with radius $BG$.

(e) Let $H$ be the intersection of this circle with the line $(AB)$ on the right of $B$.

Then

$$BH = BG = CD.$$

(f) Use your Proposition I.2 tool again to copy the segment $EF$ starting at the point $H$.

In GeoGebra, click in the order

$$H,\ E,\ F.$$

This produces a point $I$ such that

$$HI = EF.$$

(g) Draw the circle centered at $H$ with radius $HI$.

(h) Draw the circle centered at $B$ with radius $AB$.

(i) Let $K$ be an intersection point of these two circles that does not lie on the line $(AB)$.

(j) Draw the segments $BK$, $KH$, and $BH$.

Then the triangle $BKH$ has the required side lengths:

$$BK = AB, \qquad BH = CD, \qquad KH = EF.$$

So $\triangle BKH$ is the triangle we wanted.

Once the construction is complete, you can create a new GeoGebra tool with:

- inputs: $A, B, C, D, E, F$;
- outputs: the three sides $BK$, $BH$, and $KH$.

You can then test the tool on many different configurations.

A particularly important thing to observe is what happens when the triangle inequality fails. If one of the three given segments is longer than the sum of the other two, then the two circles in the construction will no longer meet in a point above the line, and the construction breaks down. This is exactly what should happen: in that case, no triangle with those three side lengths exists.

You may use the following GeoGebra activity to see a working version of this Proposition I.22 tool.

# 8 Week 2 Lecture 2: ASA, parallel lines, and angle sums in triangles

*A discussion of the ASA and AAS congruence theorems, parallel lines and Postulate 5, basic properties of parallel lines, the construction of a parallel line through a point, and the angle sum in a triangle.*

## Learnt in section

In section, we discussed the following points.

- one more Euclid-style proof using a rhombus;
- angles made by intersecting straight lines, especially Propositions I.13–I.15;
- angle and side comparisons in triangles, especially Propositions I.16–I.21;
- constructing a triangle with three given sides, Proposition I.22;
- constructing an angle equal to a given angle, Proposition I.23.

In particular:

- Proposition I.13 says that when a straight line stands on a straight line, the adjacent angles sum to two right angles.
- Proposition I.14 says that if two adjacent angles sum to two right angles, then the outer sides lie on a straight line.
- Proposition I.15 says that vertically opposite angles are equal.
- Propositions I.16–I.21 compare sides and angles in triangles.
- Proposition I.20 is the triangle inequality.

## Goals for today

Today we will discuss:

(a) the ASA and AAS theorems (Proposition I.26);

(b) parallel lines: definition and postulate;

(c) basic properties of parallel lines (Propositions I.27–I.30);

(d) constructing a parallel to a line through a point;

(e) the sum of the angles in a triangle (Proposition I.32).

This finishes the part of Book I on triangles and parallelism.

# The ASA and AAS theorems

## A comparison result: Propositions I.24 and I.25

Propositions I.24 and I.25 compare angles and sides in triangles.

**Proposition I.24.** If two triangles have two sides equal respectively, but the angle contained by the equal straight lines in one triangle is greater than the angle contained by the equal straight lines in the other, then the base opposite the greater angle is greater.

**Proposition I.25.** Conversely, if two triangles have two sides equal respectively, but the base of one is greater than the base of the other, then the angle contained by the equal straight lines in the triangle with greater base is greater.

## ASA theorem (Proposition I.26)

The ASA theorem states the following.

If two triangles have two angles equal respectively, and the side included between those angles equal, then the triangles coincide in all corresponding sides and angles.

In symbols, if triangles $ABC$ and $DEF$ satisfy

$$\angle B = \angle E, \qquad BC = EF, \qquad \angle C = \angle F,$$

then the corresponding remaining parts also agree:

$$AB = DE, \qquad AC = DF, \qquad \angle A = \angle D.$$

**Remark.** As Euclid does, we will sometimes write

$$ABC = DEF$$

to mean that the two triangles coincide in all corresponding sides and angles.

**Proof idea**

We prove this by contradiction.

Assume, for instance, that

$$AB > DE.$$

Then by Proposition I.3, we can construct a point $G$ on the segment $BA$ such that

$$BG = DE.$$

Now compare triangles $GBC$ and $DEF$. We have

$$BG = DE, \qquad BC = EF, \qquad \angle GBC = \angle DEF.$$

Hence by Proposition I.4,

$$\angle GCB = \angle DFE.$$

But by assumption,

$$\angle DFE = \angle ACB.$$

So

$$\angle GCB = \angle ACB.$$

This is impossible, since $G$ lies on the segment $BA$, and so triangle $GBC$ is properly contained in triangle $ABC$, forcing

$$\angle ACB > \angle GCB.$$

This contradiction shows that $AB$ cannot be greater than $DE$.

Proceeding in the same way, one shows that $DE$ cannot be greater than $AB$. Hence

$$AB = DE.$$

Then Proposition I.4 gives the remaining equalities:

$$AC = DF, \qquad \angle A = \angle D.$$

## AAS theorem

A closely related version is the AAS theorem: if two angles and a non-included side agree, then the two triangles are equal.

For example, if

$$\angle B = \angle E, \qquad \angle C = \angle F, \qquad AC = DF,$$

then

$$ABC = DEF.$$

**Remark.** Instead of assuming $AC = DF$, one could also assume

$$AB = DE.$$

So there are two symmetric versions of AAS.

### Application

Consider an isosceles triangle $ABC$ with

$$AB = AC.$$

Assume the bisector of $\angle B$ meets $AC$ at $E$, and the bisector of $\angle C$ meets $AB$ at $D$.

Show that

$$CD = BE \qquad \text{and} \qquad CE = BD.$$

**Step 1: diagram.** Draw the triangle together with the two bisectors.

**Step 2: identify triangles where ASA applies.**

Since $AB = AC$, Proposition I.5 gives

$$\angle ABC = \angle ACB.$$

But

$$\angle ABC = \angle EBD + \angle DBC = 2\angle DBC$$

because $BD$ bisects $\angle ABC$, and similarly

$$\angle ACB = 2\angle ECB$$

because $CE$ bisects $\angle ACB$.

So from $\angle ABC = \angle ACB$, we conclude

$$\angle DBC = \angle ECB.$$

Now compare triangles $EBC$ and $DCB$. We have

$$\angle EBC = \angle DCB, \qquad BC = BC, \qquad \angle ECB = \angle DBC.$$

Hence by ASA,

$$\triangle EBC = \triangle DCB.$$

Therefore

$$DB = EC \qquad \text{and} \qquad BE = DC.$$

## Parallel lines

### Definition

**Definition 23.** Two straight lines are said to be parallel if they do not intersect.

### Postulate 5

**Postulate 5.** If a straight line falling on two straight lines makes the interior angles on the same side sum to less than two right angles, then the two straight lines, if extended indefinitely, meet on that side on which the angles are less than two right angles.

**Remark.** This is an axiom of plane geometry, so one should include in the statement that the lines lie in a plane.

**Historical remark.** For about two thousand years, mathematicians tried either to prove Postulate 5 from Postulates 1–4 or to replace it by seemingly weaker assumptions. One famous reformulation is Playfair's axiom.

**Playfair's axiom.** Given a line $\ell$ and a point $P$ not on $\ell$, there exists a unique line $\ell'$ through $P$ parallel to $\ell$.

We will see that Euclid's postulates imply Playfair's axiom. The converse, is more subtle and was established later.

## Basic propositions about parallel lines

We now summarize Propositions I.27–I.30.

Suppose a line falls on two lines $(AB)$ and $(CD)$.

- If $(AB) \parallel (CD)$, then alternate angles are equal.
- If $(AB) \parallel (CD)$, then the exterior angle equals the interior opposite angle.
- If $(AB) \parallel (CD)$, then the interior angles on the same side sum to two right angles.
- Conversely, each of the properties above implies that the two lines are parallel.

### Proof of one implication

Let us prove that if two lines are parallel, then alternate angles are equal.

Assume

$$(AB) \parallel (CD).$$

We want to show that the relevant alternate angles, say $\angle 1$ and $\angle 2$, are equal.

Assume by contradiction that

$$\angle 1 > \angle 2.$$

Now observe that

$$\angle 1 + \angle 3 = 2 \text{ right angles.}$$

Hence

$$\angle 2 + \angle 3 < \angle 1 + \angle 3 = 2 \text{ right angles.}$$

Therefore, by Postulate 5, the two lines would meet on that side. This contradicts the assumption that they are parallel.

By the same argument, $\angle 2 > \angle 1$ is impossible. Hence

$$\angle 1 = \angle 2.$$

So the alternate angles are equal.

### Proposition I.30

**Proposition I.30.** Straight lines parallel to the same straight line are parallel to one another.

**Proof sketch.** Assume that $(AB)$ and $(CD)$ are both parallel to $(EF)$. Euclid then considers a line intersecting all three lines and applies Proposition I.29 twice to conclude that the same alternate angles are equal, and then uses Proposition I.27 to deduce that

$$(AB) \parallel (CD).$$

**Remark.** There is a gap here: Euclid simply fixes a line intersecting the three given lines, but this existence is not justified from the earlier postulates alone. This will later be repaired by strengthening the axiomatic framework.

**Another remark.** If $\ell$ is a line and $P$ is a point not on $\ell$, there is at most one line through $P$ parallel to $\ell$.

Indeed, if there were two such lines $\ell'$ and $\ell''$, both through $P$ and both parallel to $\ell$, then Proposition I.30 would imply

$$\ell' \parallel \ell''.$$

But $\ell'$ and $\ell''$ meet at $P$, contradiction.

## Constructing a parallel line through a given point

**Proposition I.31.** Given a line $\ell$ and a point $P$ not on $\ell$, we can construct a line $\ell'$ through $P$ parallel to $\ell$.

**Activity.** Follow the construction and create the line using GeoGebra Classic. You should use the tools created so far, such as Proposition I.23.

## Sum of the angles in a triangle

**Proposition I.32.** In any triangle, the sum of the three interior angles equals two right angles.

Moreover, the exterior angle equals the sum of the two remote interior angles.

So if the angles of a triangle are labeled $\angle 1$, $\angle 2$, and $\angle 3$, and $\angle 4$ is an exterior angle, then

$$\angle 1 + \angle 2 + \angle 3 = 2 \text{ right angles}$$

and

$$\angle 4 = \angle 1 + \angle 3.$$

**Proof.** Read Proposition I.32 at home.

# 9 Homework 2

*Homework 2 for MATH 128A.*

**MATH 128A: Homework 2** **Due date: Thursday, April 16**

**Instructions.** You will submit your solutions on Gradescope, which you can access through our Canvas course page by clicking on the assignment.

You are welcome to submit your homework either handwritten or in LaTeX. If you choose to submit handwritten work, please use pen rather than pencil, and make sure that both your handwriting and your scans are as readable as possible in order to facilitate grading.

Please also feel free to attend office hours if you would like hints on the homework or if you would like to discuss more advanced material. Also, you are highly encouraged to work in groups!

You will see that our Ximera course page proposes you to see the answers, but this will only become available once I publish the solution.

**Problem 10** ***Parallelograms (45 points total).***

*A quadrilateral is a figure determined by four distinct points $A, B, C, D$ such that no three lie on the same straight line, and the segments $AB$, $BC$, $CD$, and $DA$ intersect only at their endpoints.*

*A quadrilateral $ABCD$ is called a* parallelogram *if its opposite sides are parallel, that is,*

$$(AB) \parallel (CD) \qquad \text{and} \qquad (BC) \parallel (AD).$$

***Hint:*** *For these exercises, you should always start with a diagram. Then, you may rely on:*

- *the congruence theorems I.4 (SAS), I.8 (SSS), and I.26 (AAS/ASA),*
- *the angle equality I.15 (vertical angles),*
- *the angle properties of parallel lines I.27, I.28, and I.29.*

***How we will grade Problem 1.*** *We will give some credit for a correct initial diagram, some credit for the method, some credit for the correctness of the argument, and some credit for proving the desired conclusion. In particular, we will look at how well you use the information given in the problem together with postulates and previously proved propositions to deduce new consequences.*

*If you use a proposition, make sure to explain why its hypotheses are satisfied before applying it.*

(a) ***(15 points)*** *Show that if $ABCD$ is a parallelogram, then*

$$AB = CD \qquad \text{and} \qquad BC = AD.$$

***Solution:*** GeoGebra Link

*Draw the diagonal $AC$.*

*Since $ABCD$ is a parallelogram, we have $(AB) \parallel (CD)$ and $(BC) \parallel (AD)$. Because the line $AC$ meets the parallel lines $(AB)$ and $(CD)$, Proposition I.29 gives*

$$\angle BAC = \angle DCA.$$

*Likewise, since the line $AC$ meets the parallel lines $(BC)$ and $(AD)$, Proposition I.29 gives*

$$\angle BCA = \angle DAC.$$

*Also, the side $AC$ is common to triangles $BAC$ and $DCA$.*

*Therefore triangles $BAC$ and $DCA$ have two equal angles and the included side equal, so by Proposition I.26 (ASA) they are congruent. Hence the corresponding sides are equal:*

$$AB = CD \qquad \text{and} \qquad BC = AD.$$

*This is what we wanted to prove.*

(b) ***(15 points)*** *Let $ABCD$ be a parallelogram. Show that the diagonals $AC$ and $BD$ intersect at their midpoint.*

*You may assume that $AC$ and $BD$ intersect at a point $E$, and you need to show*

$$AE = EC \qquad \text{and} \qquad BE = ED.$$

***Solution:*** GeoGebra Link

*Let $E$ be the intersection point of the diagonals $AC$ and $BD$.*

*Consider triangles $AEB$ and $CED$. Since the diagonals intersect, the angles $\angle AEB$ and $\angle CED$ are vertical angles, so by Proposition I.15,*

$$\angle AEB = \angle CED.$$

*Since $(AB) \parallel (CD)$ and the line $BD$ meets them, Proposition I.29 gives*

$$\angle ABE = \angle CDE.$$

*From part (a), opposite sides of a parallelogram are equal, so*

$$AB = CD.$$

*Therefore triangles $AEB$ and $CED$ have two equal angles and one corresponding side equal, so by Proposition I.26 (AAS) they are congruent. Hence corresponding sides are equal:*

$$AE = EC \qquad \text{and} \qquad BE = ED.$$

*Thus the diagonals bisect one another, so they intersect at their midpoint.*

(c) ***(15 points)*** *(Converse) Let $ABCD$ be a quadrilateral such that*

$$AB = CD \qquad \text{and} \qquad BC = AD.$$

*Show that $ABCD$ is a parallelogram.*

***Solution:*** GeoGebra Link

*Draw the diagonal $AC$.*

*In triangles $ABC$ and $CDA$, we are given*

$$AB = CD \qquad \text{and} \qquad BC = AD,$$

*and the side $AC$ is common to both triangles. Therefore, by Proposition I.8 (SSS), triangles $ABC$ and $CDA$ are congruent.*

*Hence corresponding angles are equal:*

$$\angle BAC = \angle DCA \qquad \text{and} \qquad \angle BCA = \angle DAC.$$

*Now the line $AC$ meets the lines $(AB)$ and $(CD)$, and the alternate interior angles $\angle BAC$ and $\angle DCA$ are equal. Therefore, by Proposition I.27, the lines $(AB)$ and $(CD)$ are parallel.*

*Similarly, the line $AC$ meets the lines $(BC)$ and $(AD)$, and the alternate interior angles $\angle BCA$ and $\angle DAC$ are equal. Therefore, again by Proposition I.27,*

$$(BC) \parallel (AD).$$

*Since both pairs of opposite sides are parallel, $ABCD$ is a parallelogram.*

**Problem 11** ***Reading for the Pythagorean Theorem (25 points).***

*Read Propositions I.34, I.35, I.37, I.41, I.46, and I.47 in Book I of Euclid's* Elements.

***Important.*** *In these propositions, Euclid uses the word "equal" to refer to* area. *Two figures are said to be equal if they can be subdivided into smaller pieces which are themselves equal.*

*Choose* three *propositions among*

*I.34, I.35, I.37, I.41, I.46, I.47,*

*and for each of your choices, draw a* summary diagram *that helps you remember the statement.*

***How we will grade Problem 2.*** *We will give full credit for a clean diagram with clear markings introducing the relevant notations, together with a brief piece of text stating the hypotheses and the conclusion, in the style used during Week 2 Lecture 1.*

***Remark:*** *You are encouraged to be creative. For example, you may use different colors (or shading) to highlight the different regions in each figure. If you do not want to use color, you may instead distinguish the regions by using different patterns.*

*For instance, Proposition I.34 asserts that in a parallelogram $ABCD$, the triangles $ABC$ and $CDA$ are equal, in the sense that they have the same area. You might highlight the region $ABC$ in blue and the region $CDA$ in red, and then summarize the proposition by writing something like*

$$ABCD \text{ parallelogram} \overset{I.34}{\Longrightarrow} \text{(blue region)} = \text{(red region)}.$$

*If you prefer not to use color, you could instead mark one region with one pattern and the other region with a different pattern, and write the same kind of conclusion.*

***Solution:*** *Since this problem asks for diagrams, below I write the short text that can accompany three summary diagrams.*

***Choice 1: Proposition I.34.***

Hypotheses: *$ABCD$ is a parallelogram, and the diagonal $AC$ is drawn.*

Conclusion: *The diagonal cuts the parallelogram into two equal triangles:*

$$\triangle ABC = \triangle CDA \qquad \text{(equal in area)}.$$

*In the diagram, one can shade $\triangle ABC$ one way and $\triangle CDA$ another way.* GeoGebra Link

***Choice 2: Proposition I.41.***

Hypotheses: *A parallelogram and a triangle stand on the same base and lie between the same parallels.*

Conclusion: *The parallelogram has twice the area of the triangle:*

$$\text{area of parallelogram} = 2 \cdot \text{area of triangle}.$$

*In the diagram, one can mark the whole parallelogram with one pattern and the triangle with another, and then write that the parallelogram is double the triangle.* GeoGebra Link

***Choice 3: Proposition I.47.***

Hypotheses: *$ABC$ is a right triangle with right angle at $A$, and squares are constructed on the three sides.*

Conclusion: *The square on the side opposite the right angle equals the sum of the squares on the two legs:*

$$\text{square on } BC = \text{square on } AB + \text{square on } AC.$$

*In the diagram, one can draw the three squares and mark the two smaller ones with different patterns, indicating that together they equal the large square.* *GeoGebra Link*

**Problem 12** ***Straight-edge and compass constructions (30 points total).***

*There will be a compass-and-straightedge construction problem on the midterm, to be done by hand (no GeoGebra).*

*One thing you may have noticed is that Euclid does not hesitate to reuse constructions that were established earlier. This is perfectly natural, but some of these previously established constructions require many steps. There is another kind of exercise, closer in spirit to a puzzle, which asks you to carry out a construction in as few moves as possible using only a straightedge and compass.*

*We first need to explain how moves are counted.*

- *Drawing a new line counts as one move.*
- *Drawing a new circle counts as one move.*
- *Extending a line already drawn on the paper does* not *count as a move.*
- *Choosing points at random on the paper does* not *count as a move.*
- *Naming intersection points does* not *count as a move.*

*Thus, only actual uses of the straightedge to draw a new line and actual uses of the compass to draw a new circle count as moves.*

*The goal in the following exercises is to complete each construction in as few moves as possible. You will still receive full credit if your construction is correct but does not use the minimum possible number of moves.*

*We will sometimes give a* par *value for a construction. This is the typical number of moves an experienced constructor might use. By thinking harder, you may sometimes do better.*

*For example, the construction of an equilateral triangle on a given segment $AB$ takes 4 moves in this counting system:*

(a) *draw the circle with center $A$ and radius $AB$;*

(b) *draw the circle with center $B$ and radius $BA$;*

(c) *let $C$ be an intersection point of the two circles, and draw the segment $AC$;*

(d) *draw the segment $BC$.*

*Then $ABC$ is the required triangle.*

*In particular, the segment $AB$ is considered to be already given: you should draw it on the paper, but you should not count it as a move.*

***How we will grade Problem 3.*** *We will give points for the following:*

- *all the lines and circles used in the construction are shown;*
- *the diagram has clear markings;*
- *the number of moves is stated;*
- *a short proof is provided;*
- *the construction achieves the required goal.*

Using this convention, do the following constructions in as few moves as possible.

(a) **(10 points)** Given an angle $\angle BAC$, construct its bisector. (par = 4)

**Solution: Construction (4 moves).** GeoGebra Link

(i) Draw a circle centered at $A$ meeting the rays $AB$ and $AC$ at points $D$ and $E$.

(ii) Draw a circle centered at $D$ with radius $DE$.

(iii) Draw a circle centered at $E$ with radius $ED$.

(iv) Let $F$ be an intersection point of the two circles from steps 2 and 3. Draw the line $AF$.

**Number of moves:** 4.

**Proof.** Since $AD = AE$ (they are radii of the circle centered at $A$), and since $DF = EF$ (they are radii of the circles centered at $D$ and $E$), and since $AF$ is common, triangles $ADF$ and $AEF$ are congruent by Proposition I.8. Therefore

$$\angle DAF = \angle FAE.$$

But $D$ lies on the ray $AB$ and $E$ lies on the ray $AC$, so

$$\angle BAF = \angle FAC.$$

Hence the line $AF$ bisects the angle $\angle BAC$.

(b) **(10 points)** Given a segment $AB$, construct its midpoint. (par = 3)

**Solution: Construction (3 moves).** GeoGebra Link

(i) Draw the circle centered at $A$ with radius $AB$.
(ii) Draw the circle centered at $B$ with radius $BA$.
(iii) Let $C$ and $D$ be the two intersection points of these circles. Draw the line $CD$, and let $M$ be the intersection point of $CD$ with $AB$.

**Number of moves:** 3.

**Proof.** Because $C$ and $D$ lie on circles with equal radii centered at $A$ and $B$, we have

$$AC = CB = BD = DA.$$

In particular, $ACBD$ is a parallelogram thanks to Problem 1 (c). Thus, by Problem 1 (b), its diagonals intersect at their midpoint. Hence, $E$ is the midpoint of $AB$.

(c) **(10 points)** Given a line $\ell$ and a point $A$ on $\ell$, construct the line perpendicular to $\ell$ through $A$. (par = 4; possibly 3 according to Hartshorne)

**Solution: Construction (4 moves).** GeoGebra Link

(i) Draw a circle centered at $A$ meeting the line $\ell$ at two points $B$ and $C$.
(ii) Draw a circle centered at $B$ with radius $BC$.
(iii) Draw a circle centered at $C$ with radius $CB$.
(iv) Let $D$ be an intersection point of the circles from steps 2 and 3. Draw the line $AD$.

**Number of moves:** 4.

**Proof.** By Construction, $A$ is the midpoint of $BC$, so $AB = BC$. Then, $D$ is at the intersection of two circles with equal radii, so $CD = BD$. Finally, the side $AD$ is common to both $BAD$ and $CAD$. Thus, by the SSS Theorem, $\angle BAD = \angle CAD$, and, since $A$, $B$, and $C$ lie on the same line, $\angle BAD$ and $\angle CAD$ are right angles. Hence, since the line $(AD)$ is perpendicular to $\ell$.

**Problem 13** ***Reflection (2 bonus points).***

*Reflect on the first three weeks of the course.*

(a) *What did you find most challenging so far?*

(b) *How did you find the pace of the lectures: too fast, too slow, or just right?*

(c) *Is there something you wish we did more of, or less of?*

*Please answer in a few complete sentences. Your responses will be anonymous.*

# 10 Week 3 Lecture 1: Angles in triangles, RASS, and the center of a circle

*A discussion of the sum of the angles in a triangle, polygons and convexity, the RASS congruence theorem, an application to chords in a circle, and the construction of the center of a circle.*

## Goals for today

Today we will discuss:

(a) the sum of the angles in a triangle (Proposition I.32);

(b) polygons, convexity, and sums of interior angles;

(c) the RASS congruence theorem;

(d) an application to chords in a circle;

(e) locating the center of a circle.

## Sum of the angles in a triangle

**Proposition I.32.** In any triangle, the sum of the three interior angles equals two right angles. Moreover, an exterior angle equals the sum of the two remote interior angles.

Let the three interior angles of a triangle be denoted by $\angle 1$, $\angle 2$, and $\angle 3$, and let $\angle 4$ be an exterior angle. Then:

$$\angle 1 + \angle 2 + \angle 3 = 2 \text{ right angles}$$

and

$$\angle 4 = \angle 1 + \angle 3.$$

**Proof.** Let $ABC$ be a triangle. Construct a line $\ell$ through $C$ parallel to $(AB)$ (Proposition I.31). Next, fix a point $D$ on the ray $[BC)$ so that $C$ lies between $B$ and $D$. Finally, fix a point $E$ on $\ell$ on the same side of $(BC)$ as $A$.

Since $(AB) \parallel \ell$, Proposition I.29 gives

$$\angle A = \angle ACE \qquad \text{and} \qquad \angle B = \angle ECD.$$

Since $B, C, D$ are aligned, Proposition I.13 gives

$$\angle ACB + \angle ACE + \angle ECD = 2 \text{ right angles.}$$

Substituting the two equalities above, we obtain

$$\angle A + \angle B + \angle C = 2 \text{ right angles.}$$

Moreover,

$$\angle ACD = \angle ACE + \angle ECD = \angle A + \angle B.$$

Thus the exterior angle equals the sum of the two remote interior angles.

# Polygons and convexity

## Definition

A polygon is a finite broken line made of segments, where consecutive sides meet at their common endpoint, and non-consecutive sides do not intersect.

## Examples

Examples of polygons include:

- triangles;
- quadrilaterals;
- pentagons.

## Convex polygons

A polygon is said to be *convex* if every segment joining two vertices of the polygon stays inside the polygon.

In a convex polygon, diagonals stay inside the polygon. In a non-convex polygon, some diagonals may go outside.

## Exercise

Compute the sum of the interior angles for:

(a) a convex quadrilateral;

(b) a convex pentagon;

(c) a convex polygon with $n$ sides, where $n \geq 5$.

**Hint.** Cut the polygon into triangles.

For a quadrilateral, this gives

4 right angles.

**Question.** Is the same result still true for non-convex polygons?

**Remark.** Yes, but the proof is much more difficult.

# RASS congruence theorem

**Theorem (RASS).** If two right triangles have equal hypotenuse and one corresponding side equal, then the triangles are congruent.

**Remark.** This theorem is not in Euclid's *Elements*. It follows from the Pythagorean theorem (Proposition I.47), and we will see it again in the homework.

# Application to a chord

**Exercise (also Proposition III.3).** Consider a chord $AB$ on a circle of center $C$. Assume that $D$ is a point on $AB$. Show that

$$D \text{ is the midpoint of } AB \iff \angle ADC \text{ is a right angle.}$$

**Hint.**

($\Rightarrow$) Use the SSS theorem.

($\Leftarrow$) Use the RASS theorem.

# Circle and center

## Definition

A chord is a segment joining two points on a circle.

### Proposition III.1

The perpendicular bisector of a chord contains the center of the circle.

**Proof.** Let $AB$ be a chord of a circle of center $C$, and let $\ell$ be the perpendicular bisector of $AB$. We want to show that $C$ lies on $\ell$.

Assume for contradiction that $C$ does not lie on $\ell$. Let $D$ be the midpoint of $AB$.

Since $CA = CB$, triangle $ABC$ is isosceles, so the angles at $A$ and $B$ are equal.

Now compare triangles $ADC$ and $CDB$. We have

$$AD = DB, \qquad AC = CB, \qquad CD = CD.$$

Hence by SSS,

$$\triangle ADC = \triangle CDB.$$

In particular,

$$\angle ADC = \angle CDB.$$

But $A, D, B$ are aligned, so the two adjacent angles $\angle ADC$ and $\angle CDB$ sum to two right angles. Since they are equal, each must be a right angle. Therefore $CD$ is perpendicular to $AB$.

Since $D$ is the midpoint of $AB$, it follows that the line through $D$ perpendicular to $AB$ is precisely the perpendicular bisector $\ell$. Hence $C$ lies on $\ell$, contradiction.

Therefore the perpendicular bisector of a chord contains the center of the circle.

## Activity

Using only a compass and straightedge, make a circle and then "forget" where the center is. Construct the center in as few moves as possible, without lifting your compass.

# 11 Week 3 Section Worksheet

*A worksheet on constructions, the RASS theorem, and inscribed/circumscribed circle to a triangle.*

**Problem 14** ***Warm-up: constructions and congruence.***

(a) *Using only a compass and straightedge, construct the following:*

- *the midpoint of a segment,*
- *the bisector of an angle,*
- *a perpendicular to a line through a given point,*
- *a square on a given segment,*
- *a line parallel to a given line through a given point.*

(b) *Recall the triangle congruence theorems:*

$$SAS,\ SSS,\ AAS/ASA,\ RASS.$$

**Problem 15** ***Puzzle.***

*Let $A$, $B$, and $C$ be three non-collinear points.*

*Construct, using only a straightedge and compass, a circle passing through $A$, $B$, and $C$.*

***Remark.*** *This is possible in 7 moves.*

**Problem 16** ***Comparing chords via the center.***

*Let $AB$ and $CD$ be chords of a circle of center $E$.*

- *Consider the line $\ell_1$ perpendicular to $AB$ passing through $E$, and call $F$ the intersection point with $AB$.*
- *Consider the line $\ell_2$ perpendicular to $CD$ passing through $E$, and call $G$ the intersection point with $CD$.*

*Show that*

$$EF = EG \quad \text{if and only if} \quad AB = CD.$$

**Problem 17** ***Kites and perpendicular diagonals.***

*Let $ABCD$ be a quadrilateral such that*

$$AB = AD \quad \text{and} \quad BC = CD.$$

*Show that the diagonals $AC$ and $BD$ are perpendicular.*

**Problem 18** ***Angle bisectors in a triangle.***

*Show that all the angle bisectors in a triangle intersect at a common point.*

***Idea.*** *Construct the angle bisectors at $\angle A$ and $\angle B$, and let $D$ be their intersection point.*

***Goal.*** *Show that $DC$ bisects $\angle C$.*

***Hint.*** *Construct the perpendicular lines through $D$ to:*

- *$BC$,*
- *$AB$,*
- *$AC$.*

*Using AAS, show that:*

$$FD = ED, \qquad ED = GD.$$

*Conclude using*

$$FD = GD.$$

**Problem 19** ***Bonus: classical constructions.***

*Using only a straightedge and compass, construct:*

- *the circumscribed circle to a given triangle (par = 7),*
- *the inscribed circle to a given triangle (par = 13).*

*How many moves did you use?*

# 12 Week 3 Lecture 2: Tangents and angles in circles

*A discussion of tangent lines, tangent circles, the relation between tangency and perpendicularity to the radius, and the main angle theorems in circles.*

## Goals for today

Today we will discuss:

(a) tangents to circles and tangent circles;

(b) the relation between tangency and perpendicularity to the radius;

(c) the alignment between the centers of tangent circles and the tangency point;

(d) the angle at the center and the angle at the circumference;

(e) angles in the same segment and the angle in a semicircle;

(f) the tangent-chord theorem (Proposition III.32).

## Tangents

### Definition

We say that a line *touches* or is *tangent to* a circle if it intersects the circle at a single point.

We also say that two circles *touch* or are *tangent to each other* if they intersect in a single point.

There are two possibilities:

- the circles are tangent *externally*;
- the circles are tangent *internally*.

## A basic fact about tangent circles

**Proposition III.11–III.12.** If two circles are tangent, then their centers and the point of tangency are aligned.

**Proof.** See Propositions III.11 and III.12.

## Tangency and perpendicularity

**Proposition.** Consider a line $\ell$ intersecting a circle of center $C$ at a point $A$. Then

$$\ell \text{ is tangent to the circle at } A \quad \Longleftrightarrow \quad \ell \perp CA.$$

**Proof.** We only prove the converse implication. For the direct implication, see Proposition III.16, and also Proposition III.18.

Assume that

$$\ell \perp CA.$$

We want to prove that $A$ is the only intersection point of $\ell$ with the circle.

Proceed by contradiction and assume that $\ell$ intersects the circle at both $A$ and $B$.

Since

$$CA = CB,$$

triangle $CAB$ is isosceles. Hence the angles at $A$ and $B$ are equal. But $\angle A$ is a right angle, since $\ell \perp CA$. Therefore $\angle B$ is also a right angle.

So

$$\angle A + \angle B = 2 \text{ right angles.}$$

But then

$$\angle A + \angle B + \angle C > 2 \text{ right angles,}$$

which contradicts Proposition I.32.

This contradiction shows that $A$ is the only point where $\ell$ meets the circle. Therefore $\ell$ is tangent to the circle at $A$.

## Application

Consider a circle $\mathcal{C}$, and assume that $AB$ and $AC$ are tangent to $\mathcal{C}$ at $B$ and $C$, respectively. Show that

$$AB = AC.$$

# Angles in circles

## Angle at the center and angle at the circumference

**Proposition III.20.** Consider a circle of center $C$, a chord $AB$, and a point $D$ on the circle on the same side of the line $(AB)$ as $C$. Then

$$2\angle ADB = \angle ACB.$$

**Proof.** We need to show that

$$\angle ACB = 2\angle ADB.$$

Draw the radius $CD$, and split the central angle:

$$\angle ACB = \angle ACD + \angle DCB.$$

Since

$$CA = CD,$$

triangle $ACD$ is isosceles, so

$$\angle CDA = \angle DAC.$$

By Proposition I.32 in triangle $ACD$,

$$\angle CDA + \angle DAC + \angle ACD = 2 \text{ right angles.}$$

Since the first two angles are equal, this gives

$$2\angle CDA + \angle ACD = 2 \text{ right angles.}$$

Equivalently,

$$\angle ACD = 2\angle ADC.$$

Similarly, since

$$CD = CB,$$

triangle $CDB$ is isosceles, so the same argument gives

$$\angle DCB = 2\angle DBC.$$

Therefore

$$\angle ACB = \angle ACD + \angle DCB = 2\angle ADC + 2\angle DBC = 2(\angle ADC + \angle DBC) = 2\angle ADB.$$

### Application: equal angles in the same segment

**Proposition III.21.** Consider a chord $AB$ of a circle of center $C$, and points $D$ and $E$ on the circle on the same side of $C$ with respect to the line $(AB)$. Show that

$$\angle ADB = \angle AEB.$$

### Application: angle in a semicircle

**Proposition III.31.** Let $AB$ be a diameter of a circle and $C$ a point on the circle. Show that

$$\angle ACB$$

is a right angle.

## The tangent-chord theorem

**Proposition III.32.** Let $\ell$ be tangent to a circle $\mathcal{C}$ at $A$, and let $AB$ be a chord. Then the angle between the tangent $\ell$ and the chord $AB$ is equal to the angle in the alternate segment.

In the notation of the figure:

$$\angle ADB = \angle EAB \qquad \text{and} \qquad \angle FAB = \angle BCA.$$

**Proof.** Thanks to Proposition III.21, we may assume without loss of generality that $AD$ is a diameter of the circle.

We prove first that

$$\angle ADB = \angle EAB.$$

By Proposition I.32 applied to triangle $ADB$,

$$\angle ADB + \angle DBA + \angle BAD = 2 \text{ right angles.}$$

But since $AD$ is a diameter, Proposition III.31 gives

$$\angle DBA = \text{ right angle.}$$

Hence

$$\angle ADB + \angle BAD = \text{ right angle.} \tag{1}$$

On the other hand, since $\ell$ is tangent at $A$, Proposition III.18 gives

$$\angle EAD = \text{ right angle.} \tag{2}$$

Also,

$$\angle EAD = \angle EAB + \angle BAD. \tag{3}$$

Combining (1), (2), and (3), we get

$$\angle ADB + \angle BAD = \angle EAB + \angle BAD.$$

Therefore

$$\angle ADB = \angle EAB.$$

We leave it to the reader, using Proposition III.32 in the same spirit, to show that

$$\angle FAB = \angle BCA.$$

# 13 Homework 3

*Homework 3 for MATH 128A.*

**MATH 128A: Homework 3** **Due date: Thursday, April 23**

**Instructions.** You will submit your solutions on Gradescope, which you can access through our Canvas course page by clicking on the assignment.

You are welcome to submit your homework either handwritten or in LaTeX. If you choose to submit handwritten work, please use pen rather than pencil, and make sure that both your handwriting and your scans are as readable as possible in order to facilitate grading.

Please also feel free to attend office hours if you would like hints on the homework or if you would like to discuss more advanced material. Also, you are highly encouraged to work in groups!

You will see that our Ximera course page proposes you to see the answers, but this will only become available once I publish the solution.

**Problem 20** ***Proofs in circle geometry (60 points total).***

***Hint:*** *For these exercises, you should always start with a diagram. Then, you may rely on:*

- *the congruence theorems I.4 (SAS), I.8 (SSS), I.26 (ASA/AAS), and RASS,*
- *Proposition I.32 on the angle-sum in a triangle,*
- *the angle properties of circles discussed in class, especially III.21 and III.32,*
- *the fact that, if two circles are tangent, then their centers and point of tangency are aligned (III.11 and III.12).*

***How we will grade Problem 1.*** *We will give some credit for a correct initial diagram, some credit for the method, some credit for the correctness of the argument, and some credit for proving the desired conclusion. In particular, we will look at how well you use the information given in the problem together with previously proved propositions to deduce new consequences.*

*If you use a proposition, make sure to explain why its hypotheses are satisfied before applying it.*

(a) ***(20 points)*** *Show that the sum of opposite angles of a quadrilateral inscribed in a circle is equal to two right angles.*

***Idea.*** *Draw a circle, place four points $A$, $B$, $C$, and $D$ on the circumference of the circle so that $ABCD$ is a quadrilateral.*

***Goal.*** *Show that $\angle A + \angle C = 2$ right angles.*

***Hint.*** *Draw the diagonals $AC$ and $BD$, and play with the following properties:*

- *the sum of the angles in a triangle is two right angles (Proposition I.32),*
- *given a chord and two points on the circumference on the side of the center, you obtain equal angles (Proposition III.21).*

(b) ***(20 points)*** *Show that all the angle bisectors in a triangle intersect at a common point.*

***Idea.*** *Draw a triangle $ABC$ and draw the angle bisectors at $\angle A$ and $\angle B$, and let $D$ be their intersection point (you may assume without proof that this intersection point exists and is inside the triangle).*

***Goal.*** *Show that $DC$ bisects $\angle C$.*

***Hint.*** *Construct the perpendicular lines through $D$ to:*

- *$AB$ and call $E$ the intersection point,*
- *$BC$ and call $F$ the intersection point,*
- *$AC$ and call $G$ the intersection point.*

*Using AAS, show that*

$$FD = ED \qquad \text{and} \qquad ED = GD.$$

*Deduce that*

$$FD = GD.$$

*Then use RASS to show that $DC$ bisects $\angle C$.*

***Remark.*** *If you draw the circle with center $D$ and radius $DE$, you obtain the inscribed circle to the triangle $ABC$. You can try (though I won't grade it), to draw the inscribed circle to a given triangle in as little moves as possible using only a straightedge and a compass, without lifting the compass. (A typical number of moves is* par $= 13$.*)*

(c) ***(20 points)*** *Assume two circles are tangent externally at a point $A$. Fix points $B$ and $C$ on one of the circles, create the lines $(AB)$ and $(AC)$, and call $D$ and $E$ their respective intersection points with the other circle.*

*Show that*

$$(BC) \parallel (DE).$$

***Hint.*** *Use Proposition III.32 and the fact that the centers and tangency point are aligned.*

**Problem 21** ***Construction (10 points).***

*Using only a straightedge and compass, and without lifting the compass, construct the circumscribed circle of a given triangle. (A typical number of moves is* par $= 7$*.)*

***Idea.*** *Draw a triangle $ABC$. You need to create a point $D$ such that the circle with center $D$ and radius $DA$ also passes by $B$, and $C$.*

***Hint.*** *Remember how we found the center of a circle in class using III.1.*

***How we will grade Problem 2.*** *We will give points for the following:*

- *all the lines and circles used in the construction are shown;*
- *the diagram has clear markings;*
- *the number of moves is stated;*
- ***no proof required this time;***
- *the construction achieves the required goal.*

**Problem 22** ***Reading: the construction of the pentagon (30 points total).***

***Instructions.*** *Read the attached excerpt from Chapter 1.4 of Hartshorne on the construction of the regular pentagon.*

***A remark on this problem.*** *The construction of the regular pentagon is one of the most beautiful constructions in Euclid's* Elements*, and appears in Book IV. Up to a few additional results, namely Propositions II.11 and III.36–III.37, you now have essentially all the tools needed to read Hartshorne's discussion of the pentagon construction (which is attached to this HW in pdf form and also in a screen-readable form on the course website).*

*An important part of becoming a mathematician is learning how to read mathematics on your own. This is especially important if you are interested in research, or if you are interested in teaching and need to read references in order to prepare classes. In this problem, you are asked to read the attached extract from Chapter 1.4 of Hartshorne, to carry out the constructions described in Problems 4.1 and 4.2, and also to provide summary diagrams for Proposition II.11 and for either Proposition III.36 or Proposition III.37.*

*By the end of this problem, you will have constructed a regular pentagon using only a straightedge and compass. While you will not be graded on the proofs of Propositions II.11 and III.36–III.37, I strongly encourage you to read them as well. As you will see, these proofs take time and rely on several earlier results from Books II and III.*

***Important.*** *For these constructions, you are allowed to use Tool I.2. In other words, you are allowed to keep your compass open when you lift it. For example,*

*if a point $A$ and a segment $BC$ are given, you may open your compass to the length $BC$, lift it, place one end at $A$, and draw a circle centered at $A$ with radius equal to $BC$.*

***How we will grade Problem 3.*** *We will give credit for:*

- *clear and labeled constructions,*
- *the presence of the relevant auxiliary lines and circles from the construction,*
- *evidence that you followed the construction described in the reading,*
- *the correctness and clarity of your summary diagrams.*

(a) ***(10 points)*** *Follow the steps described in Problem 4.1 of the reading to create an isosceles triangle whose base angles are twice the summit angle.*

***Remark.*** *This is also Proposition IV.10 in the* Elements.

(b) ***(10 points)*** *As Hartshorne mentions, Propositions II.11 and III.36–III.37 play an important role in Proposition IV.10. Propose a summary diagram for Proposition II.11 and for either Proposition III.36 or Proposition III.37.*

(c) ***(10 points)*** *Follow the steps described by Hartshorne in Problem 4.2 to construct a regular pentagon.*

**Problem 23** ***Reflection (2 bonus points).***

*Reflect on the course so far.*

(a) *What did you find most challenging in the material from the last two weeks?*

(b) *Which type of exercise is helping you most: proofs, constructions, or reading Euclid and Hartshorne?*

(c) *Is there something you wish we did more of, or less of?*

*Please answer in a few complete sentences. Your responses will be anonymous.*

# 14 Homework 3 Solutions

*Solutions to Homework 3 for MATH 128A.*

**Problem 24** ***Proofs in circle geometry.***

(a) ***Solution:*** GeoGebra Link *Draw the diagonals $AC$ and $BD$.*

*By Proposition III.21, angles standing on the same chord are equal. In particular,*

$$\angle DAC = \angle DBC \qquad \text{and} \qquad \angle CAB = \angle CDB.$$

*Thus,*

$$\angle DAB = \angle DAC + \angle CAB = \angle DBC + \angle CDB.$$

*In triangle $BCD$, Proposition I.32 gives*

$$\angle DBC + \angle BCD + \angle CDB = 2 \text{ right angles.}$$

*Therefore,*

$$\angle DAB + \angle BCD = 2 \text{ right angles.}$$

(b) ***Solution:*** GeoGebra Link *Let $D$ be the intersection of the angle bisectors of $\angle A$ and $\angle B$.*

*Construct perpendiculars:*

$$DE \perp AB, \quad DF \perp BC, \quad DG \perp AC.$$

*Since $AD$ bisects $\angle A$,*

$$\angle EAD = \angle DAG,$$

*and both $\angle AED$ and $\angle AGD$ are right angles. Hence, by AAS,*

$$\triangle AED \cong \triangle AGD,$$

*so $DE = DG$.*

*Similarly, since $BD$ bisects $\angle B$,*

$$\triangle BED \cong \triangle BFD,$$

*so $DE = DF$.*

*Thus $DF = DG$.*

*Now consider right triangles $DCF$ and $DCG$. We have*

$$DF = DG, \quad DC = DC.$$

*By RASS,*

$$\triangle DCF \cong \triangle DCG,$$

*so*

$$\angle FCD = \angle DCG.$$

*Hence $DC$ bisects $\angle C$.*

(c) ***Solution:*** GeoGebra Link *Let two circles be tangent externally at $A$. Let $B, C$ be points on the first circle, and let the lines $(AB)$ and $(AC)$ meet the second circle again at $D$ and $E$, respectively.*

*Let $O_1$ and $O_2$ be the centers of the two circles. Since the circles are tangent at $A$, the points $O_1, A, O_2$ are collinear. Hence the line through $A$ perpendicular to $O_1A$ (equivalently, to $O_2A$) is tangent to both circles at $A$.*

*Let $t$ denote this common tangent at $A$.*

*By Proposition III.32 (the angle between a tangent and a chord equals the angle in the opposite arc), applied to the first circle, we obtain*

$$\angle(t, AB) = \angle ACB.$$

*Applying Proposition III.32 to the second circle, we obtain*

$$\angle(t, AD) = \angle AED.$$

*Since $AB$ and $AD$ lie on the same straight line, the angles $\angle(t, AB)$ and $\angle(t, AD)$ are equal. Hence*

$$\angle ACB = \angle AED.$$

*Since $AC$ is a transversal intersecting the lines $(BC)$ and $(DE)$, this implies*

$$BC \parallel DE.$$

**Problem 25** GeoGebra Link ***Construction.***

***Solution:*** *To construct the circumscribed circle of triangle $ABC$:*

(a) *Draw a circle centered at $A$ with radius $AB$, and a circle centered at $B$ with radius $BA$. Let $E$ and $F$ be their intersection points. Draw the line $EF$ (this is the perpendicular bisector of $AB$).*

(b) *Draw a circle centered at $A$ with radius $AC$, and a circle centered at $C$ with radius $CA$. Let $G$ and $H$ be their intersection points. Draw the line $GH$ (this is the perpendicular bisector of $AC$).*

(c) *Let $D$ be the intersection point of the lines $EF$ and $GH$.*

(d) *Draw the circle centered at $D$ with radius $DA$.*

***Number of moves:***

- *4 circles (two for each perpendicular bisector),*
- *2 lines (the two perpendicular bisectors),*
- *1 circle (center $D$, radius $DA$).*

*Total: 7 moves.*

**Problem 26** ***Reading: the construction of the pentagon.***

(a) ***Solution:*** GeoGebra Link *Following Hartshorne Problem 4.1, we construct an isosceles triangle $ABC$ such that*

$$AB = AC \qquad \text{and} \qquad \angle B = \angle C = 2\angle A.$$

(b) ***Solution:*** ***Proposition II.11 (summary).*** GeoGebra Link

*A segment $AB$ is divided at a point $H$ such that*

$$AB \cdot BH = AH^2.$$

***Proposition III.36 (summary).*** GeoGebra Link

*From a point outside a circle, a tangent and a secant are drawn. Then*

$$(\text{tangent})^2 = (\text{whole secant})(\text{external part}).$$

(c) ***Solution:*** GeoGebra Link *Following Hartshorne Problem 4.2, we construct five equally spaced points on a circle and join consecutive points to obtain a regular pentagon.*

# Part III

# Hilbert's Axioms

## 15 Week 4 Lecture 1: Hilbert's axioms, models, and parallel properties

*An introduction to Hilbert's axiomatic approach to Euclidean geometry, the axioms of incidence, models of incidence geometry, the Euclidean/elliptic/hyperbolic parallel properties, and isomorphisms of models.*

### Hilbert's axioms

Hilbert (1899) proposed additional axioms in order to establish Euclidean geometry as a rigorous mathematical theory.

Recall Euclid's proof of Proposition I.1:

- take distinct points $A$ and $B$;
- the circles centered at $A$ and $B$ of radius $AB$ intersect at a point $C$;
- then $ABC$ is an equilateral triangle.

Why do the circles intersect? For example, in the model where the points are the integers and there is only one line, namely the whole set itself, circles may fail to intersect.

So Euclid's proof relies on several hidden assumptions. Hilbert's goal was to introduce a new set of axioms making Euclid's *Elements* fully rigorous.

### Goals for today

Today we will:

(a) re-introduce axiomatic theories and models;

(b) introduce the axioms of incidence;

(c) introduce the Euclidean, elliptic, and hyperbolic parallel properties;

(d) study various models of incidence geometry;

(e) define isomorphisms of models.

# 1. Axioms of incidence and models

To introduce the axioms of incidence, we use three primitive terms:

- points,
- lines,
- incident.

**(I1)** For any two distinct points $A$ and $B$, there exists a unique line $\ell$ incident with both $A$ and $B$.

**(I2)** For every line $\ell$, there exist at least two distinct points incident with $\ell$.

**(I3)** There exist three distinct points with the property that no line is incident with all three of them. Such points are called *non-collinear*.

## What is a model?

A model for an axiomatic theory containing the primitive words *points*, *lines*, and *incidence* can be rigorously defined as:

- a set $P$ of points,
- a set $L$ of lines,
- an incidence subset $R \subseteq P \times L$.

**Remark.** The role of the incidence subset is to tell us which points and lines are incident. In set theory, $R$ is called a *relation*.

**Notation.** If $p \in P$ and $\ell \in L$ are a point and a line, we write

$$p \, I \, \ell$$

(read "$p$ is incident with $\ell$") when

$$(p, \ell) \in R.$$

An *incidence geometry* is any model for which axioms **I1**, **I2**, and **I3** are satisfied.

**Remark.** Points, lines, and incidence can be anything.

## Example 1

Let

$$P = \{\text{straightedge}, \text{chalk}\}, \qquad L = \{\text{table}\},$$

and let

$$R = \{(\text{straightedge}, \text{table}), (\text{chalk}, \text{table})\}.$$

Equivalently, we may say:

$$\text{straightedge } I \text{ table} \qquad \text{and} \qquad \text{chalk } I \text{ table}.$$

Then **I1** and **I2** are satisfied, but **I3** is not.

## Example 2

Let

$P = \{\text{all the words in the English language}\}, \qquad L = \{\text{all the books written in English}\}.$

Define incidence by declaring that

$$p \, I \, \ell \quad \text{if the word } p \text{ appears at least once in the book } \ell.$$

Question: are **I1**, **I2**, and **I3** satisfied?

## Example 3

Let

$$P = \{\varnothing, \{1, 2\}, \star\}, \qquad L = \{\varnothing\}, \qquad R = \varnothing.$$

So there is one line and no incidences.

## Example 4

Let

$$P = \{A, B, C\}, \qquad L = \big\{\{A, B\}, \{B, C\}, \{A, C\}\big\}.$$

Define incidence by

$$p \, I \, \ell \iff p \in \ell.$$

In this example, incidence is given by the usual set-theoretic notion of belonging.
For example,

$$A \, I \, \{A, B\} \qquad \text{and} \qquad B \, I \, \{A, B\}.$$

This example may be represented by a triangle diagram: bold dots for points and strokes for lines.

Question: is $(P, L, R)$ an incidence geometry? In other words, does it satisfy **I1**, **I2**, and **I3**?

### Activity 1

Show that, assuming only axioms **I1** and **I3**, the following properties hold.

(i) There exist three distinct and non-concurrent lines. In other words, there exist distinct lines $\ell, \ell', \ell''$ such that there is no point $P$ satisfying

$$P \, I \, \ell, \qquad P \, I \, \ell', \qquad P \, I \, \ell''.$$

(ii) If $\ell$ and $\ell'$ are lines and $P \neq Q$ are distinct points such that

$$P \, I \, \ell, \qquad P \, I \, \ell', \qquad Q \, I \, \ell, \qquad Q \, I \, \ell',$$

then

$$\ell = \ell'.$$

(iii) For every line $\ell$, there exists at least one point not lying on it (that is, not incident with it).

## 2. Parallel properties

**Definition.** We say that an incidence geometry satisfies:

(i) the *Euclidean parallel property* if, for every line $\ell \in L$ and every point $P \in P$ such that $P \not I \ \ell$, there exists a unique line $\ell'$ such that

$$P \, I \, \ell' \qquad \text{and} \qquad \ell \parallel \ell';$$

(ii) the *elliptic parallel property* if there exist no parallel lines;

(iii) the *hyperbolic parallel property* if, for every line $\ell$ and every point $P$ such that $P \not I \, \ell$, there exist at least two distinct lines parallel to $\ell$ and incident with $P$.

### Activity 2

(i) In Example 4, determine whether the Euclidean, elliptic, or hyperbolic parallel property is satisfied. Or maybe none of these are satisfied.

(ii) Consider the following model:

- $P = \{\text{points on a sheet of paper}\}$,
- $L = \{\text{circles of positive radius}\}$,
- $p \, I \, \ell$ if $p \in \ell$ (standard incidence).

Is this an incidence geometry? What parallel property is satisfied?

## 3. Isomorphisms between models

### Example 5

Let

$$L = \{a, b, c\},$$

a set with three elements, and let

$$P = \big\{\{a, b\}, \{b, c\}, \{a, c\}\big\},$$

the set of all subsets of $L$ with two elements.

Define incidence by

$$p \, I \, \ell \iff \ell \in p.$$

Here lines can be subsets of points. For example,

$$\{a, b\} \, I \, a \qquad \text{and} \qquad \{a, b\} \, I \, b,$$

since $a \in \{a, b\}$ and $b \in \{a, b\}$. But

$$\{a, b\} \not I \, c$$

since $c \notin \{a, b\}$.

This example also has a triangle diagram. Examples 4 and 5 look very similar. We say that they are *isomorphic*.

**Definition.** Two models $(P, L, R)$ and $(P', L', R')$ are *isomorphic* if we can find:

- a bijective map $f : P \to P'$,
- a bijective map $g : L \to L'$,

such that for every $p \in P$ and $\ell \in L$, we have

$$(p, \ell) \in R \quad \Longleftrightarrow \quad \big(f(p), g(\ell)\big) \in R'.$$

In other words, two models are isomorphic if we can map their points and lines while preserving incidence.

## Example

Consider Examples 4 and 5 and define

$$f(A) = \{a, b\}, \qquad f(B) = \{b, c\}, \qquad f(C) = \{a, c\},$$

and

$$g(\{A, B\}) = b, \qquad g(\{B, C\}) = c, \qquad g(\{A, C\}) = a.$$

Then $f$ and $g$ are bijective and preserve incidence. Therefore Examples 4 and 5 are isomorphic.

# 16 Week 4 Section Worksheet

*A worksheet on incidence geometry and parallel properties.*

We recall the axioms of incidence geometry:

- **(I1)** Given two distinct points, there is a unique line incident with both of them.
- **(I2)** Given a line, there exist two distinct points incident with it.
- **(I3)** There exist three non-collinear points.

**Problem 27** *Assume only the incidence axioms **I1** and **I3**.*

*Show that the following statements can already be deduced:*

*(i) For every point, there exists at least one line not passing through it.*

*(ii) For every point $P$, there exist at least two distinct lines passing through $P$.*

**Problem 28** *In each of the following models, determine which of the incidence axioms **I1**, **I2**, and **I3** are satisfied and which are not. Then determine whether the model has the* elliptic, Euclidean, *or* hyperbolic *parallel property.*

*(i) "Points" are lines in Euclidean three-dimensional space, "lines" are planes in Euclidean three-dimensional space, and "incidence" is the usual relation of a line lying in a plane.*

*(ii) Same as in (i), except that we restrict ourselves to lines and planes passing through a fixed ordinary point $O$.*

*(iii) "Points" are points on the unit sphere in $\mathbb{R}^3$, "lines" are great circles (that is, intersections of the sphere with planes in $\mathbb{R}^3$ passing through $0$), and "incidence" is the standard one: a point is incident to a line if it lies on it.*

**Problem 29** *(i1) Can you find a model with only 2 points satisfying **I1** and **I2**?*

*(i2) Do you think it is possible to find two such models which are not isomorphic? Prove your claim.*

*(ii1) Can you find a model with only 3 points satisfying **I1**, **I2**, and **I3**?*

*(ii2) Do you think it is possible to find two such models which are not isomorphic? Prove your claim.*

**Problem 30** *Show that axioms **I1**, **I2**, and **I3** are independent. In other words, for each axiom, find a model satisfying the other two but not that one.*

**Problem 31** ***If time permits.***

*(i) Find a model of incidence geometry (i.e. satisfying axioms **I1**, **I2**, and **I3**) which does not satisfy any of the elliptic, Euclidean, or hyperbolic parallel properties.*

***Hint.** There is such a model with 5 points.*

*(ii) How many non-isomorphic models of incidence geometry with 4 points can you find? For each model, determine which parallel property (elliptic, Euclidean, hyperbolic), or lack of parallel property, it satisfies.*

*(iii) Same question with 5 points.*

# 17 Week 4 Lecture 2

*Affine planes.*

## Affine Planes

## 1. Activity

We recall the axioms of incidence geometry:

- **(I1)** Given 2 distinct points, there is a unique line incident with both of them.
- **(I2)** Given a line, there exist two distinct points incident with it.
- **(I3)** There exists 3 non-collinear points.

(i) Can you find two non-isomorphic models with only 2 points satisfying **(I1)** and **(I2)**?

(ii) Assume that $(\mathcal{P}, \mathcal{L}, \mathcal{R})$ is an incidence geometry with 3 points only.
Show it is isomorphic to the model described in Example 4.

We say that there is a **unique incidence geometry with three points up to isomorphisms**.

## Solutions

### (i) Example of a model satisfying I1 and I2

Let

$$\mathcal{P} = \{1, 2\}, \qquad \mathcal{L} = \{\{1, 2\}\}.$$

$$P \, I \, \ell \iff P \in \ell \quad \text{(standard incidence)}.$$

**I1** and **I2** are clearly satisfied.

**Unique (up to isomorphism)?**

Assume $(\mathcal{P}', \mathcal{L}', \mathcal{R}')$ is a model for axioms **I1** and **I2** and $\mathcal{P}'$ has only two elements.

Call $x$ and $y$ the two elements of $\mathcal{P}'$.

By **I1**, there exists $\ell \in \mathcal{L}'$ such that

$$x \, I \, \ell \quad \text{and} \quad y \, I \, \ell.$$

**Are there any other lines?**

Assume $\ell' \in \mathcal{L}'$ is a line.

By **I2**, it is incident with at least 2 points.

Since we only have 2 points, it is incident with $x$ and $y$.

But $\ell$ is the only line incident with $x$ and $y$.

Therefore $\ell' = \ell$.

**Conclusion**

$$\mathcal{P}' = \{x, y\}, \quad \mathcal{L}' = \{\ell\}, \quad \mathcal{R}' = \{(x, \ell), (y, \ell)\}.$$

This model is isomorphic to $(\mathcal{P}, \mathcal{L}, \mathcal{R})$.

Indeed define

$$f(1) = x, \quad f(2) = y, \qquad g(\{1, 2\}) = \ell.$$

## (ii) The case of three points

Name our three points $x, y, z$.

We need to determine $\mathcal{L}$ and $\mathcal{R}$, i.e. the lines and incidences.

By **I1**, there exists a unique line

$$\ell_{xy} \text{ incident with } x \text{ and } y,$$

$$\ell_{yz} \text{ incident with } y \text{ and } z,$$

$$\ell_{xz} \text{ incident with } x \text{ and } z.$$

If two of these three lines were equal, then $x, y, z$ are collinear, which contradicts **I3**.

For the same reason, none of these lines is incident to 3 points.

Therefore $\ell_{xy}, \ell_{yz}, \ell_{xz}$ are distinct and only incident to 2 points each.

**Are these all the lines?**

Assume $\ell \in \mathcal{L}$.

Then $\ell$ is incident with 2 points at least by **I2** and cannot be incident with 3.

So $\ell \in \{\ell_{xy}, \ell_{yz}, \ell_{xz}\}$.

**Conclusion**

$$\mathcal{L} = \{\ell_{xy}, \ell_{yz}, \ell_{xz}\}.$$

Now defining

$$f(x) = A, \quad f(y) = B, \quad f(z) = C,$$
$$g(\ell_{xy}) = \{A, B\}, \quad g(\ell_{yz}) = \{B, C\}, \quad g(\ell_{xz}) = \{A, C\},$$

gives an isomorphism. (Note that there are others!)

## 2. Affine planes

**Definition 1.** An affine plane is an incidence geometry $(\mathcal{P}, \mathcal{L}, \mathcal{R})$ satisfying the Euclidean parallel property.

**Example questions**

(a) Any affine planes with only 3 points?

(b) An affine plane with 4 points?

Example:

- 4 points
- 6 lines
- each line incident to only 2 points

## Affine planes over a field

**Definition**

Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{Q}$ (or $\mathbb{C}$, or any field).

Points are elements of $\mathbb{F}^2$, i.e. ordered pairs $(x, y)$ where $x, y \in \mathbb{F}$.

Lines are subsets of $\mathbb{F}^2$ of the form

$$\ell = \{(x, y) \in \mathbb{F}^2 : ax + by + c = 0\}$$

for some $(a, b, c) \in \mathbb{F}^3$ with $(a, b) \neq (0, 0)$.

$$P \, I \, \ell \iff P \in \ell.$$

**Proposition 1.** $(\mathbb{F}^2, \mathcal{L}, \mathcal{R})$ is an affine plane.

***Proof*** **(I1)** Assume $(x_1, y_1) \neq (x_2, y_2)$.

Finding a line

$$\ell = \{ax + by + c = 0\}$$

such that $P_1, P_2 \in \ell$ amounts to solving:

$$\begin{cases} ax_1 + by_1 + c = 0 \\ ax_2 + by_2 + c = 0 \end{cases}$$

This system can be rewritten in matrix form as

$$\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The matrix

$$M = \begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{pmatrix}$$

has rank 2 since $(x_1, y_1) \neq (x_2, y_2)$.

Therefore the homogeneous system

$$M \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

has a nontrivial solution $(a, b, c) \neq (0, 0, 0)$.

We claim that necessarily $(a, b) \neq (0, 0)$. Indeed, if $a = b = 0$, then the equations reduce to

$$c = 0,$$

which contradicts $(a, b, c) \neq (0, 0, 0)$.

Thus we obtain a line

$$\ell = \{ax + by + c = 0\}$$

passing through $P_1$ and $P_2$.

Moreover, since $\operatorname{rank}(M) = 2$, the space of solutions is 1-dimensional.

Thus any two solutions $(a, b, c)$ and $(a', b', c') \neq (0, 0, 0)$ are proportional:

$$(a', b', c') = \lambda(a, b, c)$$

for some $\lambda \neq 0$.

But proportional coefficients define the same line, so all solutions correspond to the same line.

Therefore there exists a unique line incident with $P_1$ and $P_2$.

**(I2)** Next HW or section.

**(I3)** Homework 4.

**Euclidean parallel property**

Let

$$\ell = \{ax + by + c = 0\}, \quad P = (x_0, y_0) \notin \ell.$$

Consider another line

$$m = \{dx + ey + f = 0\}.$$

We analyze the system:

$$\begin{cases} ax + by + c = 0 \\ dx + ey + f = 0 \end{cases}$$

Number of solutions:

- 0 solutions $\Rightarrow \ell \parallel m$
- 1 solution $\Rightarrow \ell \nparallel m$
- more than 1 solution $\Rightarrow \ell = m$

Compute the determinant:

$$\Delta = \begin{vmatrix} a & b \\ d & e \end{vmatrix} = ae - bd.$$

If $\Delta = 0$, then $(a, b)$ and $(d, e)$ are proportional.

Thus

$$m = \{ax + by + d = 0\}$$

for some $d$.

Hence lines parallel to $\ell$ take the form:

$$\{ax + by + d = 0\}, \quad d \neq c,$$

and one can show with similar arguments that such a line is indeed parallel to $\ell$.

Now we want a line $m \parallel \ell$ passing through $P$.

Solve:

$$ax_0 + by_0 + d = 0.$$

Thus

$$d = -ax_0 - by_0.$$

Since $P \notin \ell$, we have $d \neq c$.

Thus $m \parallel \ell$ and it is unique.

■

## Remark (for algebra students)

For every $n \geq 1$ and prime $p$, there exists a field $\mathbb{F}_{p^n}$.

Thus there exists an affine plane with exactly

$$(p^n)^2 = p^{2n}$$

elements.

# 18 Homework 4 Solutions

*Solutions to Homework 4 for MATH 128A.*

**Problem 32** ***Independence of axiom I1.***

*Give an example of a model*

$$(\mathcal{P}, \mathcal{L}, \mathcal{R})$$

*satisfying axioms* ***I2*** *and* ***I3****, but not axiom* ***I1****.*

***Solution:*** *Let*

$$\mathcal{P} = \{A, B, C, D\}$$

*and let*

$$\mathcal{L} = \{\ell, m\}.$$

*Define the incidence relation by*

$$R(\ell) = \{A, B\}, \qquad R(m) = \{C, D\}.$$

*Axiom* ***I2*** *is satisfied because each line contains at least two distinct points.*

*Axiom* ***I3*** *is also satisfied. For example, the three points $A, C, D$ are not all incident with a common line. Indeed, $C$ and $D$ lie on $m$, but $A$ does not lie on $m$. The only other line is $\ell$, and $C, D$ do not lie on $\ell$.*

*However, axiom* ***I1*** *is not satisfied. For example, the two distinct points $A$ and $C$ do not have any line incident with both of them. Therefore this model satisfies* ***I2*** *and* ***I3****, but not* ***I1****.*

**Problem 33** ***Basic properties of affine planes.***

*Consider an affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{R})$, that is, assume axioms* ***I1****,* ***I2****, and* ***I3*** *are satisfied, and assume also the Euclidean parallel property:*

*Given a line $\ell \in \mathcal{L}$ and a point $p \in \mathcal{P}$ such that $p$ is not incident with $\ell$, there exists a unique line $\ell' \in \mathcal{L}$ such that $\ell \parallel \ell'$ and $p \, I \, \ell'$.*

*Show that if two lines $\ell$ and $m$ are distinct and not parallel, then there exists a unique point $p \in \mathcal{P}$ such that $p \, I \, \ell$ and $p \, I \, m$.*

***Solution:*** *Since $\ell$ and $m$ are not parallel, by definition they meet. Therefore there exists at least one point $p \in \mathcal{P}$ such that*

$$p \, I \, \ell \qquad \text{and} \qquad p \, I \, m.$$

*We now prove uniqueness. Suppose that $p$ and $q$ are two points incident with both $\ell$ and $m$. We want to show that $p = q$.*

*Assume, for contradiction, that $p \neq q$. Then by axiom **I1**, there exists a unique line incident with both $p$ and $q$. But both $\ell$ and $m$ are incident with $p$ and $q$. Therefore, by uniqueness in axiom **I1**, we must have*

$$\ell = m.$$

*This contradicts the assumption that $\ell$ and $m$ are distinct.*

*Hence $p = q$. Therefore $\ell$ and $m$ meet in a unique point.*

**Problem 34** ***The standard affine plane over a field.***

*Let $\mathbb{F} = \mathbb{R}$ (or $\mathbb{Q}$, or $\mathbb{C}$, or any field of your choice). Consider the affine plane $\mathbb{F}^2$ defined in Week 4 Lecture 2. Prove that this affine plane satisfies axiom **I3**.*

**Solution:** In the standard affine plane $\mathbb{F}^2$, the points are the elements of $\mathbb{F}^2$, and the lines are the subsets of the form

$$\{(x, y) \in \mathbb{F}^2 : ax + by + c = 0\},$$

where $(a, b, c) \in \mathbb{F}^3$ and

$$(a, b) \neq (0, 0).$$

Axiom **I3** says that there exist three non-collinear points. We claim that the three points

$$P = (0, 0), \qquad Q = (1, 0), \qquad R = (0, 1)$$

are not collinear.

Suppose, for contradiction, that $P, Q, R$ are collinear. Then there exists a line

$$ax + by + c = 0,$$

with $(a, b, c) \in \mathbb{F}^3$ and $(a, b) \neq (0, 0)$, containing all three points.

Since $P = (0, 0)$ lies on the line, we have

$$a \cdot 0 + b \cdot 0 + c = 0,$$

so

$$c = 0.$$

Since $Q = (1, 0)$ lies on the line, we have

$$a \cdot 1 + b \cdot 0 + c = 0,$$

so

$$a + c = 0.$$

Since $c = 0$, this gives

$$a = 0.$$

Since $R = (0, 1)$ lies on the line, we have

$$a \cdot 0 + b \cdot 1 + c = 0,$$

so

$$b + c = 0.$$

Since $c = 0$, this gives

$$b = 0.$$

Therefore

$$a = b = c = 0.$$

In particular,

$$(a, b) = (0, 0),$$

which contradicts the requirement that $(a, b) \neq (0, 0)$.

Hence $P, Q, R$ are not collinear. Therefore the standard affine plane $\mathbb{F}^2$ satisfies axiom **I3**.

**Problem 35** ***Non-isomorphic models with three points.***

*Find two non-isomorphic models with exactly three points satisfying axioms **I1** and **I2**. You may give a picture to define your models, but you also need to prove that your two models are not isomorphic.*

***Solution:*** *Let*

$$\mathcal{P} = \{A, B, C\}.$$

*We define two models.*

***Model 1.*** *Let*

$$\mathcal{L}_1 = \{\ell\}$$

*with*

$$R(\ell) = \{A, B, C\}.$$

*Thus all three points lie on a single line.*

*This model satisfies **I1**: any two distinct points lie on the unique line $\ell$. It also satisfies **I2**: the line $\ell$ contains at least two distinct points.*

***Model 2.*** *Let*

$$\mathcal{L}_2 = \{\ell_{AB}, \ell_{AC}, \ell_{BC}\},$$

*where*

$$R(\ell_{AB}) = \{A, B\}, \qquad R(\ell_{AC}) = \{A, C\}, \qquad R(\ell_{BC}) = \{B, C\}.$$

*This model satisfies **I1**: any two distinct points lie on exactly one of the three lines. It also satisfies **I2**: each line contains exactly two distinct points. We now prove that the two models are not isomorphic.*

*Suppose, for contradiction, that the two models are isomorphic. Then there exist two bijections*

$$f : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$$

*and*

$$g : \mathcal{L}_1 \longrightarrow \mathcal{L}_2$$

*such that incidence is preserved. In other words, for every point $P \in \mathcal{P}_1$ and every line $r \in \mathcal{L}_1$,*

$$P \, I_1 \, r \qquad \Longleftrightarrow \qquad f(P) \, I_2 \, g(r).$$

*In particular, an isomorphism includes a bijection*

$$g : \mathcal{L}_1 \longrightarrow \mathcal{L}_2$$

*between the set of lines of the first model and the set of lines of the second model. Therefore the two models must have the same number of lines.*

*However, Model 1 has exactly one line:*

$$\mathcal{L}_1 = \{\ell\},$$

*whereas Model 2 has exactly three lines:*

$$\mathcal{L}_2 = \{\ell_{AB}, \ell_{AC}, \ell_{BC}\}.$$

*Thus*

$$\#\mathcal{L}_1 = 1 \qquad \text{and} \qquad \#\mathcal{L}_2 = 3.$$

*This is impossible if there is a bijection*

$$g : \mathcal{L}_1 \longrightarrow \mathcal{L}_2.$$

*Therefore the two models are not isomorphic.*

**Problem 36** ***Isomorphisms and parallel properties.***

*Let $\mathcal{M} = (\mathcal{P}, \mathcal{L}, \mathcal{R})$ and $\mathcal{M}' = (\mathcal{P}', \mathcal{L}', \mathcal{R}')$ be two isomorphic models of incidence geometry. Show that if $\mathcal{M}$ satisfies the elliptic parallel property, then $\mathcal{M}'$ also satisfies the elliptic parallel property.*

***Solution:*** *Let*

$$\varphi : \mathcal{P} \longrightarrow \mathcal{P}', \qquad \Phi : \mathcal{L} \longrightarrow \mathcal{L}'$$

*be an isomorphism of incidence geometries. Thus $\varphi$ and $\Phi$ are bijections, and incidence is preserved:*

$$P \, I \, \ell \qquad \Longleftrightarrow \qquad \varphi(P) \, I' \, \Phi(\ell).$$

*Assume that $\mathcal{M}$ satisfies the elliptic parallel property. This means that any two lines in $\mathcal{M}$ meet.*

*We want to prove that $\mathcal{M}'$ also satisfies the elliptic parallel property. Let $\ell', m' \in \mathcal{L}'$. Since $\Phi$ is a bijection, there exist lines $\ell, m \in \mathcal{L}$ such that*

$$\Phi(\ell) = \ell', \qquad \Phi(m) = m'.$$

*Since $\mathcal{M}$ satisfies the elliptic parallel property, the lines $\ell$ and $m$ meet. Therefore there exists a point $P \in \mathcal{P}$ such that*

$$P \, I \, \ell \qquad \text{and} \qquad P \, I \, m.$$

*By preservation of incidence,*

$$\varphi(P) \, I' \, \Phi(\ell) = \ell' \qquad \text{and} \qquad \varphi(P) \, I' \, \Phi(m) = m'.$$

*Thus $\ell'$ and $m'$ meet.*

*Since $\ell'$ and $m'$ were arbitrary, every pair of lines in $\mathcal{M}'$ meets. Therefore $\mathcal{M}'$ satisfies the elliptic parallel property.*

**Problem 37** ***Reflection.***

*Reflect on the first four weeks of lecture.*

(a) *What have you enjoyed the most so far?*

(b) *Is there something you wish we did more of?*

(c) *How will you prepare for the midterm? If you are completing this homework after the midterm, how did you prepare?*

(d) *Are there any points you find, or found, difficult?*

(e) *What is your plan to improve on these points?*

*Please answer in a few complete sentences. Your responses will be anonymous.*

**Problem 38** ***Challenge problems.***

*(a)* ***(Classifying three-point models)*** *Show that any model with exactly three points satisfying axioms **I1** and **I2** is necessarily isomorphic to one of the two models you found previously in problem 4.*

*(b)* ***(Quadrangles in affine planes)*** *Show that any affine plane contains a quadrangle, that is, a set of four points no three of which are collinear.*

(c) ***(Finite affine planes †)*** *Consider an affine plane, and let $\ell$ and $m$ be two distinct lines. Show that the set of points incident with $\ell$ is in bijection with the set of points incident with $m$. Deduce that if one of the lines has $n$ points, where $n \geq 2$ is an integer, then the affine plane has $n^2$ points. How many lines does the affine plane have? Are there any affine planes with exactly 5 points?*

(d) ***(A minimality question †)*** *Suppose that in a model of incidence geometry every line has at least three distinct points incident with it.*

(i) *What is the least number of points and the least number of lines such a model can have?*

(ii) *Suppose further that the model has the Euclidean parallel property, that is, suppose it is an affine plane. Show that 9 is now the least possible number of points, and 12 is the least possible number of lines.*

# 19 Week 5 Lecture 1

*Affine planes and the number of points on lines.*

## Affine Planes

In the previous lecture, we defined affine planes as incidence geometries satisfying the Euclidean parallel property.

**Definition 2.** An **affine plane** is an incidence geometry

$$\alpha = (\mathcal{P}, \mathcal{L}, \mathcal{R})$$

satisfying the Euclidean parallel property.

Recall that, if $P \in \mathcal{P}$ and $\ell \in \mathcal{L}$, we write

$$P \, I \, \ell$$

to mean that the point $P$ is incident with the line $\ell$.

## 1. Activity

(1) Show that in an affine plane, the following property holds:

For every pair of distinct and non-parallel lines $\ell$ and $m$, there exists a unique point $P$ incident with both lines.

In other words, if $\ell \neq m$ and $\ell$ is not parallel to $m$, then there exists a unique point $P \in \mathcal{P}$ such that

$$P \, I \, \ell \qquad \text{and} \qquad P \, I \, m.$$

(2) Let

$$\alpha = (\mathcal{P}, \mathcal{L}, \mathcal{R})$$

be an affine plane. Given a line $\ell \in \mathcal{L}$, we denote

$$R(\ell) := \{P \in \mathcal{P} : P \, I \, \ell\}.$$

The goal is to show that, given any two lines $\ell$ and $m$, there exists a bijection

$$f : R(\ell) \longrightarrow R(m).$$

## 2. Intersections of non-parallel lines

We first prove the property from Activity 1.

**Proposition 2.** Let $\alpha = (\mathcal{P}, \mathcal{L}, \mathcal{R})$ be an affine plane. Let $\ell, m \in \mathcal{L}$ be two distinct lines.

If $\ell$ and $m$ are not parallel, then there exists a unique point $P \in \mathcal{P}$ such that

$$P \, I \, \ell \qquad \text{and} \qquad P \, I \, m.$$

***Proof*** Since $\ell$ and $m$ are not parallel, by definition there exists at least one point $P \in \mathcal{P}$ such that

$$P \, I \, \ell \qquad \text{and} \qquad P \, I \, m.$$

We now prove uniqueness. Suppose that $P$ and $Q$ are two points incident with both $\ell$ and $m$. If $P \neq Q$, then by axiom **I1**, there is a unique line incident with both $P$ and $Q$.

But both $\ell$ and $m$ are incident with $P$ and $Q$. Therefore $\ell = m$, contradicting the assumption that $\ell \neq m$.

Hence $P = Q$. Therefore the point of intersection is unique. ■

## 3. All lines have the same number of points

Let

$$\alpha = (\mathcal{P}, \mathcal{L}, \mathcal{R})$$

be an affine plane. For a line $\ell$, recall that

$$R(\ell) = \{P \in \mathcal{P} : P \, I \, \ell\}.$$

**Proposition 3.** Given any two lines $\ell, m \in \mathcal{L}$, there exists a bijection

$$f : R(\ell) \longrightarrow R(m).$$

In particular, all lines in an affine plane have the same number of points.

***Proof*** If $\ell = m$, then the identity map

$$\mathrm{id} : R(\ell) \longrightarrow R(m)$$

is a bijection. So assume from now on that

$$\ell \neq m.$$

**Step 1: Choose two special points.**

We first show that there exist two distinct points $a, b \in \mathcal{P}$ such that

$$a \, I \, \ell, \qquad a \not I m,$$

and

$$b \not I \ell, \qquad b \, I \, m.$$

By axiom **I2**, there exist two distinct points on $\ell$. If both of these points were also on $m$, then by axiom **I1** the lines $\ell$ and $m$ would be equal, contradicting the assumption that $\ell \neq m$. Therefore at least one of these two points lies on $\ell$ but not on $m$. Call this point $a$.

Similarly, by axiom **I2**, there exist two distinct points on $m$. If both of these points were also on $\ell$, then by axiom **I1** the lines $m$ and $\ell$ would be equal, again contradicting $\ell \neq m$. Therefore at least one of these two points lies on $m$ but not on $\ell$. Call this point $b$.

In particular, $a \neq b$.

**Step 2: Define an auxiliary line.**

Let $n$ be the unique line incident with both $a$ and $b$. This line exists and is unique by axiom **I1**.

We claim that

$$n \neq \ell \qquad \text{and} \qquad n \neq m.$$

Indeed, if $n = \ell$, then $b \, I \, \ell$, contradicting the choice of $b$. Similarly, if $n = m$, then $a \, I \, m$, contradicting the choice of $a$.

**Step 3: Intersections of $n$ with $\ell$ and $m$.**

Since $n$ is incident with $a$, and $a \, I \, \ell$, the lines $n$ and $\ell$ meet at $a$. Since $n \neq \ell$, they meet in exactly one point by the first problem. Hence

$$R(n) \cap R(\ell) = \{a\}.$$

Similarly,

$$R(n) \cap R(m) = \{b\}.$$

**Step 4: Construct the map.**

Let $x \in R(\ell)$.

If $x = a$, we will define

$$f(a) = b.$$

Now assume that

$$x \in R(\ell) \qquad \text{and} \qquad x \neq a.$$

Since $x \in R(\ell)$, $x \neq a$, and

$$R(n) \cap R(\ell) = \{a\},$$

we have

$$x \not I n.$$

Therefore, by the Euclidean parallel property, there exists a unique line $n_x$ such that

$$x \, I \, n_x \qquad \text{and} \qquad n_x \parallel n.$$

We now show that $m$ and $n_x$ meet in a unique point. First, note that

$$n_x \neq m.$$

Indeed, if $n_x = m$, then $m \parallel n$, because $n_x \parallel n$. But this is impossible, since $m$ and $n$ meet at the point $b$.

Next, we show that $m$ is not parallel to $n_x$. Suppose, for contradiction, that

$$m \parallel n_x.$$

Since $n_x \parallel n$, both $m$ and $n$ would be lines through the point $b$ parallel to $n_x$. Also, $m \neq n$, by step 2. This contradicts the uniqueness part of the Euclidean parallel property.

Therefore $m$ and $n_x$ are distinct and not parallel. By the previous proposition, they meet in a unique point. Let $y \in \mathcal{P}$ be this unique point:

$$y \, I \, m \qquad \text{and} \qquad y \, I \, n_x.$$

We define

$$f(x) = y.$$

Thus

$$f : R(\ell) \longrightarrow R(m)$$

is given by

$$f(x) = \begin{cases} b, & \text{if } x = a, \\ \text{the unique point in } R(m) \cap R(n_x), & \text{if } x \neq a. \end{cases}$$

**Step 5: The constructed map $f : R(\ell) \longrightarrow R(m)$ is bijective.**

Next HW. ■

**Corollary 1.** In an affine plane, if one line is incident with exactly $n$ points (where $n \geq 2$ is an integer), then all lines are incident with exactly $n$ points.

# 20 Mock Midterm Exam

*A worksheet to help you prepare for the midterm.*

**Problem 39** *For each question, circle the unique correct answer. No justification is required.*

(i) *Which of the following statements is true?*

- (a) *If two lines are parallel, then they intersect at exactly one point.*
- (b) *If two lines are both perpendicular to a third line, then they are parallel.*
- (c) *If two lines intersect, then they are parallel.*
- (d) *If two lines are not parallel, then they are perpendicular.*

(ii) *In the six parts of a Euclidean proposition, which part usually states what is given?*

- (a) *Exposition.*
- (b) *Specification.*
- (c) *Construction.*
- (d) *Conclusion.*

(iii) *Which of the following assertions is* ***false****?*

- (a) *Vertical angles are equal.*
- (b) *The perpendicular bisector of a segment passes through its midpoint.*
- (c) *If two lines are parallel, then alternate interior angles are equal.*
- (d) *If two segments have the same length, then they lie on the same line.*

**Problem 40** *This exercise contains four "Euclid-style" proof questions.*

(i) *Let $ABCD$ be a quadrilateral such that*

$$AB \parallel CD \qquad \text{and} \qquad AB = CD.$$

*Prove that*

$$AD \parallel BC \qquad \text{and} \qquad AD = BC.$$

(ii) *Let $ABCD$ be a quadrilateral such that*

$$AB = AD \qquad \text{and} \qquad CB = CD.$$

*Assume that the diagonals $AC$ and $BD$ meet at a point $E$. Prove that*

$$AC \perp BD.$$

(iii) *Let $AB$ be a diameter of a circle, and let $C$ be another point on the circle. Prove that*

$$\angle ACB$$

*is a right angle.*

(iv) *Let $ABCD$ be a cyclic quadrilateral (i.e. four points on a circle) such that*

$$AB = CD.$$

*Prove that*

$$AD \parallel BC.$$

***Hint.*** *Draw the diagonals $AC$ and $BD$, and let $E$ be their point of intersection. First show that the triangles $\triangle CDE$ and $\triangle ABE$ are congruent. Then show that*

$$\angle DAE = \angle BCE,$$

*and conclude.*

**Problem 41** *Using only straightedge and compass, perform the following constructions and state how many moves you used. No justification is required.*

(i) *Construct the midpoint of a given segment $AB$.*

(ii) *Construct a square on a given segment $AB$.*

(iii) *Construct the perpendicular to a given line $l$ passing through a given point $P$ not on the line.*

(iv) *Construct a circle passing through three given non-collinear points $A$, $B$, and $C$.*

(v) *Construct a line tangent to a given circle and passing through a given point $A$ outside the circle.*

**Problem 42** *Assume the incidence axioms **I1**, **I2**, and **I3**. Prove the following.*

(i) *Two distinct lines intersect in at most one point.*

(ii) *If $P$ is a point and $l$ is a line not passing through $P$, then there exist at least two distinct lines through $P$ which meet $l$.*

(iii) *There exist at least three distinct lines.*

# 21 Homework 5

*Homework 5 for MATH 128A.*

**MATH 128A: Homework 5** **Due date: Thursday, May 7**

**Instructions.** You will submit your solutions on Gradescope, which you can access through our Canvas course page by clicking on the assignment.

You are welcome to submit your homework either handwritten or in LaTeX. If you choose to submit handwritten work, please use pen rather than pencil, and make sure that both your handwriting and your scans are as readable as possible in order to facilitate grading.

I encourage you to attend office hours if you would like hints on the homework or if you would like to discuss more advanced material. Also, you are highly encouraged to work in groups!

**Problem 43** ***(20 points) A transitivity property for parallel lines.***

*Let $\alpha = (\mathcal{P}, \mathcal{L}, \mathcal{R})$ be an* ***affine plane*** *and recall that two lines are called parallel if they do not meet.*

*Let $\ell, m, n \in \mathcal{L}$ be three lines and assume that $\ell \parallel m$ and $m \parallel n$. Show that*

$$\ell = n \qquad \text{or} \qquad \ell \parallel n.$$

*In other words, show that if two lines are both parallel to the same line $m$, then they are either equal or parallel to each other.*

***Hint.*** *Use the Euclidean parallel property.*

**Problem 44** ***(80 points) Affine planes and bijections between points on lines.***

*Let $\alpha = (\mathcal{P}, \mathcal{L}, \mathcal{R})$ be an* ***affine plane*** *and recall that, for a line $\ell \in \mathcal{L}$, we defined*

$$R(\ell) = \{P \in \mathcal{P} : P \,\mathrm{I}\, \ell\}$$

*as the set of points incident with $\ell$. In class, we started proving that if $\ell$ and $m$ are any two lines in an affine plane, then there exists a bijection*

$$f : R(\ell) \longrightarrow R(m).$$

*The goal of this problem is to complete the proof. In what follows, we assume that $\ell$ and $m$ are* ***distinct lines****.*

***Recap from W5L1:*** *In class, we chose two points $a \in \mathcal{P}$ and $b \in \mathcal{P}$ such that*

$$a \in R(\ell), \quad a \notin R(m), \quad b \in R(m), \quad b \notin R(\ell).$$

*Then, we introduced $n$, the unique line through $a$ and $b$ and showed that*

$$R(n) \cap R(\ell) = \{a\} \qquad \text{and} \qquad R(n) \cap R(m) = \{b\}.$$

*We defined a map*

$$f : R(\ell) \longrightarrow R(m)$$

*as follows.*

- *First, define*
$$f(a) = b.$$
- *Now let $x \in R(\ell)$ with $x \neq a$. By the Euclidean parallel property, there exists a unique line $n_x$ such that*
$$x \,\mathrm{I}\, n_x \qquad \text{and} \qquad n_x \parallel n.$$
*The line $n_x$ is distinct and not parallel to $m$ (see the notes of W5L1 for more details), so $n_x$ and $m$ meet in a unique point. We define $f(x)$ to be this unique point:*
$$R(n_x) \cap R(m) = \{f(x)\}.$$

*Thus*

$$f(x) = \begin{cases} b, & \text{if } x = a, \\ \text{the unique point in } R(n_x) \cap R(m), & \text{if } x \neq a. \end{cases}$$

***Important:*** *In the problem, you may use anything from the recap and anything we proved during W5L1.*

(1) ***Injectivity.*** *Let $x_1, x_2 \in R(\ell)$, and assume*

$$\boxed{f(x_1) = f(x_2)}$$

*We want to show that $x_1 = x_2$.*

(a) *(10 points) Assume that $x_1 = a$. Show that $x_2 = a$.*
*Hint: Use the fact that $f(a) = b$, and that $R(n) \cap R(m) = \{b\}$.*

(b) *(10 points) Now assume that $x_1 \neq a$. Show that $x_2 \neq a$ and conclude that $x_1 = x_2$.*

(2) ***Surjectivity.***

*Let $y \in R(m)$. We want to show that there exists $x \in R(\ell)$ such that $f(x) = y$.*

(a) *(10 points) First assume that $y = b$. Find $x \in R(\ell)$ such that $f(x) = y$.*

***From now on, we assume that $y \neq b$.***

(b) *(10 points) Explain why* $y \notin R(n)$.

(c) *(10 points) By the Euclidean parallel property, there exists a unique line* $n_y$ *such that*

$$y \,\mathrm{I}\, n_y \qquad \text{and} \qquad n_y \parallel n.$$

*Explain why* $n_y$ *is distinct and not parallel to* $\ell$.

*Hint: You may proceed by contradiction and use Problem 1.*

(d) *(10 points) Since* $n_y$ *is distinct and not parallel to* $\ell$*, the lines* $n_y$ *and* $\ell$ *meet in a unique point. Let* $x$ *be this point:*

$$R(n_y) \cap R(\ell) = \{x\}.$$

*Show that* $x \neq a$.

(e) *(10 points) Show that*

$$n_y = n_x.$$

*Hint: Use the Euclidean parallel property.*

(f) *(10 points) Conclude that*

$$f(x) = y.$$

***Conclusion.*** *Since* $f$ *is both injective and surjective,* $f$ *is a bijection. Therefore* $R(\ell)$ *and* $R(m)$ *have the same number of points.*

**Problem 45** ***Reflection on the midterm exam (2 bonus points).***

- *How did your midterm exam go? (Did you lose time on certain questions? Did you find one question particularly challenging?)*
- *If you could go back, is there something you would have done differently during the exam or while preparing for the exam?*
- *What did you learn from this experience, and how will you use it to prepare more effectively for the final exam?*

*Please answer in a few complete sentences. Your responses will be anonymous.*

***Comment.*** *Since there was only one lecture last week, this week's homework is a little shorter. You can use this as an opportunity to catch up with the material and submit regrade requests for past homework assignments.*

# 22 Homework 5 Solutions

*Solutions to Homework 5 for MATH 128A.*

**Problem 46** ***A transitivity property for parallel lines.***

*Let $\alpha = (\mathcal{P}, \mathcal{L}, \mathcal{R})$ be an* ***affine plane*** *and recall that two lines are called parallel if they do not meet.*

*Let $\ell, m, n \in \mathcal{L}$ be three lines and assume that $\ell \parallel m$ and $m \parallel n$. Show that*

$$\ell = n \qquad \text{or} \qquad \ell \parallel n.$$

*In other words, show that if two lines are both parallel to the same line $m$, then they are either equal or parallel to each other.*

***Solution:*** *Assume that $\ell \parallel m$ and $m \parallel n$. We want to show that $\ell = n$ or $\ell \parallel n$.*

*We distinguish two cases. First, if $\ell \parallel n$, then we are done.*

*Now assume that $\ell$ is not parallel to $n$. Then $\ell$ and $n$ meet, so there exists a point $x \in \mathcal{P}$ such that $x \,\mathrm{I}\, \ell$ and $x \,\mathrm{I}\, n$. Since $\ell \parallel m$, we have $x \not\mathrm{I}\, m$. Thus $x$ is a point not incident with $m$. Moreover, both $\ell$ and $n$ are lines through $x$ parallel to $m$. By the Euclidean parallel property, we conclude $\ell = n$.*

*Hence, in all cases, $\ell = n$ or $\ell \parallel n$.*

**Problem 47** ***Affine planes and bijections between points on lines.***

*Let $\alpha = (\mathcal{P}, \mathcal{L}, \mathcal{R})$ be an* ***affine plane*** *and recall that, for a line $\ell \in \mathcal{L}$, we defined*

$$R(\ell) = \{P \in \mathcal{P} : P \,\mathrm{I}\, \ell\}$$

*as the set of points incident with $\ell$. In class, we started proving that if $\ell$ and $m$ are any two lines in an affine plane, then there exists a bijection*

$$f : R(\ell) \longrightarrow R(m).$$

*The goal of this problem is to complete the proof. In what follows, we assume that $\ell$ and $m$ are* ***distinct lines****.*

***Solution:*** *We use the notation and construction from the problem statement. Recall that $a \in R(\ell)$, $a \notin R(m)$, $b \in R(m)$, $b \notin R(\ell)$, and $n$ is the unique line through $a$ and $b$. We also know that $R(n) \cap R(\ell) = \{a\}$ and $R(n) \cap R(m) = \{b\}$.*

*The map $f : R(\ell) \longrightarrow R(m)$ is defined by $f(a) = b$, and, if $x \in R(\ell)$ with $x \neq a$, then $n_x$ is the unique line through $x$ parallel to $n$, and $f(x)$ is the unique point in $R(n_x) \cap R(m)$. We prove that $f$ is injective and surjective.*

***Injectivity.*** *Let $x_1, x_2 \in R(\ell)$, and assume that $f(x_1) = f(x_2)$. We want to prove that $x_1 = x_2$.*

*First assume that $x_1 = a$. Then $f(x_1) = f(a) = b$. Since $f(x_1) = f(x_2)$, it follows that $f(x_2) = b$. We claim that $x_2 = a$.*

*Suppose, for contradiction, that $x_2 \neq a$. Then $n_{x_2}$ is defined, and $f(x_2)$ is the unique point in $R(n_{x_2}) \cap R(m)$. Since $f(x_2) = b$, we have $b \,\mathrm{I}\, n_{x_2}$. But $b \,\mathrm{I}\, n$, and $n_{x_2} \parallel n$. Therefore $n_{x_2}$ and $n$ meet at $b$, contradicting the fact that $n_{x_2} \parallel n$. Hence $x_2 = a$. Therefore, if $x_1 = a$, then $x_2 = a$, and so $x_1 = x_2$.*

*Now assume that $x_1 \neq a$. We first show that $x_2 \neq a$. Indeed, if $x_2 = a$, then by the previous case, applied with the roles of $x_1$ and $x_2$ reversed, we would have $x_1 = a$. This contradicts our assumption that $x_1 \neq a$. Therefore $x_2 \neq a$.*

*Since both $x_1 \neq a$ and $x_2 \neq a$, the lines $n_{x_1}$ and $n_{x_2}$ are defined. Let $y = f(x_1) = f(x_2)$. Then $y \,\mathrm{I}\, n_{x_1}$ and $y \,\mathrm{I}\, n_{x_2}$. Also, $n_{x_1} \parallel n$ and $n_{x_2} \parallel n$. By the first problem, we have $n_{x_1} = n_{x_2}$.*

*Observe that $n_{x_1}$ is distinct from $\ell$ (since $a \not\mathrm{I}\, n_{x_1}$) and not parallel to $\ell$ (since $x_1 \,\mathrm{I}\, \ell$). Thus $n_{x_1}$ intersects $\ell$ in a unique point $x_1$. For the same reason, $n_{x_2}$ intersects $\ell$ in a unique point $x_2$. Since $n_{x_1} = n_{x_2}$, we conclude that $x_1 = x_2$.*

***Surjectivity.*** *Let $y \in R(m)$. We want to show that there exists $x \in R(\ell)$ such that $f(x) = y$.*

*First assume that $y = b$. Then we can take $x = a$, since $f(a) = b = y$.*

*From now on, assume that $y \neq b$. Since $y \in R(m)$, and since $R(n) \cap R(m) = \{b\}$, we have $y \not\mathrm{I}\, n$. By the Euclidean parallel property, there exists a unique line $n_y$ such that $y \,\mathrm{I}\, n_y$ and $n_y \parallel n$.*

*We claim that $n_y$ is distinct and not parallel to $\ell$. First, $n_y \neq \ell$. Indeed, if $n_y = \ell$, then $n_y$ and $n$ would meet at $a$, because $a \,\mathrm{I}\, \ell$ and $a \,\mathrm{I}\, n$. This contradicts $n_y \parallel n$. Therefore $n_y \neq \ell$.*

*Next, suppose for contradiction that $n_y \parallel \ell$. Since $n_y \parallel n$, Problem 1 implies that $\ell = n$ or $\ell \parallel n$. But $\ell \neq n$ (since $b \not\mathrm{I}\, \ell$), and $\ell$ meets $n$ at $a$. Thus $\ell$ is not parallel to $n$. This is a contradiction. Hence $n_y$ is not parallel to $\ell$.*

*Since $n_y$ is distinct and not parallel to $\ell$, the lines $n_y$ and $\ell$ meet in a unique point. Let $x$ be this point, so $R(n_y) \cap R(\ell) = \{x\}$.*

*We claim that $x \neq a$. Suppose, for contradiction, that $x = a$. Then $a \,\mathrm{I}\, n_y$. But $a \,\mathrm{I}\, n$, and $n_y \parallel n$. This contradicts the fact that parallel lines do not meet. Therefore $x \neq a$.*

*Since $x \in R(\ell)$ and $x \neq a$, the line $n_x$ is defined as the unique line through $x$ parallel to $n$. But $n_y$ also passes through $x$, because $x \,\mathrm{I}\, n_y$, and $n_y \parallel n$. By uniqueness in the Euclidean parallel property, we get $n_y = n_x$.*

*Now $y \,\mathrm{I}\, n_y$, $y \in R(m)$, and $n_y = n_x$. Hence $y \in R(n_x) \cap R(m)$. By definition, $f(x)$ is the unique point in $R(n_x) \cap R(m)$. Therefore $f(x) = y$. Thus $f$ is*

*surjective.*

*Since $f$ is both injective and surjective, $f$ is a bijection.*

# 23 Week 6 Lecture 1

*Counting points in affine planes, projective planes, and the first idea of projective completion.*

## 1. Counting points in affine planes

In this entire section, let

$$\alpha = (\mathcal{P}, \mathcal{L}, \mathcal{R})$$

be an affine plane.

Given a line $\ell \in \mathcal{L}$, we recall the notation

$$R(\ell) := \{p \in \mathcal{P} : p \, I \, \ell\}.$$

This is the set of points incident with $\ell$.

During last class and this week's homework, we proved the following proposition.

**Proposition 4.** Let $\ell, m \in \mathcal{L}$ be two lines. Then there exists a bijective map

$$f : R(\ell) \longrightarrow R(m).$$

In particular, all lines in an affine plane are incident with the same number of points.

Using the proposition above, we will prove the following theorem.

**Theorem 1.** Let $n \geq 2$ be an integer, and assume that for some line $\ell \in \mathcal{L}$, we have

$$\#R(\ell) = n.$$

In other words, assume that one line is incident with exactly $n$ points.

Then

$$\#\mathcal{P} = n^2.$$

That is, there are exactly $n^2$ points in our affine plane. The integer $n$ is called the **order** of the affine plane.

***Proof*** By axiom **I3**, there exists a point

$$p \in \mathcal{P}$$

such that

$$p \not I \, \ell.$$

Choose a point $q \in R(\ell)$. Let $m$ be the unique line incident with $p$ and $q$. This line exists and is unique by axiom **I1**.

Since $p \not I \ell$, we have

$$m \neq \ell.$$

Moreover, the point $q$ is the unique intersection point of $\ell$ and $m$. Indeed, if there were another point $q' \neq q$ incident with both $\ell$ and $m$, then $\ell$ and $m$ would be two distinct lines incident with both $q$ and $q'$, contradicting axiom **I1**.

Since $\#R(\ell) = n$, write the points of $\ell$ as

$$R(\ell) = \{q, q_1, \ldots, q_{n-1}\}.$$

For each $1 \leq i \leq n-1$, define $m_i$ to be the unique line parallel to $m$ and passing through $q_i$.

This is possible by the Euclidean parallel property, because

$$q_i \not I\, m.$$

Indeed, $m \cap \ell = \{q\}$, and $q_i \neq q$.

We now observe that

$$R(m_i) \cap R(\ell) = \{q_i\}.$$

Indeed, $q_i \in R(m_i) \cap R(\ell)$. Moreover, since $m_i \parallel m$, we have $m_i \neq \ell$, and $m_i$ cannot meet $\ell$ in any point other than $q_i$.

Let $i \neq j$. Suppose, for contradiction, that there exists a point $x \in \mathcal{P}$ such that

$$x \, I \, m_i \qquad \text{and} \qquad x \, I \, m_j.$$

Since $m_i \parallel m$, we have $x \not I\, m$. Indeed, if $x \, I \, m$, then $m_i$ and $m$ would meet at $x$, contradicting $m_i \parallel m$.

Therefore $m_i$ and $m_j$ are both lines through $x$ parallel to $m$. By the Euclidean parallel property, there is a unique line through $x$ parallel to $m$. Hence

$$m_i = m_j.$$

But $m_i$ and $m_j$ meet $\ell$ at the unique points $q_i$ and $q_j$, respectively. Hence

$$q_i = q_j.$$

This contradicts the fact that $q_i$ and $q_j$ are distinct. Therefore the lines $m_i$ and $m_j$ do not meet.

Therefore the lines

$$m, m_1, \ldots, m_{n-1}$$

are pairwise parallel.

By the proposition from last class, each of these lines has exactly $n$ points. Therefore the union

$$R(m) \cup R(m_1) \cup \cdots \cup R(m_{n-1})$$

contains exactly $n^2$ points.

We now prove that this union is all of $\mathcal{P}$.

Suppose, for contradiction, that there exists a point $x \in \mathcal{P}$ such that

$$x \not I\, m$$

and, for every $1 \leq i \leq n-1$,

$$x \not I\, m_i.$$

Let $t$ be the unique line incident with $q$ and $x$. This line exists and is unique by axiom **I1**.

We claim first that

$$t \neq m.$$

Indeed, if $t = m$, then $x\, I\, m$, contradicting the choice of $x$.

We also claim that $t$ is not parallel to $m$. Indeed, both $t$ and $m$ are incident with $q$, so they meet at $q$.

Now fix $1 \leq i \leq n-1$. We claim that

$$t \neq m_i \qquad \text{and} \qquad t \text{ is not parallel to } m_i.$$

First, $t \neq m_i$, because $x \not I\, m_i$, whereas $x\, I\, t$.

Second, suppose for contradiction that

$$t \parallel m_i.$$

Since $m_i \parallel m$, both $t$ and $m$ would be lines through $q$ parallel to $m_i$. This contradicts the uniqueness part of the Euclidean parallel property, because $t \neq m$.

Therefore $t$ and $m_i$ are distinct and not parallel. Hence they meet in a unique point. Call this point $x_i$:

$$x_i\, I\, t \qquad \text{and} \qquad x_i\, I\, m_i.$$

Thus the line $t$ is incident with the points

$$q, x_1, \ldots, x_{n-1}, x.$$

These $n+1$ points are all distinct. Indeed, the points $x_i$ lie on the distinct parallel lines $m_i$, the point $q$ lies on $m$, and $x$ was chosen to lie on none of the lines

$$m, m_1, \ldots, m_{n-1}.$$

Therefore $t$ is incident with at least $n+1$ points. But by the proposition from last class, every line is incident with exactly $n$ points. This is a contradiction.

Hence no such point $x$ exists, and therefore

$$\mathcal{P} = R(m) \cup R(m_1) \cup \cdots \cup R(m_{n-1}).$$

Since the lines

$$m, m_1, \ldots, m_{n-1}$$

are pairwise parallel and each contains exactly $n$ points, this union contains exactly

$$n \cdot n = n^2$$

points.

Therefore

$$\#\mathcal{P} = n^2.$$

■

## 2. Projective planes

**Definition 3.** A **projective plane** is a model of incidence geometry

$$(\mathcal{P}, \mathcal{L}, \mathcal{R})$$

with the elliptic parallel property, and such that every line has at least three distinct points incident with it.

Recall that the **elliptic parallel property** says that any two distinct lines meet. In other words, if

$$\ell, m \in \mathcal{L} \qquad \text{and} \qquad \ell \neq m,$$

then there exists a point $p \in \mathcal{P}$ such that

$$p \, I \, \ell \qquad \text{and} \qquad p \, I \, m.$$

**Example.** The Fano plane is a projective plane with 7 points and 7 lines.

## 3. Activity: A point avoiding two lines

**Problem 48** *Show that the following statement holds in any projective plane:*

*For every pair of distinct lines $\ell, m \in \mathcal{L}$, there exists a point $P \in \mathcal{P}$ such that*

$$P \not I\, \ell \qquad \text{and} \qquad P \not I\, m.$$

*In other words, show that in a projective plane, given two distinct lines, there exists a point incident with neither of these two lines.*

***Proof*** Let $\ell, m \in \mathcal{L}$ be two distinct lines.

By the elliptic parallel property, there exists a point $x \in \mathcal{P}$ such that

$$x\, I\, \ell \qquad \text{and} \qquad x\, I\, m.$$

Since we are in a projective plane, every line is incident with at least three distinct points. Therefore, choose distinct points

$$x, a, b \in R(\ell)$$

and distinct points

$$x, c, d \in R(m).$$

Observe that $a, c, d$ are distinct. Indeed, if one of $c$ or $d$ were equal to $a$, then the two distinct lines $\ell$ and $m$ would have two distinct points in common, namely $x$ and $a$. This would contradict axiom **I1**. Similarly, $b, c, d$ are distinct.

By axiom **I1**, there exists a unique line $n$ incident with $a$ and $c$.

We note that

$$n \neq \ell.$$

Indeed, if $n = \ell$, then $c\, I\, \ell$. Since $c\, I\, m$ and $c \neq x$, the lines $\ell$ and $m$ would have two distinct points in common, contradicting axiom **I1**.

Similarly,

$$n \neq m.$$

Indeed, if $n = m$, then $a\, I\, m$, again contradicting the fact that $\ell$ and $m$ are distinct and already meet at $x$.

Since $n \neq \ell$, the point $b$ is not incident with $n$. Otherwise $n$ and $\ell$ would both contain $a$ and $b$, so $n = \ell$.

Similarly, since $n \neq m$, the point $d$ is not incident with $n$.

Also,

$$x \not I\, n.$$

Indeed, if $x \, I \, n$, then $n$ and $\ell$ would both contain $x$ and $a$, so $n = \ell$, a contradiction.

The line $n$ must be incident with at least three points. It is already incident with $a$ and $c$, so there exists a point

$$P \in R(n)$$

such that

$$P \neq a \qquad \text{and} \qquad P \neq c.$$

We claim that

$$P \not I \, \ell \qquad \text{and} \qquad P \not I \, m.$$

Indeed, if $P \, I \, \ell$, then the lines $n$ and $\ell$ would both contain $a$ and $P$. Since $P \neq a$, axiom **I1** would imply

$$n = \ell,$$

a contradiction.

Similarly, if $P \, I \, m$, then the lines $n$ and $m$ would both contain $c$ and $P$. Since $P \neq c$, axiom **I1** would imply

$$n = m,$$

also a contradiction.

Thus

$$P \not I \, \ell \qquad \text{and} \qquad P \not I \, m.$$

■

## 4. Activity: All projective lines have the same number of points

Let

$$\pi = (\mathcal{P}, \mathcal{L}, \mathcal{R})$$

be a projective plane.

Given a line $\ell \in \mathcal{L}$, define

$$R(\ell) := \{P \in \mathcal{P} : P \, I \, \ell\}.$$

This is the set of points incident with $\ell$.

**Problem 49** *Show that if $\ell$ and $m$ are distinct lines, then there is a bijective map*

$$f : R(\ell) \longrightarrow R(m).$$

*In other words, show that all lines in a projective plane have the same number of points on them.*

***Proof*** If $\ell = m$, then the identity map

$$\mathrm{id} : R(\ell) \longrightarrow R(m)$$

is a bijection.

So assume that

$$\ell \neq m.$$

By the previous activity, there exists a point $P \in \mathcal{P}$ such that

$$P \not I\, \ell \qquad \text{and} \qquad P \not I\, m.$$

Let $x \in R(\ell)$. By axiom **I1**, there exists a unique line $\ell_{xP}$ incident with both $x$ and $P$:

$$x\, I\, \ell_{xP} \qquad \text{and} \qquad P\, I\, \ell_{xP}.$$

Since $P \not I\, m$, the line $\ell_{xP}$ is not equal to $m$.

Thus, by the elliptic parallel property, $\ell_{xP}$ and $m$ meet in a unique point. Define

$$f(x) := \text{the unique point in } R(\ell_{xP}) \cap R(m).$$

Equivalently,

$$f(x) = \ell_{xP} \cap m.$$

This defines a map

$$f : R(\ell) \longrightarrow R(m).$$

You will prove in section that this map is bijective. ■

# 5. Projective completion

## Goal

The goal is to construct a projective plane from an affine plane.

Let

$$\alpha = (\mathcal{P}, \mathcal{L}, \mathcal{R})$$

be an affine plane.

The idea is the following: think of train rails disappearing toward the horizon. Parallel lines do not meet inside the affine plane, but they appear to meet at a common point at infinity.

For example, every line parallel to a fixed line $\ell$ should meet at one common point at infinity. Every line parallel to another line $m$ should also meet at a common point at infinity, but a different one if

$$m \nparallel \ell.$$

Thus, in the projective completion, each parallel class of affine lines will give one new point at infinity.

# 24 Week 6 Section Worksheet: Counting in Affine and Projective Planes

*In this worksheet, we study several counting arguments in affine and projective planes. We first prove that any two lines in a projective plane have the same number of points. We then deduce the standard formula for the number of points in a finite projective plane. Finally, we prove that in a finite affine plane of order $n$, every point lies on exactly $n+1$ lines.*

## Notation

Let $\mathcal{M} = (\mathcal{P}, \mathcal{L}, I)$ be an incidence geometry. If $\ell \in \mathcal{L}$, we write

$$R(\ell) = \{p \in \mathcal{P} : p \, I \, \ell\}$$

for the set of points incident with $\ell$.

Recall that in a projective plane, any two distinct lines meet in a unique point.

## Problem 1: A bijection between two projective lines

Let $\mathcal{M} = (\mathcal{P}, \mathcal{L}, I)$ be a projective plane. The goal of this exercise is to prove the following statement:

*If $\ell$ and $m$ are distinct lines, then there is a bijective map*

$$f : R(\ell) \longrightarrow R(m).$$

Below, we introduce a map. Your goal will be to prove that this map is bijective.

Assume that

$$\ell \neq m.$$

By yesterday's activity, there exists a point $P \in \mathcal{P}$ such that

$$P \not I \, \ell \qquad \text{and} \qquad P \not I \, m.$$

Let $x \in R(\ell)$. By axiom **I1**, there exists a unique line $n_x \in \mathcal{L}$ incident with both $x$ and $P$:

$$x \, I \, n_x \qquad \text{and} \qquad P \, I \, n_x.$$

Since $P \not I\, m$, the line $n_x$ is not equal to $m$.

Thus, by the projective plane axiom that any two distinct lines meet in a unique point, $n_x$ and $m$ meet in a unique point. Define

$$f(x) := \text{the unique point in } R(n_x) \cap R(m).$$

This defines a map

$$f : R(\ell) \longrightarrow R(m).$$

(a) Show that $f$ is injective. That is, show that if

$$f(x_1) = f(x_2),$$

then

$$x_1 = x_2.$$

(b) Show that $f$ is surjective. That is, show that for every $y \in R(m)$, there exists $x \in R(\ell)$ such that

$$f(x) = y.$$

## Problem 2: Counting points in a finite projective plane

Let $\mathcal{M} = (\mathcal{P}, \mathcal{L}, I)$ be a projective plane. Suppose that there exists a line

$$\ell \in \mathcal{L}$$

such that $R(\ell)$ contains exactly $n + 1$ points.

The goal of this exercise is to prove that

$$|\mathcal{P}| = n^2 + n + 1.$$

If you want to challenge yourself, I recommend trying this on your own first! Then, you may look at the questions below for hints.

(a) Let $p \in \mathcal{P}$ be a point such that

$$p \not I\, \ell.$$

Explain why such a point exists.

(b) For each point $x \in R(\ell)$, let $r_x$ be the unique line incident with both $p$ and $x$:

$$p\, I\, r_x \qquad \text{and} \qquad x\, I\, r_x.$$

Show that if $x_1, x_2 \in R(\ell)$ and $x_1 \neq x_2$, then

$$r_{x_1} \neq r_{x_2}.$$

(c) Show that every line incident with $p$ is equal to $r_x$ for some $x \in R(\ell)$.

(d) Deduce that exactly $n+1$ lines are incident with $p$.

(e) By Problem 1, every line has the same number of points. Deduce that every line contains exactly $n+1$ points.

(f) For each $x \in R(\ell)$, explain why

$$R(r_x) \setminus \{p\}$$

has exactly $n$ points.

(g) Show that the sets

$$R(r_x) \setminus \{p\}, \qquad x \in R(\ell),$$

form a partition of

$$\mathcal{P} \setminus \{p\}.$$

(h) Conclude that

$$|\mathcal{P}| = 1 + (n+1)n = n^2 + n + 1.$$

## Problem 3: Counting lines through a point in an affine plane

Let $\mathcal{A} = (\mathcal{P}, \mathcal{L}, I)$ be a finite affine plane of order $n$. This means that every line is incident with exactly $n$ points. You may also use the fact that

$$|\mathcal{P}| = n^2.$$

Fix a point $p \in \mathcal{P}$. The goal of this exercise is to prove that exactly $n+1$ lines are incident with $p$.

Let

$$r_1, \ldots, r_k$$

be the distinct lines incident with $p$.

(a) Show that every point $q \in \mathcal{P} \setminus \{p\}$ lies on one of the lines

$$r_1, \ldots, r_k.$$

(b) Show that if $i \neq j$, then

$$\big(R(r_i) \setminus \{p\}\big) \cap \big(R(r_j) \setminus \{p\}\big) = \varnothing.$$

(c) Deduce that the sets

$$R(r_1) \setminus \{p\}, \ldots, R(r_k) \setminus \{p\}$$

form a partition of $\mathcal{P} \setminus \{p\}$.

(d) Explain why each set

$$R(r_i) \setminus \{p\}$$

has exactly $n - 1$ points.

(e) Deduce that

$$n^2 - 1 = k(n - 1).$$

(f) Conclude that

$$k = n + 1.$$

Therefore exactly $n + 1$ lines are incident with $p$.

## Bonus exercises

The following questions are optional challenge problems.

(a) Show that, up to isomorphism, there exists a unique affine plane with 4 points, i.e. a unique affine plane of order 2.

(b) Show that, up to isomorphism, there exists a unique affine plane with 9 points, i.e. a unique affine plane of order 3.

(c) Construct an affine plane of order 4 using a diagram. That is, construct an affine plane with 16 points.

(d) Construct an affine plane of order 5 using a diagram. That is, construct an affine plane with 25 points.

(e) Can there exist an affine plane of order 6, i.e. an affine plane with 36 points?

For this last question, look up the *Bruck–Ryser–Chowla Theorem* and think about how it can be used.

# 25 Week 6 Lecture 2

*Projective completion, points at infinity, and models of the real projective plane.*

## 1. Projective completion

### Goal

Our goal is to construct a projective plane from an affine plane.

Let

$$\alpha = (\mathcal{P}, \mathcal{L}, \mathcal{R})$$

be an affine plane.

The motivating idea is perspective drawing. Think of train tracks disappearing toward the horizon. Parallel lines do not meet inside the affine plane, but they appear to meet at a point at infinity.

For example, every line parallel to a fixed line $\ell$ should meet at a common point at infinity. Similarly, every line parallel to another line $m$ should meet at another common point at infinity. If

$$m \nparallel \ell,$$

then these should be different points at infinity.

How do we formalize this? We begin by defining an equivalence relation on the set of lines.

## 2. Parallel classes

**Definition 4.** Let $S$ be a set. A relation $\sim$ on $S$ is called an **equivalence relation** if it satisfies the following three properties:

(a) **Reflexivity.** For every $x \in S$,

$$x \sim x.$$

(b) **Symmetry.** For every $x, y \in S$, if

$$x \sim y,$$

then

$$y \sim x.$$

(c) **Transitivity.** For every $x, y, z \in S$, if

$$x \sim y \qquad \text{and} \qquad y \sim z,$$

then

$$x \sim z.$$

Given two lines $\ell, \ell' \in \mathcal{L}$, we say that $\ell$ is equivalent to $\ell'$, and we write

$$\ell \sim \ell',$$

if

$$\ell = \ell' \qquad \text{or} \qquad \ell \parallel \ell'.$$

In other words, $\ell \sim \ell'$ if the two lines are equal or parallel.

**Proposition 5.** The relation $\sim$ is an equivalence relation on $\mathcal{L}$.

***Proof*** We must check reflexivity, symmetry, and transitivity.

**Reflexivity.** For every line $\ell \in \mathcal{L}$, we have

$$\ell = \ell.$$

Therefore

$$\ell \sim \ell.$$

**Symmetry.** Suppose that

$$\ell \sim \ell'.$$

Then either $\ell = \ell'$, or $\ell \parallel \ell'$.
If $\ell = \ell'$, then $\ell' = \ell$, so $\ell' \sim \ell$.
If $\ell \parallel \ell'$, then by symmetry of the notion of parallelism,

$$\ell' \parallel \ell,$$

and therefore $\ell' \sim \ell$.

**Transitivity.**
See HW5.

■

# 3. Equivalence classes

Something very useful about equivalence relations is that they allow us to define equivalence classes.

**Definition 5.** Given a line $\ell \in \mathcal{L}$, we define

$$[\ell] := \{\ell' \in \mathcal{L} : \ell' \sim \ell\}.$$

This is the **equivalence class** of $\ell$.

In this context, it is also called the **parallel class of** $\ell$.

Thus $[\ell]$ consists of all lines parallel to $\ell$, together with $\ell$ itself.

By transitivity, if

$$\ell \sim \ell',$$

then

$$[\ell] = [\ell'].$$

We denote by

$$\boxed{\mathcal{L}_{/\sim} := \{[\ell] : \ell \in \mathcal{L}\}}$$

the set of all equivalence classes. This is called the **quotient set** of $\mathcal{L}$ by the equivalence relation $\sim$.

**Proposition 6.** The set of lines $\mathcal{L}$ is partitioned into equivalence classes:

$$\mathcal{L} = \bigsqcup_{A \in \mathcal{L}_{/\sim}} A.$$

***Proof*** First, every line belongs to some equivalence class. Indeed, if $\ell \in \mathcal{L}$, then

$$\ell \in [\ell].$$

Therefore

$$\mathcal{L} = \bigcup_{A \in \mathcal{L}_{/\sim}} A.$$

It remains to prove that this union is disjoint. Suppose that

$$A, B \in \mathcal{L}_{/\sim} \qquad \text{and} \qquad A \cap B \neq \emptyset.$$

Since $A$ and $B$ are equivalence classes, there exist lines $\ell, m \in \mathcal{L}$ such that

$$A = [\ell], \qquad B = [m].$$

Choose $r \in A \cap B$. Then

$$r \sim \ell \qquad \text{and} \qquad r \sim m.$$

By symmetry and transitivity,

$$\ell \sim m.$$

Therefore

$$[\ell] = [m],$$

and hence

$$A = B.$$

Thus two equivalence classes are either equal or disjoint, so

$$\mathcal{L} = \bigsqcup_{A \in \mathcal{L}_{/\sim}} A.$$

■

## 4. Example: the affine plane with four points

Consider the affine plane with the smallest possible number of points: the affine plane of order 2. It has 4 points and 6 lines.

Label the four points by

$$0, 1, 2, 3.$$

We write $\ell_{ij}$ for the line incident with $i$ and $j$.

**Problem 50** *Find the equivalence classes of parallel lines and describe the quotient set*

$$\mathcal{L}_{/\sim}.$$

**Solution:** There are three pairs of parallel lines:

$$\ell_{12} \parallel \ell_{03},$$

$$\ell_{01} \parallel \ell_{23},$$

and

$$\ell_{02} \parallel \ell_{13}.$$

Therefore

$$[\ell_{12}] = [\ell_{03}] = \{\ell_{12}, \ell_{03}\},$$
$$[\ell_{01}] = [\ell_{23}] = \{\ell_{01}, \ell_{23}\},$$

and

$$[\ell_{02}] = [\ell_{13}] = \{\ell_{02}, \ell_{13}\}.$$

Thus there are three equivalence classes, each containing two lines. Hence

$$\mathcal{L}_{/\sim} = \big\{[\ell_{03}], [\ell_{01}], [\ell_{02}]\big\}.$$

Equivalently, we could also write

$$\mathcal{L}_{/\sim} = \big\{[\ell_{12}], [\ell_{23}], [\ell_{13}]\big\}.$$

This is because

$$[\ell_{12}] = [\ell_{03}], \qquad [\ell_{23}] = [\ell_{01}], \qquad [\ell_{13}] = [\ell_{02}].$$

We say that $\ell$ is a **representative** of the equivalence class $[\ell]$. For example, in

$$[\ell_{03}] = \{\ell_{12}, \ell_{03}\},$$

both $\ell_{12}$ and $\ell_{03}$ are representatives of the same equivalence class.

Observe that

$$\mathcal{L} = \{\ell_{01}, \ell_{23}\} \sqcup \{\ell_{02}, \ell_{13}\} \sqcup \{\ell_{03}, \ell_{12}\}.$$

This is precisely the partition of $\mathcal{L}$ into parallel classes.

## 5. Points at infinity

We now return to our affine plane

$$\alpha = (\mathcal{P}, \mathcal{L}, \mathcal{R}).$$

We have decomposed the set of affine lines into disjoint equivalence classes:

$$\mathcal{L}_{/\sim} = \{[\ell] : \ell \in \mathcal{L}\}.$$

We will say that $\ell$ and all lines parallel to $\ell$ meet at infinity in a common point, which we call

$$[\ell].$$

Thus each equivalence class of parallel lines becomes one new point at infinity.

Lines in different equivalence classes meet in different points at infinity. The horizon should connect all of these points at infinity. This gives the idea of the **line at infinity**.

# 6. The projective completion

We now define the projective completion of an affine plane.

**Definition 6.** Let

$$\alpha = (\mathcal{P}, \mathcal{L}, \mathcal{R})$$

be an affine plane. Its **projective completion** is the incidence geometry

$$\alpha^* = (\mathcal{P}^*, \mathcal{L}^*, \mathcal{R}^*)$$

defined as follows.

## The new set of points

We define

$$\mathcal{P}^* := \mathcal{P} \sqcup \mathcal{L}_{/\sim}.$$

The points of $\mathcal{P}$ are called **ordinary points** (aka old points), and the points of

$$\mathcal{L}_{/\sim}$$

are called **points at infinity** (aka new points).

Thus we are adding one new point for each parallel class of affine lines.

## The new set of lines

We define

$$\mathcal{L}^* := \mathcal{L} \sqcup \{\ell_\infty\}.$$

Here $\ell_\infty$ is a new line, called the **line at infinity** (aka new line). It represents the horizon. Lines $\ell \in \mathcal{L}$ will be referred to as **ordinary lines** (aka old lines).

## The new incidence relation

What matters now is the new incidence relation $\mathcal{R}^*$.

**(i) Old incidences are kept.**

If

$$p \in \mathcal{P} \qquad \text{and} \qquad \ell \in \mathcal{L},$$

then

$$p \, I^* \, \ell$$

if and only if

$$p \, I \, \ell.$$

**(ii) Incidences involving points at infinity.**

Let $[\ell] \in \mathcal{L}_{/\sim}$. Then $[\ell]$ is incident with every affine line in its equivalence class. That is, if $m \in \mathcal{L}$, then

$$[\ell] \, I^* \, m$$

if and only if

$$m \in [\ell].$$

Equivalently,

$$[\ell] \, I^* \, m \qquad \Longleftrightarrow \qquad m = \ell \text{ or } m \parallel \ell.$$

An old line $\ell \in \mathcal{L}$ is incident to precisely one point at infinity, namely $[\ell]$.

**(iii) The line at infinity.**

Every point at infinity is incident with the line at infinity:

$$[\ell] \, I^* \, \ell_\infty$$

for every

$$[\ell] \in \mathcal{L}_{/\sim}.$$

No ordinary point is incident with $\ell_\infty$.

This completes the definition of the projective completion.

## 7. Example: completing the order-2 affine plane

Return to the affine plane with four points.

We found that

$$\mathcal{L}_{/\sim} = \big\{[\ell_{01}], [\ell_{02}], [\ell_{03}]\big\}.$$

Thus the projective completion has three new points:

$$[\ell_{01}], \qquad [\ell_{02}], \qquad [\ell_{03}].$$

The point $[\ell_{01}]$ is incident with both

$$\ell_{01} \qquad \text{and} \qquad \ell_{23},$$

since

$$\ell_{01} \parallel \ell_{23}.$$

Similarly,

$$[\ell_{02}]\, I^*\, \ell_{02} \qquad \text{and} \qquad [\ell_{02}]\, I^*\, \ell_{13},$$

and

$$[\ell_{03}]\, I^*\, \ell_{03} \qquad \text{and} \qquad [\ell_{03}]\, I^*\, \ell_{12}.$$

Finally, we add the line at infinity

$$\ell_\infty,$$

which is incident with all three points at infinity:

$$[\ell_{01}], \qquad [\ell_{02}], \qquad [\ell_{03}].$$

**Corollary 2.** The Fano plane is the projective completion of the affine plane with four points.

# 8. Why is the projective completion projective?

**Question 51** *Why is the projective completion of an affine plane a projective plane?*

This will be a HW6 problem, but we prove Axiom 1 for inspiration.

Let

$$\alpha^* = (\mathcal{P}^*, \mathcal{L}^*, \mathcal{R}^*)$$

be the projective completion of an affine plane

$$\alpha = (\mathcal{P}, \mathcal{L}, \mathcal{R}).$$

We want to show that $\alpha^*$ is a projective plane. We must verify the incidence axioms and the elliptic parallel property.

## Axiom I1

Let

$$p, q \in \mathcal{P}^* \qquad \text{with} \qquad p \neq q.$$

We must show that there exists a unique line

$$\ell \in \mathcal{L}^*$$

incident with both $p$ and $q$.

**Case 1:$(p, q) \in \mathcal{P}$, i.e., $p$ and $q$ are ordinary points (aka old points)**

By axiom **I1** in the affine plane, there exists a unique affine line

$$\ell \in \mathcal{L}$$

such that

$$p \, I \, \ell \qquad \text{and} \qquad q \, I \, \ell.$$

Therefore

$$p \, I^* \, \ell \qquad \text{and} \qquad q \, I^* \, \ell.$$

Moreover, neither $p$ nor $q$ is incident with $\ell_\infty$, because ordinary points are not incident with the line at infinity.

If another affine line $\ell' \in \mathcal{L}$ were incident with both $p$ and $q$, i.e.:

$$p \; I^* \; \ell' \quad \& \quad q \; I^* \; \ell',$$

then we would have:

$$p \; I \; \ell' \quad \& \quad q \; I \; \ell'.$$

By axiom **I1** in the affine plane, this implies $\ell' = \ell$ .

Thus the unique line through $p$ and $q$ is $\ell$.

**Case 2:** $p, q \in \mathcal{L}_{/\sim}$

Suppose that $p$ and $q$ are points at infinity. Write

$$p = [\ell], \qquad q = [m],$$

with

$$[\ell] \neq [m].$$

By definition, both $p$ and $q$ are incident with

$$\ell_\infty.$$

We claim that no affine line is incident with both $p$ and $q$. Indeed, each affine line belongs to exactly one equivalence class, so it has exactly one point at infinity.

Therefore the unique line incident with both $p$ and $q$ is

$$\ell_\infty.$$

**Case 3:** $p \in \mathcal{P}$ **and** $q \in \mathcal{L}_{/\sim}$

Suppose that $p$ is an ordinary point and

$$q = [\ell]$$

is a point at infinity.

If

$$p \, I \, \ell,$$

then $\ell$ itself is incident with both $p$ and $[\ell]$ for the new incident relation, i.e., $[\ell] \; I^* \; \ell$ and $p \; I^* \; \ell$.

If

$$p \not I \; \ell,$$

then by the Euclidean parallel property, there exists a unique affine line $\ell'$ through $p$ parallel to $\ell$. Then

$$[\ell] = [\ell']$$

and hence

$$[\ell] \, I^* \, \ell'.$$

So $\ell'$ is incident with both $p$ and $[\ell]$ for the new incident relation.

In either case, there exists a line incident with both $p$ and $q$.

For uniqueness, suppose that $\ell'$ and $\ell''$ are affine lines incident with both $p$ and $[\ell]$. Then both $\ell'$ and $\ell''$ pass through $p$, and both belong to the parallel class $[\ell]$. Hence they are both parallel to $\ell$. By the Euclidean parallel property, there is a unique line through $p$ parallel to $\ell$. Therefore

$$\ell' = \ell''.$$

Thus **I1** holds in the projective completion.

# 26 Homework 6

*Homework 6 for MATH 128A.*

**MATH 128A: Homework 6** **Due date: Thursday, May 14**

**Instructions.** You will submit your solutions on Gradescope, which you can access through our Canvas course page by clicking on the assignment.

You are welcome to submit your homework either handwritten or in LaTeX. If you choose to submit handwritten work, please use pen rather than pencil, and make sure that both your handwriting and your scans are as readable as possible in order to facilitate grading.

I encourage you to attend office hours if you would like hints on the homework or if you would like to discuss more advanced material. Also, you are highly encouraged to work in groups!

**Problem 52** ***(33 points) The affine plane of order*** 3 ***and its projective completion.***

*The goal of this problem is to draw the affine plane of order* 3*, determine its parallel classes, and then construct its projective completion.*

(a) ***(11 points)*** *Draw an affine plane of order* 3*, i.e. an affine plane with* 9 *points.*

*Hint: You may arrange the* 9 *points in a* $3 \times 3$ *grid. The lines should have exactly* 3 *points each.*

(b) ***(11 points)*** *Determine the parallel classes of lines in the affine plane you drew in part* (a)*. In other words, group the lines into families of mutually parallel lines.*

*Hint: There should be* 4 *parallel classes, and each parallel class should contain* 3 *lines.*

(c) ***(11 points)*** *Draw the projective completion of the affine plane of order* 3 *you drew in part* (a)*.*

*Hint: Add one point at infinity for each parallel class of lines. Then extend each affine line by adding its corresponding point at infinity. Finally, add one new line at infinity incident with all the points at infinity.*

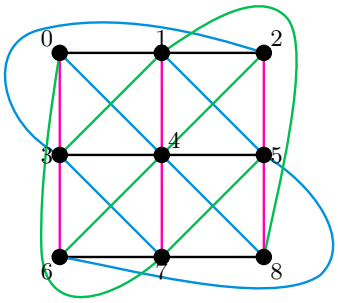
**Solution:** **(a)** Label the 9 points as follows:

$$\begin{matrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{matrix}$$

One affine plane of order 3 is obtained by taking the following 12 lines:

$$\begin{aligned}
&\ell_{012} = \{0,1,2\}, \quad \ell_{345} = \{3,4,5\}, \quad \ell_{678} = \{6,7,8\},\\
&\ell_{036} = \{0,3,6\}, \quad \ell_{147} = \{1,4,7\}, \quad \ell_{258} = \{2,5,8\},\\
&\ell_{048} = \{0,4,8\}, \quad \ell_{156} = \{1,5,6\}, \quad \ell_{237} = \{2,3,7\},\\
&\ell_{246} = \{2,4,6\}, \quad \ell_{138} = \{1,3,8\}, \quad \ell_{057} = \{0,5,7\}.
\end{aligned}$$

Each line contains exactly 3 points.




**(b)** The parallel classes are

$$\{\ell_{012}, \ell_{345}, \ell_{678}\},$$

$$\{\ell_{036}, \ell_{147}, \ell_{258}\},$$

$$\{\ell_{048}, \ell_{156}, \ell_{237}\},$$

and

$$\{\ell_{246}, \ell_{138}, \ell_{057}\}.$$

Thus there are 4 parallel classes, each containing 3 lines.

**(c)** To construct the projective completion, we add one point at infinity for each parallel class. We name these points by choosing one line from each class:

$$[\ell_{012}], \qquad [\ell_{036}], \qquad [\ell_{048}], \qquad [\ell_{246}].$$

Here

$$\begin{aligned}
&[\ell_{012}] = [\ell_{345}] = [\ell_{678}],\\
&[\ell_{036}] = [\ell_{147}] = [\ell_{258}],\\
&[\ell_{048}] = [\ell_{156}] = [\ell_{237}],
\end{aligned}$$

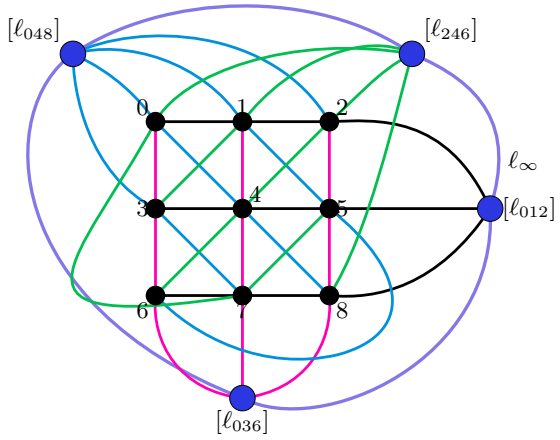
and

$$[\ell_{246}] = [\ell_{138}] = [\ell_{057}].$$

Then we add the line at infinity

$$\ell_\infty = \{[\ell_{012}], [\ell_{036}], [\ell_{048}], [\ell_{246}]\}.$$

In the picture below, the colors match the parallel classes above: black for $\{\ell_{012}, \ell_{345}, \ell_{678}\}$, magenta for $\{\ell_{036}, \ell_{147}, \ell_{258}\}$, blue for $\{\ell_{048}, \ell_{156}, \ell_{237}\}$, and green for $\{\ell_{246}, \ell_{138}, \ell_{057}\}$.




Therefore the projective completion has $9+4 = 13$ points and $12+1 = 13$ lines.

**Problem 53** ***(33 points) Counting lines through a point in an affine plane.***

*Let $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{R})$ be a finite affine plane of order $n$. This means that every line is incident with exactly $n$ points. You may also use the fact that $|\mathcal{P}| = n^2$.*

*For a line $r \in \mathcal{L}$, recall that $R(r) := \{q \in \mathcal{P} : q\, I\, r\}$ denotes the set of points incident with $r$.*

*Fix a point $p \in \mathcal{P}$. The goal of this exercise is to prove that exactly $n+1$ lines are incident with $p$. Let $r_1, \ldots, r_k$ be the distinct lines incident with $p$. In other words, we assume that $p$* ***is incident with exactly*** *$k$* ***lines****, where $k \geq 1$ is an integer we need to determine. Our goal is to show that $k = n+1$.*

(a) ***(5 points)*** *Show that every point $q \in \mathcal{P} \setminus \{p\}$ lies on one of the lines $r_1, \ldots, r_k$.*

(b) ***(5 points)*** *Show that if $i \neq j$, then*

$$\big(R(r_i) \setminus \{p\}\big) \cap \big(R(r_j) \setminus \{p\}\big) = \varnothing.$$

(c) ***(5 points)*** *Deduce that the sets $R(r_1) \setminus \{p\}, \ldots, R(r_k) \setminus \{p\}$ form a partition of $\mathcal{P} \setminus \{p\}$.*

(d) ***(6 points)*** *Explain why each set $R(r_i) \setminus \{p\}$ has exactly $n-1$ points.*

(e) ***(6 points)*** *Deduce that $n^2 - 1 = k(n-1)$.*

(f) ***(6 points)*** *Conclude that $k = n+1$.*

*Therefore exactly $n+1$ lines are incident with $p$.*

**Solution:** Let $r_1, \ldots, r_k$ be the distinct lines incident with $p$.

**(a)** Let $q \in \mathcal{P} \setminus \{p\}$. By axiom **I1**, there is a unique line through $p$ and $q$. Since $r_1, \ldots, r_k$ are all the lines incident with $p$, this line must be one of the $r_i$. Hence $q$ lies on one of the lines $r_1, \ldots, r_k$.

**(b)** Suppose $i \neq j$, and suppose for contradiction that

$$q \in \big(R(r_i) \setminus \{p\}\big) \cap \big(R(r_j) \setminus \{p\}\big).$$

Then $q \neq p$, and both $p$ and $q$ are incident with $r_i$ and with $r_j$. By axiom **I1**, there is a unique line through the two distinct points $p$ and $q$. Therefore $r_i = r_j$, contradicting $i \neq j$. Hence

$$\big(R(r_i) \setminus \{p\}\big) \cap \big(R(r_j) \setminus \{p\}\big) = \varnothing.$$

**(c)** By part (a), every point of $\mathcal{P} \setminus \{p\}$ lies in one of the sets

$$R(r_1) \setminus \{p\}, \ldots, R(r_k) \setminus \{p\}.$$

By part (b), these sets are pairwise disjoint. Therefore they form a partition of $\mathcal{P} \setminus \{p\}$.

**(d)** Since the affine plane has order $n$, every line has exactly $n$ points. Thus each $R(r_i)$ has $n$ points. Since $p \in R(r_i)$, removing $p$ leaves exactly $n-1$ points:

$$|R(r_i) \setminus \{p\}| = n - 1.$$

**(e)** Since $|\mathcal{P}| = n^2$, we have

$$|\mathcal{P} \setminus \{p\}| = n^2 - 1.$$

By part (c), $\mathcal{P} \setminus \{p\}$ is partitioned into $k$ disjoint sets, each of size $n-1$. Therefore

$$n^2 - 1 = k(n-1).$$

**(f)** Since

$$n^2 - 1 = (n-1)(n+1),$$

we get

$$k(n-1) = (n-1)(n+1).$$

Since $n \geq 2$, we have $n - 1 \neq 0$, so we can divide by $n-1$. Hence

$$k = n + 1.$$

Therefore exactly $n+1$ lines are incident with $p$.

**Problem 54** ***(34 points) The projective completion of an affine plane.***

*Let $\alpha = (\mathcal{P}, \mathcal{L}, \mathcal{R})$ be an affine plane, and let $\alpha^* = (\mathcal{P}^*, \mathcal{L}^*, \mathcal{R}^*)$ be its projective completion, as introduced during W6L2. Recall that the points of $\alpha^*$ are the ordinary points of $\mathcal{P}$, together with one point at infinity $[\ell]$ for each parallel class of affine lines. The lines of $\alpha^*$ are the affine lines in $\mathcal{L}$, together with one new line $\ell_\infty$, called the line at infinity.*

*The goal of this problem is to show that $\alpha^*$ satisfies axiom **I2**, axiom **I3**, and the elliptic parallel property.*

(a) ***(5 points) Axiom I2 for affine lines.*** *Show that every old line $\ell \in \mathcal{L}$, is incident with at least three distinct points of $\mathcal{P}^*$.*

(b) ***(5 points) Axiom I2 for the line at infinity.*** *Show that the line at infinity $\ell_\infty$ is incident with at least three distinct points of $\mathcal{P}^*$.*

*Hint: Use axiom **I3** in the affine plane to find three non-collinear ordinary points. Use these points to create three distinct parallel classes.*

(c) ***(4 points)*** *Deduce that every line of $\alpha^*$ is incident with at least three distinct points.*

(d) ***(5 points)*** *Show that $\alpha^*$ satisfies axiom **I3**; that is, show that there exist three points in $\mathcal{P}^*$ which are not all incident with one common line.*

(e) ***(5 points)*** *Show that any two distinct old lines $\ell, m \in \mathcal{L}$ meet in $\alpha^*$.*

*Hint: Distinguish between the case where $\ell$ and $m$ are parallel in the affine plane and the case where they aren't.*

(f) ***(5 points)*** *Show that every old line $\ell \in \mathcal{L}$ meets the line at infinity $\ell_\infty$.*

(g) ***(5 points)*** *Conclude that $\alpha^*$ satisfies the elliptic parallel property; that is, any two distinct lines in $\mathcal{L}^*$ meet.*

**Solution:** Recall that in the projective completion $\alpha^*$, each old affine line $\ell \in \mathcal{L}$ is extended by adding the point at infinity $[\ell]$ corresponding to its parallel class. The new line $\ell_\infty$ is incident exactly with the points at infinity.

**(a)** Let $\ell \in \mathcal{L}$ be an old affine line. By axiom **I2** in the affine plane, $\ell$ is incident with at least two distinct ordinary points. In the projective completion, $\ell$ is also incident with the point at infinity $[\ell]$. This point is not an ordinary point, so $\ell$ is incident with at least three distinct points of $\mathcal{P}^*$.

**(b)** By axiom **I3** in the affine plane, there exist three ordinary points $A, B, C \in \mathcal{P}$ which are not all incident with one common affine line.

Let $\ell$ be the unique affine line incident with $A$ and $B$, let $m$ be the unique affine line incident with $A$ and $C$, and let $n$ be the unique affine line incident with $B$ and $C$.

We first claim that $\ell, m, n$ belong to three distinct parallel classes. Indeed, $\ell$ and $m$ both contain $A$. If $\ell = m$, then this one line would contain $A$, $B$, and $C$, contradicting the choice of $A, B, C$. Hence $\ell \neq m$. Since $\ell$ and $m$ are distinct lines meeting at $A$, they are not parallel, and therefore they cannot belong to the same parallel class.

Similarly, $\ell$ and $n$ are distinct lines meeting at $B$, so they are not parallel and cannot belong to the same parallel class. Also, $m$ and $n$ are distinct lines meeting at $C$, so they are not parallel and cannot belong to the same parallel class.

Thus $\ell, m, n$ determine three distinct parallel classes. Therefore the corresponding points at infinity

$$[\ell], \qquad [m], \qquad [n]$$

are three distinct points of $\mathcal{P}^*$. All points at infinity are incident with the line at infinity $\ell_\infty$, so $\ell_\infty$ is incident with at least three distinct points.

**(c)** Every line of $\alpha^*$ is either an old affine line or the line at infinity. By parts (a) and (b), every such line is incident with at least three distinct points.

**(d)** By axiom **I3** in the affine plane, there exist three ordinary points $A, B, C \in \mathcal{P}$ which are not all incident with one common affine line. These same three points are also points of $\mathcal{P}^*$.

They are not all incident with $\ell_\infty$, because $\ell_\infty$ contains only points at infinity. They are also not all incident with any old line, because that would mean they were collinear in the original affine plane. Therefore $A, B, C$ are three points of $\mathcal{P}^*$ which are not all incident with one common line. Hence $\alpha^*$ satisfies axiom **I3**.

**(e)** Let $\ell, m \in \mathcal{L}$ be two distinct old affine lines. If $\ell$ and $m$ are not parallel in the affine plane, then they meet in an ordinary point, so they also meet in $\alpha^*$.

If $\ell$ and $m$ are parallel in the affine plane, then they belong to the same parallel class. Hence

$$[\ell] = [m].$$

In the projective completion, both $\ell$ and $m$ are incident with this common point at infinity. Therefore they meet in $\alpha^*$.

Thus any two distinct old lines meet in $\alpha^*$.

**(f)** Let $\ell \in \mathcal{L}$ be an old affine line. In the projective completion, $\ell$ is incident with the point at infinity $[\ell]$. By definition, $[\ell]$ is also incident with the line at infinity $\ell_\infty$. Therefore $\ell$ meets $\ell_\infty$.

**(g)** Let $L, M \in \mathcal{L}^*$ be two distinct lines. There are three cases.

If both $L$ and $M$ are old affine lines, then they meet by part (e). If one of them is an old affine line and the other is $\ell_\infty$, then they meet by part (f). The remaining case, where both are $\ell_\infty$, cannot occur because $L$ and $M$ are distinct.

Therefore any two distinct lines in $\mathcal{L}^*$ meet. Hence $\alpha^*$ satisfies the elliptic parallel property.

**Problem 55** ***Weekly reflection (2 bonus points).***

*Please answer in a few complete sentences. Your responses will be anonymous.*

- *What is one idea from this week that you found interesting, surprising, or useful?*
- *What is one idea from this week that still feels confusing?*
- *What, if anything, makes it difficult for you to ask questions during lecture?*
- *What, if anything, makes it difficult for you to attend class, section, or office hours?*
- *Is there something I could do to make it easier for you to ask questions, attend section, or come to office hours?*

**Problem 56** ***Challenge problems (not graded).***

(a) ***(The Transylvania lottery and the Fano plane †)*** *In the* ***Transylvania lottery****, each ticket consists of three distinct numbers chosen from* $\{1, 2, \ldots, 14\}$*. The winning ticket also consists of three distinct numbers.* ***A ticket wins if at least two of its numbers match two of the winning numbers.*** *Using Fano planes, show that one can buy* 14 *tickets and be certain of winning.*

(b) ***(Uniqueness of the affine plane of order*** 3 ***†)*** *Show that, up to isomorphism, there exists a unique affine plane with* 9 *points, i.e. a unique affine plane of order* 3*.*

(c) ***(Constructing an affine plane of order*** 4 ***†)*** *Construct an affine plane of order* 4 *using a diagram. That is, construct an affine plane with* 16 *points.*

# 27 Week 7 Lecture 1: Visualizing Projective Completions

*Visualizing the real projective plane and introducing homogeneous coordinates.*

## 1. Visualizing the real projective plane

Let

$$\mathcal{A} = (\mathbb{R}^2, \mathcal{L}_{\mathbb{R}^2}, I_{\mathbb{R}^2})$$

be the affine plane on $\mathbb{R}^2$, also called the **real affine plane**.

The goal of this lecture is to understand how we can visualize its projective completion

$$\mathcal{A}^*,$$

also called the **real projective plane**.

To help visualize this construction, you can click here for the Desmos 3D animation introduced in class.

## 2. Step 1: An isomorphic model for the real affine plane

Let

$$\mathbb{S}^2_+ := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1,\ z > 0\}.$$

This is the open northern hemisphere.

We identify a point $(x, y) \in \mathbb{R}^2$ with the point $(x, y, 1) \in \mathbb{R}^3$. In other words, we identify the real affine plane with the plane $z = 1$ in $\mathbb{R}^3$. This plane is tangent to the unit sphere at the north pole

$$N = (0, 0, 1).$$

Now consider the segment from the origin $(0, 0, 0)$ to the point $(x, y, 1)$. This segment intersects the open northern hemisphere in a unique point. We define

$$f(x, y) := \frac{(x, y, 1)}{\sqrt{x^2 + y^2 + 1}}.$$

Thus we have a map

$$f : \mathbb{R}^2 \longrightarrow \mathbb{S}^2_+.$$

**Claim 1.** The map $f : \mathbb{R}^2 \to \mathbb{S}^2_+$ is bijective.

***Proof*** Left as a calculus exercise. ■

**Remark 1.** We have found a bijection from $\mathbb{R}^2$ to $\mathbb{S}^2_+$. We can therefore define lines in $\mathbb{S}^2_+$ to be the images under $f$ of lines in $\mathbb{R}^2$.

More precisely, define

$$\mathcal{L}_{\mathbb{S}^2_+} := \{f(\ell) : \ell \in \mathcal{L}_{\mathbb{R}^2}\}.$$

Since lines in $\mathbb{S}^2_+$ are subsets of $\mathbb{S}^2_+$, we use the standard incidence relation of belonging:

$$p \, I \, \ell \qquad \Longleftrightarrow \qquad p \in \ell.$$

**Claim 2.** The map

$$\mathcal{L}_{\mathbb{R}^2} \longrightarrow \mathcal{L}_{\mathbb{S}^2_+}, \qquad \ell \longmapsto f(\ell),$$

is bijective.

***Proof*** This follows from the fact that $f : \mathbb{R}^2 \to \mathbb{S}^2_+$ is bijective. ■

**Corollary 3.** The incidence geometries

$$(\mathbb{R}^2, \mathcal{L}_{\mathbb{R}^2}, I_{\mathbb{R}^2}) \qquad \text{and} \qquad (\mathbb{S}^2_+, \mathcal{L}_{\mathbb{S}^2_+}, I_{\mathbb{S}^2_+})$$

are isomorphic.

**Remark 2.** Lines in $\mathbb{S}^2_+$ are exactly the semi-great circles, that is, intersections of $\mathbb{S}^2_+$ with planes through the origin.

**Corollary 4.** The projective completion of the real affine plane is isomorphic to the projective completion of the open northern hemisphere model.

## 3. Step 2: Projective completion of the open northern hemisphere model

Let

$$\overline{\mathbb{S}^2_+} := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1, \ z \geq 0\}.$$

This is the closed northern hemisphere. It is the open northern hemisphere together with the equator

$$\mathbb{S}^1_{\text{eq}} := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1, \ z = 0\}.$$

**Observation 1.** Lines in $\mathcal{L}_{\mathbb{S}^2_+}$ are parallel if and only if, when continued to the closed northern hemisphere, they intersect the equator at the same pair of antipodal points $p$ and $-p$.

Therefore, an equivalence class of parallel lines in $\mathcal{L}_{\mathbb{S}^2_+}$ is determined by a pair

$$\{p, -p\},$$

where $p$ is a point on the equator.

Thus, as a model of the projective completion of the real affine plane, we may consider the closed northern hemisphere with antipodal points on the equator identified.

## 4. Identifying antipodal points on the equator

We now make this rigorous. We define an equivalence relation on $\overline{\mathbb{S}^2_+}$ as follows.

If $x \in \mathbb{S}^2_+$, so $x$ is in the open northern hemisphere, define

$$x \sim y \qquad \Longleftrightarrow \qquad y = x.$$

Thus points in the open hemisphere are only equivalent to themselves.

If $x \in \overline{\mathbb{S}^2_+} \setminus \mathbb{S}^2_+$, so $x$ is on the equator, define

$$x \sim y \qquad \Longleftrightarrow \qquad y = x \quad \text{or} \quad y = -x.$$

Thus points on the equator are identified with their antipodal points.

**Claim 3.** This is an equivalence relation.

We denote the quotient set by

$$\overline{\mathbb{S}^2_+}/\sim .$$

**Remark 3.** Points in the open hemisphere are alone in their equivalence classes. Points on the equator have equivalence classes of the form

$$\{p, -p\}.$$

## 5. The bijection from the projective completion to the quotient hemisphere

Let ${\mathbb{S}^2_+}^*$ be the projective completion of the open northern hemisphere model. Recall that its points are

$${\mathbb{S}^2_+}^* = \mathbb{S}^2_+ \sqcup \mathcal{L}_{\mathbb{S}^2_+}/\!\parallel .$$

Here $\mathcal{L}_{\mathbb{S}^2_+}/\!\parallel$ denotes the set of parallel classes of lines in $\mathbb{S}^2_+$.

We define a map

$$F : \mathbb{S}_+^{2\,*} \longrightarrow \overline{\mathbb{S}_+^2}/\sim$$

as follows.

If $x \in \mathbb{S}_+^2$, define

$$F(x) = [x].$$

If $[\ell]$ is a parallel class of lines in $\mathbb{S}_+^2$, then every line in this class, when continued to the closed northern hemisphere, meets the equator at the same antipodal pair $\{p, -p\}$. Define

$$F([\ell]) = [p].$$

**Claim 4.** The map

$$F : \mathbb{S}_+^{2\,*} \longrightarrow \overline{\mathbb{S}_+^2}/\sim$$

is bijective.

***Proof*** This is left as an exercise. ■

# 6. Mapping lines to lines

We now describe the corresponding lines.

Recall that the lines in $\mathbb{S}_+^{2\,*}$ are

$$\mathcal{L}^*_{\mathbb{S}_+^2} = \mathcal{L}_{\mathbb{S}_+^2} \sqcup \{\ell_\infty\}.$$

Given a line $\ell \in \mathcal{L}_{\mathbb{S}_+^2}$, let $\overline{\ell} \subseteq \overline{\mathbb{S}_+^2}$ be the continuation of $\ell$ to the closed northern hemisphere. This is a closed half-great circle, and it meets the equator at two antipodal points $p$ and $-p$.

Let

$$\pi : \overline{\mathbb{S}_+^2} \longrightarrow \overline{\mathbb{S}_+^2}/\sim, \qquad x \longmapsto [x],$$

be the quotient map. Define

$$G(\ell) := \pi(\overline{\ell}).$$

Equivalently,

$$G(\ell) = \{[x] : x \in \overline{\ell}\}.$$

Thus $G(\ell)$ is the closed half-great circle corresponding to $\ell$, with antipodal points identified on the equator.

For the line at infinity, define

$$G(\ell_\infty) := \{[x] : x \in \mathbb{S}^1_{\text{eq}}\}.$$

This is the equator with antipodal points identified.

**Claim 5.** The map $G$ is bijective, and together with $F$, it gives an isomorphism of incidence geometries.

***Proof*** This is left as an exercise. ■

**Corollary 5.** The real projective plane can be modeled as follows:

- points are elements of the closed northern hemisphere, with antipodal points on the equator identified;
- lines are closed half-great circles on the closed northern hemisphere, with antipodal points on the equator identified.

# 7. The sphere model

We can also use the whole sphere. Equip

$$\mathbb{S}^2$$

with the antipodal equivalence relation:

$$p \sim q \qquad \Longleftrightarrow \qquad q = p \quad \text{or} \quad q = -p.$$

We denote the quotient by

$$\mathbb{S}^2/\{\pm 1\}.$$

There is a bijection

$$\overline{\mathbb{S}^2_+}/\sim \longrightarrow \mathbb{S}^2/\{\pm 1\}, \qquad [p] \longmapsto [p].$$

As a result, we may define lines in $\mathbb{S}^2/\{\pm 1\}$ as the images of lines in $\overline{\mathbb{S}^2_+}/\sim$, and we get an isomorphic model of the real projective plane.

**Corollary 6.** In the sphere model of the real projective plane:

- points are points on the sphere, with antipodal points identified;
- lines are great circles, with antipodal points identified.

**Remark 4.** This is the model used by Cinderella for elliptic geometry.

**Remark 5.** Cinderella is very easy to install and use. You can download the current beta version here. We will use it to visualize the sphere model of the real projective plane.

## 8. Projective completion of $F^2$

We now describe an algebraic model of the projective completion of the affine plane over a field.

Let $F$ be a field, for example

$$F = \mathbb{R}, \quad \mathbb{Q}, \quad \mathbb{C},$$

or any finite field.

## 9. Definition of $\mathbb{P}^2(F)$

Consider

$$F^3 \setminus \{(0,0,0)\}.$$

We define an equivalence relation on this set as follows.

Given two nonzero triples

$$(x,y,z), (x',y',z') \in F^3 \setminus \{(0,0,0)\},$$

we say that

$$(x,y,z) \sim (x',y',z')$$

if there exists $\lambda \in F^\times = F \setminus \{0\}$ such that

$$(x',y',z') = \lambda(x,y,z).$$

The equivalence class of $(x,y,z)$ is denoted by

$$[x:y:z].$$

Thus

$$[x:y:z] = \{(\lambda x, \lambda y, \lambda z) : \lambda \in F^\times\}.$$

**Definition 7.** The **projective plane over** $F$ is

$$\mathbb{P}^2(F) := \big(F^3 \setminus \{(0,0,0)\}\big)/\sim .$$

**Remark 6.** The equivalence classes in $\mathbb{P}^2(F)$ correspond to lines in $F^3$ through the origin.

## 10. Lines in $\mathbb{P}^2(F)$

The points of $\mathbb{P}^2(F)$ are the equivalence classes

$$[x:y:z].$$

The lines of $\mathbb{P}^2(F)$ are subsets of the form

$$\ell_{a,b,c} := \{[x : y : z] \in \mathbb{P}^2(F) : ax + by + cz = 0\},$$

where

$$(a, b, c) \in F^3 \setminus \{(0, 0, 0)\}.$$

Incidence is the standard incidence relation of belonging:

$$p \, I \, \ell \qquad \Longleftrightarrow \qquad p \in \ell.$$

This incidence geometry is called the **projective plane over** $F$.

## 11. $\mathbb{P}^2(F)$ as a projective completion

To help visualize homogeneous coordinates and the projective completion of $F^2$, you can click here for the Desmos 3D animation introduced in class.

**Proposition 7.** The projective plane $\mathbb{P}^2(F)$ is isomorphic to the projective completion of the affine plane $F^2$.

***Proof*** Let

$$\mathcal{A} = (F^2, \mathcal{L}_{F^2}, I_{F^2})$$

be the affine plane over $F$, and let

$$\mathcal{A}^* = (\mathcal{P}^*, \mathcal{L}^*, I^*)$$

be its projective completion.

We define a map

$$f : \mathcal{P}^* \longrightarrow \mathbb{P}^2(F)$$

as follows.

For an ordinary affine point $(x, y) \in F^2$, define

$$f(x, y) := [x : y : 1].$$

Now let $[\ell]$ be a point at infinity, where $\ell$ is an affine line of the form

$$ax + by + c = 0.$$

Define

$$f([\ell]) := [-b : a : 0].$$

This is well-defined because every line parallel to $\ell$ has the form

$$ax + by + d = 0$$

for some $d \in F$, and all such lines determine the same point

$$[-b : a : 0].$$

**Claim 6.** The map $f$ is well-defined and bijective.

***Proof*** This is left as an exercise. ■

We now define a map on lines

$$g : \mathcal{L}^* \longrightarrow \mathcal{L}_{\mathbb{P}^2(F)}.$$

If $\ell \in \mathcal{L}_{F^2}$ is the affine line

$$ax + by + c = 0,$$

define

$$g(\ell) := \{[x : y : z] \in \mathbb{P}^2(F) : ax + by + cz = 0\}.$$

For the line at infinity, define

$$g(\ell_\infty) := \{[x : y : z] \in \mathbb{P}^2(F) : z = 0\}.$$

Equivalently,

$$g(\ell_\infty) = \{[x : y : 0] : (x, y) \neq (0, 0)\}.$$

**Claim 7.** The map $g$ is well-defined and bijective.

***Proof*** This is left as an exercise. ■

**Claim 8.** The maps

$$f : \mathcal{P}^* \longrightarrow \mathbb{P}^2(F)$$

and

$$g : \mathcal{L}^* \longrightarrow \mathcal{L}_{\mathbb{P}^2(F)}$$

give an isomorphism of incidence geometries.

***Proof*** This is left as an exercise. ■

■

## 12. Another isomorphic model

There is another useful model of $\mathbb{P}^2(F)$.

Define $\widehat{\mathcal{P}}$ to be the set of lines in $F^3$ through the origin. In other words, its elements are subsets of the form

$$P = \{\lambda(x, y, z) : \lambda \in F\},$$

for some

$$(x, y, z) \neq (0, 0, 0).$$

Define $\widehat{\mathcal{L}}$ to be the set of planes in $F^3$ through the origin. In other words, its elements are subsets of the form

$$L = \{(x, y, z) \in F^3 : ax + by + cz = 0\},$$

for some

$$(a, b, c) \neq (0, 0, 0).$$

Incidence is inclusion:

$$P \; I \; L \qquad \Longleftrightarrow \qquad P \subseteq L.$$

The map

$$\mathbb{P}^2(F) \longrightarrow \widehat{\mathcal{P}}, \qquad [x : y : z] \longmapsto \{\lambda(x, y, z) : \lambda \in F\},$$

is bijective.

Together with the corresponding map sending projective lines to planes through the origin, this gives an isomorphic model of $\mathbb{P}^2(F)$.

# 28 Week 7 Section Worksheet: Homogeneous Coordinates and Projective Completion

*The purpose of this worksheet is to make sure you have a solid understanding of homogeneous coordinates and to fill in some of the gaps from Section 11 of W7L1.*

## Notation

Let $F$ be $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, or any other field (if you know what that means, no problem if not!). Recall that $\mathbb{P}^2(F)$ is defined as follows. We consider $F^3 \setminus \{(0,0,0)\}$ and define

$$(x, y, z) \sim (x', y', z')$$

if there exists $\lambda \in F^\times$ such that $(x', y', z') = \lambda(x, y, z)$. The equivalence class of $(x, y, z)$ is denoted by

$$[x : y : z].$$

The points of $\mathbb{P}^2(F)$ are the equivalence classes $[x : y : z]$. The lines of $\mathbb{P}^2(F)$ are subsets of the form

$$\{[x : y : z] \in \mathbb{P}^2(F) : ax + by + cz = 0\},$$

where $(a, b, c) \neq (0, 0, 0)$. Incidence is the usual relation of belonging.

## Problem: $\mathbb{P}^2(F)$ as the projective completion of $F^2$

The goal of this problem is to prove that $\mathbb{P}^2(F)$ is isomorphic to the projective completion of the affine plane $F^2$.

### Part 1: Homogeneous coordinates

(a) Show that $\sim$ is an equivalence relation on $F^3 \setminus \{(0,0,0)\}$.
*Hint: Check reflexivity, symmetry, and transitivity.*

(b) For each of the following, answer YES or NO. Briefly justify your answer.

(i) Is $[1 : 0 : 0] = [2 : 0 : 0]$ in $\mathbb{P}^2(\mathbb{Q})$?

(ii) Is $[1:0:0] = [1:1:1]$ in $\mathbb{P}^2(\mathbb{Q})$?

(iii) Is $[1:2:3] = [2:4:6]$ in $\mathbb{P}^2(\mathbb{Q})$?

(iv) Is $[1:2:3] = [2:4:5]$ in $\mathbb{P}^2(\mathbb{Q})$?

(v) Is $[1:1:0] = [2:2:0]$ in $\mathbb{P}^2(\mathbb{Q})$?

(c) Let $[x:y:z] \in \mathbb{P}^2(F)$ with $z \neq 0$. Show that

$$[x:y:z] = \left[\frac{x}{z} : \frac{y}{z} : 1\right].$$

## Part 2: Lines in $\mathbb{P}^2(F)$

(a) Show that the expression

$$ax + by + cz = 0$$

defines a well-defined condition on projective points. In other words, show that if $[x:y:z] = [x':y':z']$, then $ax + by + cz = 0$ if and only if $ax' + by' + cz' = 0$.

(b) Describe the projective line passing through $[1:0:0]$ and $[0:0:1]$.

*Hint: Find a nonzero triple $(a, b, c)$ such that both points satisfy $ax + by + cz = 0$. There will be many solutions; a one-dimensional space of solutions to be more precise.*

(c) Describe the projective line passing through $[1:0:0]$ and $[0:1:0]$.

(d) Let $a, b, c \in F$, with $(a, b) \neq (0, 0)$, and consider the affine line

$$\ell = \{(x, y) \in F^2 : ax + by + c = 0\}.$$

We define its *projective closure* as

$$\overline{\ell} := \{[x:y:z] \in \mathbb{P}^2(F) : ax + by + cz = 0\}.$$

Show that $\overline{\ell}$ meets the line at infinity $z = 0$ in exactly one point, namely

$$[-b:a:0].$$

*Hint: Substitute $z = 0$ into $ax + by + cz = 0$.*

## Part 3: The map on points

Let $\mathcal{A} = (F^2, \mathcal{L}_{F^2}, I_{F^2})$ be the affine plane over $F$, and let $\mathcal{A}^* = (\mathcal{P}^*, \mathcal{L}^*, I^*)$ be its projective completion.

Recall that the points of $\mathcal{A}^*$ are of two types: ordinary points $(x, y) \in F^2$, and points at infinity $[\ell]$, where $\ell$ is a parallel class of affine lines.

Define a map $f : \mathcal{P}^* \to \mathbb{P}^2(F)$ as follows. If $(x, y) \in F^2$, define $f(x, y) = [x : y : 1]$. If $[\ell]$ is a point at infinity, choose a representative $\ell$ with equation $ax + by + c = 0$, and define

$$f([\ell]) = [-b : a : 0].$$

To help visualize homogeneous coordinates and the projective completion of $F^2$, you can click here for the Desmos 3D animation introduced in class.

(a) Show that $f$ is well-defined on points at infinity. In other words, show that if two affine lines are parallel, then they determine the same projective point at infinity.

*Hint: Parallel affine lines have equations $ax + by + c = 0$ and $\lambda ax + \lambda by + d = 0$, for some $\lambda \in F^\times$.*

(b) Show that $f$ is injective.

*Hint: Treat separately the cases of two ordinary points, one ordinary point and one point at infinity, and two points at infinity.*

(c) Show that $f$ is surjective.

*Hint: If $[x : y : z]$ has $z \neq 0$, scale so that $z = 1$. If $z = 0$, show that $[x : y : 0] = [-b : a : 0]$ for some $(a, b) \neq (0, 0)$.*

## Part 4: The map on lines

Recall that the lines of $\mathcal{A}^*$ are of two types: ordinary affine lines $\ell \in \mathcal{L}_{F^2}$, and the line at infinity $\ell_\infty$.

Define a map $g : \mathcal{L}^* \to \mathcal{L}_{\mathbb{P}^2(F)}$ as follows. If $\ell \subseteq F^2$ is the affine line $ax + by + c = 0$, define

$$g(\ell) = \{[x : y : z] \in \mathbb{P}^2(F) : ax + by + cz = 0\}.$$

For the line at infinity, define

$$g(\ell_\infty) = \{[x : y : z] \in \mathbb{P}^2(F) : z = 0\}.$$

(a) Show that $g$ is injective.

*Hint: If two affine lines have the same projective closure, compare their affine points with $z = 1$. Also, no affine line maps to $z = 0$.*

(b) Show that $g$ is surjective.

*Hint: Let $L$ be a projective line $Ax + By + Cz = 0$. If $(A, B) \neq (0, 0)$, show that $L$ is the image of the affine line $Ax + By + C = 0$. If $(A, B) = (0, 0)$, what is $L$?*

## Part 5: Incidence

Finally, show that the maps $f$ and $g$ preserve incidence. In other words, show that for every point $p \in \mathcal{P}^*$ and every line $\ell \in \mathcal{L}^*$,

$$p \, I^* \, \ell \qquad \Longleftrightarrow \qquad f(p) \in g(\ell).$$

(a) Check this when $p = (x, y)$ is an ordinary affine point and $\ell$ is an ordinary affine line.

(b) Check this when $p = [\ell]$ is a point at infinity and $m$ is an ordinary affine line.

*Hint: Write $\ell$ as $ax + by + c = 0$ and $m$ as $a'x + b'y + c' = 0$. The point at infinity $[\ell]$ is incident with $m$ exactly when $m$ is parallel to $\ell$.*

(c) Check this when $p = [\ell]$ is a point at infinity and the line is $\ell_\infty$.

(d) Conclude that $\mathcal{A}^*$ and $\mathbb{P}^2(F)$ are isomorphic as incidence geometries.

# 29 Week 7 Lecture 2 and Week 8 Lecture 1

*Transformations of incidence geometries, permutations, examples, and the group structure of automorphisms.*

## 1. Transformations

For this lecture, we will use the following convention: lines are subsets of the set of points. Thus, given a model

$$\mathcal{M} = (\mathcal{P}, \mathcal{L}, I),$$

we assume that every line $\ell \in \mathcal{L}$ is a subset $\ell \subseteq \mathcal{P}$, and incidence is the usual relation of belonging:

$$p\, I\, \ell \qquad \Longleftrightarrow \qquad p \in \ell.$$

This is not very restrictive for the incidence geometries we are studying. If $\mathcal{M} = (\mathcal{P}, \mathcal{L}, \mathcal{R})$ satisfies axioms **I1** and **I2**, then each line $\ell \in \mathcal{L}$ determines the set

$$R(\ell) := \{p \in \mathcal{P} : p\, \mathcal{R}\, \ell\}.$$

One can replace $\mathcal{L}$ by the set of subsets $\{R(\ell) : \ell \in \mathcal{L}\}$. This gives an isomorphic model in which lines really are sets of points.

**Definition 8.** Let $\mathcal{M} = (\mathcal{P}, \mathcal{L}, I)$ be an incidence geometry whose lines are subsets of $\mathcal{P}$. A **transformation** of $\mathcal{M}$, also called an **automorphism**, is a bijection

$$f : \mathcal{P} \to \mathcal{P}$$

such that, for every line $\ell \in \mathcal{L}$, the image $f(\ell)$ is also a line in $\mathcal{L}$. Here

$$f(\ell) := \{f(p) : p \in \ell\}.$$

So a transformation does two things:

- it is a bijection from the set of points to itself;
- it sends lines to lines.

We denote by $\mathrm{Aut}(\mathcal{M})$ the set of all transformations of $\mathcal{M}$.

**Remark 7.** An automorphism is an isomorphism from a model to itself. Intuitively, it is a way of moving the points of a model around without breaking the incidence structure.

## 2. The one-point model

Consider the model with one point

$$\mathcal{P} = \{*\}$$

and no lines.

There is only one function $\mathcal{P} \to \mathcal{P}$, namely the function sending $*$ to itself. This function is bijective. Since there are no lines, there is nothing to check about sending lines to lines. Therefore this model has exactly one transformation.

## 3. The two-point model

Consider the model $\mathcal{M}_2$ with points

$$\mathcal{P} = \{0, 1\}$$

and one line

$$\ell = \{0, 1\}.$$

The identity map is a transformation. Indeed, it is a bijection, and

$$\mathrm{id}(\ell) = \mathrm{id}(\{0,1\}) = \{\mathrm{id}(0), \mathrm{id}(1)\} = \{0,1\} = \ell.$$

Thus the identity sends the only line to a line.

There is also the map

$$\begin{array}{rcl} r : \mathcal{P} & \longrightarrow & \mathcal{P} \\ 0 & \longmapsto & 1 \\ 1 & \longmapsto & 0. \end{array}$$

This map is a bijection. Moreover,

$$r(\ell) = r(\{0,1\}) = \{r(0), r(1)\} = \{1,0\} = \{0,1\} = \ell.$$

Therefore $r$ is a transformation.

There are no other transformations. Indeed, let $f : \mathcal{P} \to \mathcal{P}$ be a transformation. Then $f$ is a bijection, hence injective. If $f(0) = 0$, then $f(1) \neq 0$, so $f(1) = 1$, and $f = \mathrm{id}$. If $f(0) = 1$, then $f(1) \neq 1$, so $f(1) = 0$, and $f = r$. Therefore

$$\mathrm{Aut}(\mathcal{M}_2) = \{\mathrm{id}, r\}.$$

## 4. Permutations

**Definition 9.** Let $\Sigma$ be a set. A **permutation** of $\Sigma$ is a bijection

$$f : \Sigma \to \Sigma.$$

We denote by $\mathrm{Perm}(\Sigma)$ the set of all permutations of $\Sigma$.

For example, if $\Sigma = \{0, 1, 2\}$, then the following are permutations:

$$\begin{array}{rcl} f : \Sigma & \longrightarrow & \Sigma \\ 0 & \longmapsto & 0 \\ 1 & \longmapsto & 1 \\ 2 & \longmapsto & 2 \end{array} \qquad \begin{array}{rcl} g : \Sigma & \longrightarrow & \Sigma \\ 0 & \longmapsto & 1 \\ 1 & \longmapsto & 0 \\ 2 & \longmapsto & 2 \end{array} \qquad \begin{array}{rcl} h : \Sigma & \longrightarrow & \Sigma \\ 0 & \longmapsto & 1 \\ 1 & \longmapsto & 2 \\ 2 & \longmapsto & 0. \end{array}$$

The set $\{0, 1, 2\}$ has 6 permutations. Indeed, to define a permutation, we have 3 choices for the image of 0. Once this is chosen, we have 2 choices left for the image of 1, because a permutation is injective. Then there is only 1 choice left for the image of 2. Thus there are $3 \cdot 2 \cdot 1 = 6$ permutations.

More generally, if $\Sigma$ has $n$ elements, then

$$|\operatorname{Perm}(\Sigma)| = n!.$$

Indeed, there are $n$ choices for the image of the first element, $n - 1$ choices for the image of the second element, $n-2$ choices for the image of the third element, and so on.

## 5. The smallest incidence geometry with three points

Let $\mathcal{M}_3$ be the model with points

$$\mathcal{P} = \{0, 1, 2\}$$

and lines

$$\{0, 1\}, \qquad \{0, 2\}, \qquad \{1, 2\}.$$

Thus the lines are exactly the subsets of $\mathcal{P}$ containing two elements.

We claim that

$$\operatorname{Aut}(\mathcal{M}_3) = \operatorname{Perm}(\{0, 1, 2\}).$$

First, every transformation is a permutation of $\{0, 1, 2\}$, because a transformation is, by definition, a bijection from the point set to itself. Thus

$$\operatorname{Aut}(\mathcal{M}_3) \subseteq \operatorname{Perm}(\{0, 1, 2\}).$$

Conversely, let $f \in \operatorname{Perm}(\{0, 1, 2\})$. We show that $f$ is a transformation. Let $\ell$ be a line. Since lines in $\mathcal{M}_3$ are exactly the two-point subsets, we can write

$$\ell = \{a, b\}$$

with $a \neq b$. Since $f$ is injective, $f(a) \neq f(b)$. Hence

$$f(\ell) = f(\{a, b\}) = \{f(a), f(b)\}$$

is a two-point subset of $\{0, 1, 2\}$. Therefore $f(\ell)$ is a line. Hence $f$ sends lines to lines, and so $f$ is a transformation.

Thus

$$\mathrm{Aut}(\mathcal{M}_3) = \mathrm{Perm}(\{0, 1, 2\}).$$

In particular,

$$|\mathrm{Aut}(\mathcal{M}_3)| = 3! = 6.$$

## 6. The affine plane of order 2

The affine plane of order 2 has 4 points. We label them

$$\mathcal{P} = \{0, 1, 2, 3\}.$$

In this model, every line contains exactly 2 points, and any two distinct points determine a unique line. Therefore the lines are exactly all two-point subsets of $\mathcal{P}$.

We claim that

$$\mathrm{Aut}(\mathcal{M}) = \mathrm{Perm}(\{0, 1, 2, 3\}).$$

Again, every transformation is a permutation of the point set. Conversely, let $f \in \mathrm{Perm}(\{0, 1, 2, 3\})$. Let $\ell$ be a line. Since every line is a two-point subset, we can write $\ell = \{a, b\}$, with $a \neq b$. Since $f$ is injective, $f(a) \neq f(b)$. Therefore

$$f(\ell) = \{f(a), f(b)\}$$

is a two-point subset of $\mathcal{P}$. Since every two-point subset is a line in the affine plane of order 2, $f(\ell)$ is a line. Therefore $f$ is a transformation.

Thus

$$\mathrm{Aut}(\mathcal{M}) = \mathrm{Perm}(\{0, 1, 2, 3\}).$$

In particular, this model has

$$4! = 24$$

transformations.

## 7. Not every permutation is a transformation

In the previous two examples, every permutation of the point set was a transformation. This is not always true.

Consider the model with points

$$\mathcal{P} = \{0, 1, 2\}$$

and lines

$$\ell = \{0, 2\}, \qquad m = \{1, 2\}.$$

Define $f : \mathcal{P} \to \mathcal{P}$ by

$$\begin{array}{rcl} f : \mathcal{P} & \longrightarrow & \mathcal{P} \\ 0 & \longmapsto & 1 \\ 1 & \longmapsto & 2 \\ 2 & \longmapsto & 0. \end{array}$$

This is a permutation of $\mathcal{P}$, but it is not a transformation. Indeed,

$$f(\ell) = f(\{0, 2\}) = \{f(0), f(2)\} = \{1, 0\} = \{0, 1\}.$$

But $\{0, 1\}$ is not a line in this model. Therefore $f$ does not send lines to lines.

This example shows that being a permutation is not enough. To be a transformation, a bijection must also preserve the incidence structure.

## 8. A five-point example

Consider the model with points

$$\mathcal{P} = \{0, 1, 2, 3, 4\},$$

and lines

$$\{0, 1\}, \quad \{2, 3, 4\}, \quad \{0, 2\}, \quad \{0, 3\}, \quad \{0, 4\}, \quad \{1, 2\}, \quad \{1, 3\}, \quad \{1, 4\}.$$

This is the model appearing in Homework 7.

The line $\{2, 3, 4\}$ is special because it is the only line containing three points. Let $F$ be a transformation. Since $F$ is bijective, the image of a three-point set is again a three-point set. Since $F$ sends lines to lines, $F(\{2, 3, 4\})$ must be a line with three points. But $\{2, 3, 4\}$ is the only such line. Therefore

$$F(\{2, 3, 4\}) = \{2, 3, 4\}.$$

Since $F$ is bijective, it sends complements to complements. The complement of $\{2, 3, 4\}$ in $\mathcal{P}$ is $\{0, 1\}$. Hence

$$F(\{0, 1\}) = \{0, 1\}.$$

This strongly restricts the possible transformations. In particular, $F$ must permute the set $\{2, 3, 4\}$, and it must also permute the set $\{0, 1\}$. This is the starting point for the Homework 7 problem.

## 9. Transformations of the real affine plane

The examples above were finite. We also know many transformations of the usual real affine plane $\mathbb{R}^2$.

For example, translations are transformations. If $v \in \mathbb{R}^2$, then

$$T_v : \mathbb{R}^2 \to \mathbb{R}^2, \qquad T_v(p) = p + v$$

is a bijection and sends lines to lines.

Rotations are also transformations. More generally, if $A \in \mathrm{GL}_2(\mathbb{R})$ and $b \in \mathbb{R}^2$, then the map

$$p \longmapsto Ap + b$$

is a transformation of the real affine plane. Later, we will study analogous transformations over $\mathbb{Q}$.

## 10. Groups

The set of transformations of a model has more structure than just being a set. We can compose transformations.

**Definition 10.** A **group** is a set $G$ together with an operation $*$ such that:

(a) **Closure:** if $g, h \in G$, then $g * h \in G$.

(b) **Neutral element:** there exists $e \in G$ such that $g * e = e * g = g$ for all $g \in G$.

(c) **Inverse:** for every $g \in G$, there exists $h \in G$ such that $g * h = h * g = e$.

(d) **Associativity:** for all $g, h, k \in G$, we have $g * (h * k) = (g * h) * k$.

**Example 1.** The set $\mathbb{Z}$, with addition, is a group. The neutral element is 0, and the inverse of $k \in \mathbb{Z}$ is $-k$.

**Example 2.** The set $\mathbb{R}$, with addition, is a group.

**Example 3.** The set $\mathbb{R} \setminus \{0\}$, with multiplication, is a group. The neutral element is 1, and the inverse of $x \neq 0$ is $1/x$.

**Example 4.** The set $\{1, -1\}$, with multiplication, is a group. The neutral element is 1, and both elements are their own inverses.

**Example 5.** The set $\mathrm{GL}_2(\mathbb{R})$ of invertible $2 \times 2$ real matrices is a group under matrix multiplication. The neutral element is the identity matrix, and the inverse of $A$ is the inverse matrix $A^{-1}$.

# 11. Automorphisms form a group

**Proposition 8.** Let $\mathcal{M} = (\mathcal{P}, \mathcal{L}, I)$ be a model satisfying axioms **I1** and **I2**. Then $\mathrm{Aut}(\mathcal{M})$, equipped with composition, is a group.

***Proof*** We check the group properties.

First, we check closure. Let $f, g \in \mathrm{Aut}(\mathcal{M})$. Since $f$ and $g$ are bijections, their composition $f \circ g$ is also a bijection. We now check that $f \circ g$ sends lines to lines. Let $\ell \in \mathcal{L}$. Since $g$ is a transformation, $g(\ell)$ is a line. Since $f$ is a transformation, $f(g(\ell))$ is also a line. But

$$f(g(\ell)) = (f \circ g)(\ell).$$

Therefore $f \circ g$ sends lines to lines, so $f \circ g \in \mathrm{Aut}(\mathcal{M})$.

The identity map is a transformation. It is bijective, and for every line $\ell$, we have $\mathrm{id}(\ell) = \ell$. Thus the identity map is the neutral element for composition.

Now let $f \in \mathrm{Aut}(\mathcal{M})$. Since $f$ is a bijection, it has an inverse map $f^{-1} : \mathcal{P} \to \mathcal{P}$. One must check that $f^{-1}$ sends lines to lines. This is one of the homework exercises, and it uses axioms **I1** and **I2**. Therefore $f^{-1} \in \mathrm{Aut}(\mathcal{M})$. By definition of inverse maps,

$$f \circ f^{-1} = \mathrm{id} \qquad \text{and} \qquad f^{-1} \circ f = \mathrm{id}\,.$$

Finally, composition of maps is associative. Indeed, if $f, g, h \in \mathrm{Aut}(\mathcal{M})$, then for every point $p \in \mathcal{P}$,

$$((f \circ g) \circ h)(p) = f(g(h(p))) = (f \circ (g \circ h))(p).$$

Hence

$$(f \circ g) \circ h = f \circ (g \circ h).$$

Therefore $\mathrm{Aut}(\mathcal{M})$, with composition, is a group. ■

**Remark 8.** In practice, saying that $\mathrm{Aut}(\mathcal{M})$ is a group means that:

- we can compose transformations and get transformations;
- the identity is a transformation;
- the inverse of a transformation is a transformation;
- parentheses do not matter when composing transformations.

# 30 Homework 7

*Homework 7 for MATH 128A.*

**MATH 128A: Homework 7** **Due date: Thursday, May 21**

**Instructions.** You will submit your solutions on Gradescope, which you can access through our Canvas course page by clicking on the assignment.

You are welcome to submit your homework either handwritten or in LaTeX. If you choose to submit handwritten work, please use pen rather than pencil, and make sure that both your handwriting and your scans are as readable as possible in order to facilitate grading.

I encourage you to attend office hours if you would like hints on the homework or if you would like to discuss more advanced material. Also, you are highly encouraged to work in groups!

**Problem 57** ***(20 points) Warm-up: transformations and inverses.***

(a) ***(10 points)*** *Let $\Sigma$ be a set. Show that if $f : \Sigma \to \Sigma$ and $g : \Sigma \to \Sigma$ are bijections, then $f \circ g : \Sigma \to \Sigma$ is also a bijection.*

(b) ***(10 points)*** *Let $\mathcal{M} = (\mathcal{P}, \mathcal{L}, \mathcal{R})$ be a model satisfying axioms **I1** and **I2**. Assume that $F : \mathcal{P} \to \mathcal{P}$ is a transformation of $\mathcal{M}$, i.e., $F$ is a bijection such that, for every line $\ell \in \mathcal{L}$, the image $F(\ell)$ is also a line in $\mathcal{L}$.*

*Show that the inverse map $F^{-1} : \mathcal{P} \to \mathcal{P}$ is also a transformation.*

*Hint: Let $\ell \in \mathcal{L}$. You need to show that $F^{-1}(\ell)$ is a line. Choose two distinct points $p, q \in \ell$, and use the fact that $F$ is a transformation.*

**Problem 58** ***(40 points) Transformations of a five-point model.***

*Consider the incidence geometry $\mathcal{M} = (\mathcal{P}, \mathcal{L}, I)$, where $\mathcal{P} = \{0, 1, 2, 3, 4\}$, and where the lines are*

$$\{0,1\}, \quad \{2,3,4\}, \quad \{0,2\}, \quad \{0,3\}, \quad \{0,4\}, \quad \{1,2\}, \quad \{1,3\}, \quad \{1,4\}.$$

*Equivalently, you can view the model using the following GeoGebra diagram:*

*GeoGebradiagram.*

*Recall that a **transformation** of $\mathcal{M}$ is a bijection $F : \mathcal{P} \to \mathcal{P}$ such that, for every line $\ell \in \mathcal{L}$, the image $F(\ell)$ is also a line in $\mathcal{L}$. The goal of this problem is to determine all transformations of $\mathcal{M}$.*

(a) ***(4 points)*** *Explain why any transformation $F$ must send the line $\{2,3,4\}$ to itself.*

*Hint: This line is special: it is the only line containing three points. What happens to the image of a three-point line under a bijection?*

(b) ***(4 points)*** *Deduce that any transformation $F$ must send the set $\{0,1\}$ to itself.*

*Hint: Since $F$ is a bijection, it must send the complement of $\{2,3,4\}$ to the complement of $\{2,3,4\}$.*

(c) ***(4 points)*** *Given a permutation $\sigma$ of $\{2,3,4\}$, define a map*

$$\begin{array}{rcl} f(\sigma):\mathcal{P} & \longrightarrow & \mathcal{P} \\ 0 & \longmapsto & 0 \\ 1 & \longmapsto & 1 \\ 2 & \longmapsto & \sigma(2) \\ 3 & \longmapsto & \sigma(3) \\ 4 & \longmapsto & \sigma(4). \end{array}$$

*Given a permutation $\eta$ of $\{0,1\}$, define a map*

$$\begin{array}{rcl} g(\eta):\mathcal{P} & \longrightarrow & \mathcal{P} \\ 0 & \longmapsto & \eta(0) \\ 1 & \longmapsto & \eta(1) \\ 2 & \longmapsto & 2 \\ 3 & \longmapsto & 3 \\ 4 & \longmapsto & 4. \end{array}$$

*Let $\sigma = (234)$, meaning that $\sigma(2) = 3$, $\sigma(3) = 4$, and $\sigma(4) = 2$. Let $\eta = (01)$, meaning that $\eta(0) = 1$ and $\eta(1) = 0$. Define $F = f(\sigma) \circ g(\eta)$. Compute $F(0)$, $F(1)$, $F(2)$, $F(3)$, and $F(4)$.*

(d) ***(4 points)*** *Let $F$ be a transformation of $\mathcal{M}$, and assume that $F(0) = 0$. Show that $F(1) = 1$.*

(e) ***(5 points)*** *Let $F$ be a transformation of $\mathcal{M}$, and assume that $F(0) = 0$. Show that there exists a permutation $\sigma$ of $\{2,3,4\}$ such that $F = f(\sigma)$.*

(f) ***(4 points)*** *Let $F$ be a transformation of $\mathcal{M}$, and assume that $F(0) = 1$. Show that $F(1) = 0$.*

(g) ***(5 points)*** *Let $F$ be a transformation of $\mathcal{M}$, and assume that $F(0) = 1$. Let $\eta = (01)$ be the permutation of $\{0,1\}$ that exchanges 0 and 1. Show that there exists a permutation $\sigma$ of $\{2,3,4\}$ such that $F = g(\eta) \circ f(\sigma)$.*

(h) ***(5 points)*** *Conclude that every transformation of $\mathcal{M}$ is of the form $g(\eta) \circ f(\sigma)$, where $\sigma$ is a permutation of $\{2,3,4\}$ and $\eta$ is a permutation of $\{0,1\}$.*

(i) ***(5 points)*** *Deduce that* $|\operatorname{Aut}(\mathcal{M})| = 12$*, i.e., there are exactly* 12 *transformations.*

**Problem 59** ***(40 points) The dual of a projective plane.***

*Let* $\Pi = (\mathcal{P}, \mathcal{L}, I)$ *be a projective plane. We define a new incidence geometry*

$$\Pi^* = (\mathcal{P}^*, \mathcal{L}^*, I^*)$$

*as follows. The points of* $\Pi^*$ *are the lines of* $\Pi$*, so* $\mathcal{P}^* := \mathcal{L}$*. The lines of* $\Pi^*$ *are the points of* $\Pi$*, so* $\mathcal{L}^* := \mathcal{P}$*. Finally, incidence is defined by reversing the original incidence relation: if* $\ell \in \mathcal{P}^* = \mathcal{L}$ *and* $p \in \mathcal{L}^* = \mathcal{P}$*, then*

$$\ell \, I^* \, p \qquad \Longleftrightarrow \qquad p \, I \, \ell.$$

*The goal of this problem is to prove that* $\Pi^*$ *is also a projective plane.*

(a) ***(7 points)*** *Show that any two distinct points of* $\Pi^*$ *are incident with a unique line of* $\Pi^*$*.*

*Hint: A point of* $\Pi^*$ *is a line of* $\Pi$*. Use the fact that two distinct lines in a projective plane meet in a unique point.*

(b) ***(6 points)*** *Let* $p \in \mathcal{P}$*. Show that there exists a line* $\ell \in \mathcal{L}$ *such that* $p \not I \, \ell$*.*

(c) ***(7 points)*** *Use the previous part to show that every point of* $\Pi$ *is incident with at least three distinct lines of* $\Pi$*.*

*Hint: Choose a line* $\ell$ *not incident with* $p$*, choose three distinct points on* $\ell$*, and join them to* $p$*.*

(d) ***(6 points)*** *Deduce that every line of* $\Pi^*$ *is incident with at least three distinct points of* $\Pi^*$*.*

(e) ***(7 points)*** *Show that there exist three points of* $\Pi^*$ *which are not collinear in* $\Pi^*$*.*

*Hint: In terms of the original projective plane* $\Pi$*, this means finding three lines which are not all incident with one common point.*

(f) ***(7 points)*** *Show that any two distinct lines of* $\Pi^*$ *meet.*

*Hint: A line of* $\Pi^*$ *is a point of* $\Pi$*. Use the projective plane axiom saying that any two distinct points of* $\Pi$ *are incident with a unique line.*

*Conclude that* $\Pi^*$ *is a projective plane. This projective plane is called the* ***dual projective plane*** *of* $\Pi$*.*

**Problem 60** ***Weekly reflection (2 bonus points).***

*Please answer in a few complete sentences. Your responses will be* ***anonymous****.*

(a) *How confident do you feel with the concepts from the past week? If you feel confident, would you be interested in trying some of the challenge problems? If you do not feel confident yet, what is one strategy you could use to improve your understanding? For example, would you consider talking to a classmate, coming to office hours, or trying extra practice problems?*

(b) *Have you used AI tools at all during this class? If yes, how have you used them? If you use AI tools, what do you do to make sure homework and worksheets are still helping you practice and prepare for the final exam? If you are not using AI, explain why you are not using it.*

**Problem 61** ***Challenge problems (not graded).***

(a) ***(Every projective plane contains an affine plane †)*** *Let $\Pi = (\mathcal{P}, \mathcal{L}, I)$ be a projective plane. Fix a line $\ell \in \mathcal{L}$. Define*

$$\mathcal{P}' = \{p \in \mathcal{P} : p \not I\, \ell\} \qquad \text{and} \qquad \mathcal{L}' = \mathcal{L} \setminus \{\ell\}.$$

*We define incidence $I'$ by restriction of the old incidence relation: if $p' \in \mathcal{P}'$ and $m' \in \mathcal{L}'$, then*

$$p'\, I'\, m' \qquad \Longleftrightarrow \qquad p'\, I\, m'.$$

*Show that $(\mathcal{P}', \mathcal{L}', I')$ is an affine plane.*

(b) ***(Automorphism groups from two special lines †)*** *Consider the following incidence geometry. There are distinct points*

$$A_1, \ldots, A_n, B_1, \ldots, B_m.$$

*The points $A_1, \ldots, A_n$ lie on one line $\ell$, and the points $B_1, \ldots, B_m$ lie on another line $m$. In addition, for each pair $(i, j)$, we declare*

$$\{A_i, B_j\}$$

*to be a line.*

*Describe the automorphism group of this model.*

(i) *What happens when $n \neq m$?*

(ii) *What happens when $n = m$?*

(iii) *Can you find a model whose automorphism group has cardinality 144?*

(iv) *How much can you generalize this construction?*

# 31 Homework 7 Solutions

*Solutions to Homework 7.*

## Problem 1

(a) Let $f : \Sigma \to \Sigma$ and $g : \Sigma \to \Sigma$ be bijections. We show that $f \circ g$ is a bijection.

First, $f \circ g$ is injective. Suppose

$$(f \circ g)(x) = (f \circ g)(y).$$

Then $f(g(x)) = f(g(y))$. Since $f$ is injective, $g(x) = g(y)$. Since $g$ is injective, $x = y$.

Second, $f \circ g$ is surjective. Let $z \in \Sigma$. Since $f$ is surjective, there exists $w \in \Sigma$ such that $f(w) = z$. Since $g$ is surjective, there exists $x \in \Sigma$ such that $g(x) = w$. Then

$$(f \circ g)(x) = f(g(x)) = f(w) = z.$$

Hence $f \circ g$ is bijective.

(b) Let $\ell \in \mathcal{L}$. We prove that $F^{-1}(\ell)$ is a line.

By **I2**, choose two distinct points $p, q \in \ell$. Since $F$ is bijective, the points

$$F^{-1}(p), \qquad F^{-1}(q)$$

are distinct. By **I1**, there is a unique line $m \in \mathcal{L}$ incident with both $F^{-1}(p)$ and $F^{-1}(q)$.

Since $F$ is a transformation, $F(m)$ is a line. It contains

$$F(F^{-1}(p)) = p \qquad \text{and} \qquad F(F^{-1}(q)) = q.$$

The line $\ell$ also contains $p$ and $q$. By uniqueness in **I1**,

$$F(m) = \ell.$$

Therefore

$$m = F^{-1}(\ell).$$

Thus $F^{-1}(\ell)$ is a line. Since $\ell$ was arbitrary, $F^{-1}$ sends lines to lines. Also, $F^{-1}$ is a bijection. Hence $F^{-1}$ is a transformation.

# Problem 2

(a) The line $\{2,3,4\}$ is the only line containing three points. If $F$ is a transformation, then $F(\{2,3,4\})$ is a line. Since $F$ is injective, the three points $F(2), F(3), F(4)$ are distinct. Thus $F(\{2,3,4\})$ is a line with three points. The only such line is $\{2,3,4\}$. Therefore

$$F(\{2,3,4\}) = \{2,3,4\}.$$

(b) Since $F$ is a bijection and

$$F(\{2,3,4\}) = \{2,3,4\},$$

it sends the complement of $\{2,3,4\}$ to the complement of $\{2,3,4\}$. Hence

$$F(\{0,1\}) = \{0,1\}.$$

(c) We have $F = f(\sigma) \circ g(\eta)$, so we first apply $g(\eta)$, then $f(\sigma)$. Since $g(\eta)$ exchanges 0 and 1, and $f(\sigma)$ fixes 0 and 1,

$$F(0) = 1, \qquad F(1) = 0.$$

Since $g(\eta)$ fixes $2,3,4$, and $f(\sigma)$ sends $2 \mapsto 3$, $3 \mapsto 4$, and $4 \mapsto 2$,

$$F(2) = 3, \qquad F(3) = 4, \qquad F(4) = 2.$$

(d) By part (b),

$$F(\{0,1\}) = \{0,1\}.$$

If $F(0) = 0$, then injectivity forces

$$F(1) = 1.$$

(e) By part (d), $F(1) = 1$. By part (a), $F$ sends $\{2,3,4\}$ to itself. Since $F$ is bijective, its restriction to $\{2,3,4\}$ is a permutation. Define

$$\sigma(i) = F(i) \qquad \text{for } i \in \{2,3,4\}.$$

Then $\sigma$ is a permutation of $\{2,3,4\}$, and $F$ agrees with $f(\sigma)$ on all five points. Therefore

$$F = f(\sigma).$$

(f) By part (b),

$$F(\{0,1\}) = \{0,1\}.$$

If $F(0) = 1$, then injectivity forces

$$F(1) = 0.$$

(g) By part (f), $F(1) = 0$. By part (a), $F$ sends $\{2,3,4\}$ to itself. Since $F$ is bijective, its restriction to $\{2,3,4\}$ is a permutation. Define

$$\sigma(i) = F(i) \qquad \text{for } i \in \{2,3,4\}.$$

Let $\eta = (01)$. Then $g(\eta) \circ f(\sigma)$ sends

$$0 \mapsto 1, \qquad 1 \mapsto 0,$$

and for $i \in \{2,3,4\}$,

$$i \mapsto \sigma(i) = F(i).$$

Thus $F$ and $g(\eta) \circ f(\sigma)$ agree on every point, so

$$F = g(\eta) \circ f(\sigma).$$

(h) Let $F$ be a transformation. By part (b),

$$F(\{0,1\}) = \{0,1\}.$$

Hence either $F(0) = 0$ or $F(0) = 1$.

If $F(0) = 0$, then by part (e),

$$F = f(\sigma) = g(\mathrm{id}) \circ f(\sigma)$$

for some permutation $\sigma$ of $\{2,3,4\}$.

If $F(0) = 1$, then by part (g),

$$F = g(\eta) \circ f(\sigma),$$

where $\eta = (01)$ and $\sigma$ is a permutation of $\{2,3,4\}$.

Therefore every transformation is of the form

$$g(\eta) \circ f(\sigma),$$

where $\eta$ is a permutation of $\{0,1\}$ and $\sigma$ is a permutation of $\{2,3,4\}$.

Conversely, every map of this form is a transformation. Indeed, it sends $\{0,1\}$ to itself, sends $\{2,3,4\}$ to itself, and sends each two-point line of the form

$$\{a,i\}, \qquad a \in \{0,1\}, \quad i \in \{2,3,4\},$$

to another line of the same form:

$$\{\eta(a), \sigma(i)\}.$$

(i) There are

$$2! = 2$$

permutations of $\{0,1\}$, and

$$3! = 6$$

permutations of $\{2,3,4\}$. For each choice of $\eta$ and $\sigma$, the map $g(\eta) \circ f(\sigma)$ is a transformation, and different choices give different maps because the restrictions to $\{0,1\}$ and $\{2,3,4\}$ determine $\eta$ and $\sigma$. Therefore

$$|\operatorname{Aut}(\mathcal{M})| = 2! \cdot 3! = 2 \cdot 6 = 12.$$

# Problem 3

Let

$$\Pi = (\mathcal{P}, \mathcal{L}, I)$$

be a projective plane, and let

$$\Pi^* = (\mathcal{P}^*, \mathcal{L}^*, I^*)$$

be the dual incidence geometry.

(a) Let $\ell, m \in \mathcal{P}^*$ be two distinct points of $\Pi^*$. By definition, $\ell$ and $m$ are two distinct lines of $\Pi$.

By the elliptic parallel property in $\Pi$, the two lines $\ell$ and $m$ meet. Hence there exists a point $p \in \mathcal{P}$ such that

$$p \, I \, \ell \qquad \text{and} \qquad p \, I \, m.$$

Since points of $\Pi$ are lines of $\Pi^*$, this point $p$ is a line of $\Pi^*$. Moreover, by definition of the dual incidence relation,

$$\ell \, I^* \, p \qquad \text{and} \qquad m \, I^* \, p.$$

Thus $\ell$ and $m$ are incident with at least one common line of $\Pi^*$.

We now prove uniqueness. Suppose that $q \in \mathcal{L}^* = \mathcal{P}$ is another line of $\Pi^*$ incident with both $\ell$ and $m$. Then, in $\Pi$,

$$q \, I \, \ell \qquad \text{and} \qquad q \, I \, m.$$

If $q \neq p$, then the two distinct points $p$ and $q$ of $\Pi$ are both incident with the two lines $\ell$ and $m$. By axiom **I1** in $\Pi$, there is a unique line through $p$ and $q$, so $\ell = m$, contradicting the assumption that $\ell$ and $m$ are distinct.

Therefore $q = p$. Hence any two distinct points of $\Pi^*$ are incident with a unique line of $\Pi^*$.

(b) Let $p \in \mathcal{P}$. Choose three non-collinear points $A, B, C$ in $\Pi$. Let

$$\ell_{AB}, \qquad \ell_{AC}, \qquad \ell_{BC}$$

be the unique lines through $A, B$, through $A, C$, and through $B, C$, respectively.

We claim that at least one of these three lines is not incident with $p$. Suppose, for contradiction, that

$$p \, I \, \ell_{AB}, \qquad p \, I \, \ell_{AC}, \qquad p \, I \, \ell_{BC}.$$

Since $\ell_{AB}$ and $\ell_{AC}$ both contain $A$, and since $A, B, C$ are not collinear, the lines $\ell_{AB}$ and $\ell_{AC}$ are distinct. Therefore their unique intersection point is $A$. Since $p$ is incident with both $\ell_{AB}$ and $\ell_{AC}$, we get

$$p = A.$$

But $p$ is also incident with $\ell_{BC}$, so $A$ is incident with $\ell_{BC}$. This means that $A, B, C$ are all incident with $\ell_{BC}$, contradicting the fact that $A, B, C$ are non-collinear.

Therefore at least one of the lines

$$\ell_{AB}, \qquad \ell_{AC}, \qquad \ell_{BC}$$

is not incident with $p$.

(c) Let $p \in \mathcal{P}$. By part (b), choose a line $\ell \in \mathcal{L}$ such that

$$p \not I \ell.$$

Since $\Pi$ is a projective plane, the line $\ell$ contains at least three distinct points. Choose distinct points

$$a, b, c \in \ell.$$

Let

$$\ell_{pa}, \qquad \ell_{pb}, \qquad \ell_{pc}$$

be the unique lines through $p, a$, through $p, b$, and through $p, c$, respectively.

We claim that these three lines are distinct. Suppose, for example, that

$$\ell_{pa} = \ell_{pb}.$$

Then this line contains both $a$ and $b$. Since $\ell$ also contains $a$ and $b$, and since $a \neq b$, axiom **I1** implies

$$\ell_{pa} = \ell.$$

But $p$ is incident with $\ell_{pa}$, so this would imply

$$p \, I \, \ell,$$

contradicting the choice of $\ell$. Therefore $\ell_{pa} \neq \ell_{pb}$. The same argument shows that

$$\ell_{pa}, \qquad \ell_{pb}, \qquad \ell_{pc}$$

are pairwise distinct.

Each of these three distinct lines is incident with $p$. Hence every point $p \in \mathcal{P}$ is incident with at least three distinct lines.

(d) Let $p \in \mathcal{L}^*$ be a line of $\Pi^*$. By definition, $p$ is a point of $\Pi$. By part (c), there are at least three distinct lines of $\Pi$ incident with $p$. These lines of $\Pi$ are points of $\Pi^*$, and they are incident with the line $p$ of $\Pi^*$. Therefore every line of $\Pi^*$ is incident with at least three distinct points.

(e) Choose three non-collinear points $A, B, C$ in $\Pi$. Let

$$\ell_{AB}, \qquad \ell_{AC}, \qquad \ell_{BC}$$

be the unique lines through $A, B$, through $A, C$, and through $B, C$, respectively. These three lines of $\Pi$ are three points of $\Pi^*$.

We claim that these three points of $\Pi^*$ are not collinear. Suppose, for contradiction, that they are collinear in $\Pi^*$. Then there exists a line of $\Pi^*$ incident with all three of them. A line of $\Pi^*$ is a point $p \in \mathcal{P}$. Therefore, in $\Pi$, the three lines

$$\ell_{AB}, \qquad \ell_{AC}, \qquad \ell_{BC}$$

are all incident with $p$.

Since $\ell_{AB}$ and $\ell_{AC}$ are distinct and both contain $A$, their unique intersection point is $A$. Since $p$ is incident with both $\ell_{AB}$ and $\ell_{AC}$, we get

$$p = A.$$

But $p$ is also incident with $\ell_{BC}$, so $A$ is incident with $\ell_{BC}$. Hence $A, B, C$ are all incident with $\ell_{BC}$, contradicting the fact that $A, B, C$ are non-collinear.

Therefore $\ell_{AB}, \ell_{AC}, \ell_{BC}$ are three non-collinear points of $\Pi^*$.

(f) Let $p, q \in \mathcal{L}^*$ be two distinct lines of $\Pi^*$. By definition, $p$ and $q$ are two distinct points of $\Pi$.

Since $\Pi$ satisfies axiom **I1**, there exists a line $\ell \in \mathcal{L}$ incident with both $p$ and $q$. This line $\ell$ of $\Pi$ is a point of $\Pi^*$.

Since

$$p \, I \, \ell \qquad \text{and} \qquad q \, I \, \ell,$$

the definition of the dual incidence relation gives

$$\ell \, I^* \, p \qquad \text{and} \qquad \ell \, I^* \, q.$$

Thus the two lines $p$ and $q$ of $\Pi^*$ meet at the point $\ell$ of $\Pi^*$.

Parts (a), (d), (e), and (f) verify the projective plane axioms for $\Pi^*$. Therefore $\Pi^*$ is a projective plane.

# 32 Week 8 Lecture 2

*Groups of transformations, with a detailed activity on the transformations of the integer line.*

## 1. Reminder: transformations form a group

Last time, we studied transformations of finite incidence geometries. We used several ideas repeatedly. For example, if a line has two points, then its image under a transformation is again a line with two points. Also, if two lines meet at a point, then their images are two lines meeting at the image of that point.

Today, we study an infinite example.

First, recall that a **group** is a set $G$ together with an operation which takes two elements of $G$ and gives back another element of $G$. The operation must satisfy the following properties:

(a) **Closure:** if $g, h \in G$, then $g * h \in G$.

(b) **Neutral element:** there is an element $e \in G$ such that $g * e = e * g = g$ for every $g \in G$.

(c) **Inverse:** for every $g \in G$, there is an element $h \in G$ such that $g * h = h * g = e$.

(d) **Associativity:** parentheses do not matter.

In our case, if $\mathcal{M}$ is a model satisfying axioms **I1** and **I2**, then $\mathrm{Aut}(\mathcal{M})$, with composition, is a group.

In practice, this means:

- if $f$ and $g$ are transformations, then $f \circ g$ is a transformation;
- the identity map is a transformation;
- if $f$ is a transformation, then $f^{-1}$ is also a transformation;
- when composing transformations, parentheses do not matter.

## 2. Other examples of groups

**Example 6.** The set $\mathbb{Z}$, with addition, is a group. The neutral element is 0, because $k + 0 = 0 + k = k$. The inverse of $k$ is $-k$, because $k + (-k) = 0$. Addition is associative.

**Example 7.** The set $\mathbb{R}$, with addition, is also a group.

**Example 8.** The set $\mathbb{R} \setminus \{0\}$, with multiplication, is a group. The neutral element is 1, and the inverse of $x \neq 0$ is $1/x$.

**Example 9.** The set $\{1, -1\}$, with multiplication, is a group. The neutral element is 1, and each element is its own inverse.

**Example 10.** The set $\mathrm{GL}_2(\mathbb{R})$ of invertible $2 \times 2$ real matrices is a group under matrix multiplication. The neutral element is the identity matrix, and the inverse of a matrix $A$ is its inverse matrix $A^{-1}$.

**Question 62** *What other examples of groups do you know?*

## 3. The integer line

We now define a model $\mathcal{M}_{\mathbb{Z}}$. The points are

$$\mathcal{P} = \mathbb{Z}.$$

The lines are the subsets of the form

$$\{k, k+1\}, \qquad k \in \mathbb{Z}.$$

Thus $\{0,1\}$, $\{1,2\}$, $\{2,3\}$, $\{-1,0\}$, and $\{-2,-1\}$ are lines.

The goal of this activity is to determine

$$\mathrm{Aut}(\mathcal{M}_{\mathbb{Z}}).$$

A transformation of $\mathcal{M}_{\mathbb{Z}}$ is a bijection $f : \mathbb{Z} \to \mathbb{Z}$ such that, for every $k \in \mathbb{Z}$, the image

$$f(\{k, k+1\}) = \{f(k), f(k+1)\}$$

is again a line. In other words, $f(k)$ and $f(k+1)$ must be consecutive integers.

## 4. First examples and non-examples

**Question 63** *Is the map exchanging* 0 *and* 3*, and fixing every other integer, a transformation?*

No. Let $f$ be this map. Then

$$f(\{2,3\}) = \{f(2), f(3)\} = \{2,0\} = \{0,2\}.$$

But $\{0,2\}$ is not a line, because 0 and 2 are not consecutive. Hence $f$ is not a transformation.

**Question 64** *Is the map exchanging* 0 *and* 1*, and fixing every other integer, a transformation?*

No. Let $f$ be this map. Then

$$f(\{1,2\}) = \{f(1), f(2)\} = \{0,2\}.$$

Again, $\{0,2\}$ is not a line. Hence $f$ is not a transformation.

## 5. Translations

For every $\ell \in \mathbb{Z}$, define

$$T_\ell : \mathbb{Z} \to \mathbb{Z}, \qquad T_\ell(k) = k + \ell.$$

We claim that $T_\ell$ is a transformation.

First, $T_\ell$ is a bijection. Its inverse is $T_{-\ell}$, since $T_{-\ell}(T_\ell(k)) = k$.

Second, $T_\ell$ sends lines to lines. Indeed,

$$T_\ell(\{k, k+1\}) = \{T_\ell(k), T_\ell(k+1)\} = \{k+\ell, k+1+\ell\}.$$

This is the line

$$\{k+\ell, (k+\ell)+1\}.$$

Therefore $T_\ell$ is a transformation.

## 6. Reflections

Define

$$s : \mathbb{Z} \to \mathbb{Z}, \qquad s(k) = -k.$$

We claim that $s$ is a transformation.

First, $s$ is a bijection, and $s^{-1} = s$. Second,

$$s(\{k, k+1\}) = \{s(k), s(k+1)\} = \{-k, -k-1\} = \{-k-1, -k\}.$$

This is a line, because $-k-1$ and $-k$ are consecutive. Thus $s$ is a transformation.

More generally, for any $a \in \mathbb{Z}$, the map

$$k \longmapsto a - k$$

is a transformation. It is the composition $T_a \circ s$.

## 7. Normalizing an arbitrary transformation

Let $f \in \mathrm{Aut}(\mathcal{M}_{\mathbb{Z}})$. Suppose

$$f(0) = m.$$

We want to reduce to the case where 0 is fixed.

Since $T_{-m}$ is a transformation, the composition

$$g := T_{-m} \circ f$$

is also a transformation. Moreover,

$$g(0) = T_{-m}(f(0)) = T_{-m}(m) = 0.$$

Now we ask: where can $g(1)$ go?

Since $\{0, 1\}$ is a line, its image under $g$ must be a line:

$$g(\{0, 1\}) = \{g(0), g(1)\} = \{0, g(1)\}.$$

The only lines containing 0 are $\{-1, 0\}$ and $\{0, 1\}$. Therefore

$$g(1) = 1 \qquad \text{or} \qquad g(1) = -1.$$

Thus every transformation can be normalized so that 0 is fixed, and then there are two cases:

$$g(0) = 0, \quad g(1) = 1,$$

or

$$g(0) = 0, \quad g(1) = -1.$$

## 8. Case 1: $g(0) = 0$ and $g(1) = 1$

We now prove that if $g \in \mathrm{Aut}(\mathcal{M}_{\mathbb{Z}})$, $g(0) = 0$, and $g(1) = 1$, then

$$g = \mathrm{id}\,.$$

### Step 1: showing that $g(2) = 2$

Since $\{1, 2\}$ is a line, its image is a line:

$$g(\{1, 2\}) = \{g(1), g(2)\} = \{1, g(2)\}.$$

Thus $g(2)$ must be adjacent to 1. Therefore

$$g(2) = 0 \qquad \text{or} \qquad g(2) = 2.$$

But $g(0) = 0$, and $g$ is injective. Since $2 \neq 0$, we cannot have $g(2) = 0$. Therefore

$$g(2) = 2.$$

### Step 2: showing that $g(3) = 3$

Since $\{2, 3\}$ is a line, its image is a line:

$$g(\{2, 3\}) = \{g(2), g(3)\} = \{2, g(3)\}.$$

Thus $g(3)$ must be adjacent to 2. Therefore

$$g(3) = 1 \qquad \text{or} \qquad g(3) = 3.$$

But $g(1) = 1$, and $g$ is injective. Since $3 \neq 1$, we cannot have $g(3) = 1$. Therefore

$$g(3) = 3.$$

## 9. Reminder: what is induction?

We now want to show that $g(k) = k$ for all nonnegative integers $k$.

When a pattern repeats, a common proof strategy is **induction**. The idea is the following. Suppose we want to prove a statement $P(n)$ for every integer $n \geq 0$. It is enough to prove:

(a) **Base case:** $P(0)$ is true.

(b) **Induction step:** assuming $P(k)$ is true, prove $P(k + 1)$.

We will use a slightly stronger version, called **strong induction**. To prove $P(n)$ for all $n \geq 0$, we prove:

(a) **Base case:** $P(0)$ is true.

(b) **Strong induction step:** assuming $P(0), P(1), \ldots, P(k)$ are true, prove $P(k + 1)$.

## 10. Step 3: showing that $g(k) = k$ for all $k \geq 0$

We prove by strong induction that

$$g(k) = k$$

for all $k \geq 0$.

The base case is $k = 0$, and it is true because $g(0) = 0$.

Now assume that $g(0) = 0, g(1) = 1, \ldots, g(k) = k$, where $k \geq 1$. We want to prove that $g(k + 1) = k + 1$.

Since $\{k, k+1\}$ is a line, its image is a line:

$$g(\{k, k+1\}) = \{g(k), g(k+1)\} = \{k, g(k+1)\}.$$

Therefore $g(k+1)$ must be adjacent to $k$. Hence

$$g(k+1) = k-1 \qquad \text{or} \qquad g(k+1) = k+1.$$

But $g(k-1) = k-1$ by the induction hypothesis, and $g$ is injective. Since $k+1 \neq k-1$, we cannot have $g(k+1) = k-1$. Therefore

$$g(k+1) = k+1.$$

By strong induction,

$$g(k) = k$$

for every $k \geq 0$.

# 33 Homework 8

*Homework 8 for MATH 128A.*

**MATH 128A: Homework 8** **Due date: Thursday, May 28**

**Instructions.** You will submit your solutions on Gradescope, which you can access through our Canvas course page by clicking on the assignment.

You are welcome to submit your homework either handwritten or in LaTeX. If you choose to submit handwritten work, please use pen rather than pencil, and make sure that both your handwriting and your scans are as readable as possible in order to facilitate grading.

I encourage you to attend office hours if you would like hints on the homework or if you would like to discuss more advanced material. Also, you are highly encouraged to work in groups!

In this homework, we study the following incidence geometry $\mathcal{M}_{\mathbb{Z}^2}$. The points are elements of $\mathbb{Z}^2$, i.e. ordered pairs $(k, \ell)$, where $k, \ell \in \mathbb{Z}$. The lines are the horizontal and vertical edges

$$\{(k,\ell),(k+1,\ell)\} \qquad \text{and} \qquad \{(k,\ell),(k,\ell+1)\},$$

where $k, \ell \in \mathbb{Z}$. Two points are called **adjacent** if they lie on one common line.

**Problem 65** ***(40 points) Examples of transformations of the square grid.***

(a) ***(7 points)*** *For $(a, b) \in \mathbb{Z}^2$, define $T_{(a,b)} : \mathbb{Z}^2 \to \mathbb{Z}^2$ by*

$$T_{(a,b)}(k,\ell) = (k+a, \ell+b).$$

*Show that $T_{(a,b)}$ is a transformation of $\mathcal{M}_{\mathbb{Z}^2}$.*

(b) ***(7 points)*** *Show that the inverse of a translation is a translation. More precisely, show that*

$$T_{(a,b)}^{-1} = T_{(-a,-b)}.$$

(c) ***(7 points)*** *Define*

$$R_x(k,\ell) = (-k,\ell), \qquad R_y(k,\ell) = (k,-\ell).$$

*Show that $R_x$ and $R_y$ are transformations of $\mathcal{M}_{\mathbb{Z}^2}$.*

(d) ***(6 points)*** *Show that each of the transformations*

$$\text{id}, \qquad R_x, \qquad R_y, \qquad R_x \circ R_y$$

*is its own inverse.*

(e) ***(7 points)*** *Define $S(k, \ell) = (\ell, k)$. Show that $S$ is a transformation of $\mathcal{M}_{\mathbb{Z}^2}$.*

(f) ***(6 points)*** *Show that $S$ is its own inverse, i.e.*

$$S^{-1} = S.$$

**Problem 66** ***(35 points) Normalizing a transformation.***

*Let $F$ be a transformation of $\mathcal{M}_{\mathbb{Z}^2}$.*

(a) ***(7 points)*** *Let $p \in \mathbb{Z}^2$. Show that $F$ sends the four points adjacent to $p$ to the four points adjacent to $F(p)$.*

*Hint: Each point has exactly four adjacent points. Since $F$ is injective and sends lines to lines, what happens to the four neighbors of $p$?*

(b) ***(7 points)*** *Show that, by composing $F$ on the left with a translation, we may assume that*

$$F(0, 0) = (0, 0).$$

*Hint: If $F(0, 0) = (a, b)$, compose $F$ with $T_{(-a,-b)}$.*

(c) ***(7 points)*** *Now assume that $F(0, 0) = (0, 0)$. Show that $F(1, 0)$ must be one of*

$$(1, 0), \quad (-1, 0), \quad (0, 1), \quad (0, -1).$$

(d) ***(7 points)*** *Show that, by composing $F$ on the left with transformations chosen from*

$$\mathrm{id}, \quad R_x, \quad R_y, \quad R_x \circ R_y, \quad S, \quad R_x \circ S, \quad R_y \circ S, \quad R_x \circ R_y \circ S,$$

*we may assume that*

$$F(0, 0) = (0, 0) \qquad \text{and} \qquad F(1, 0) = (1, 0).$$

(e) ***(7 points)*** *Now assume that $F(0, 0) = (0, 0)$ and $F(1, 0) = (1, 0)$. Show that*

$$F(0, 1) = (0, 1) \qquad \text{or} \qquad F(0, 1) = (0, -1).$$

*Then explain why, by composing with $R_y$ if necessary, we may assume that*

$$F(0, 0) = (0, 0), \qquad F(1, 0) = (1, 0), \qquad F(0, 1) = (0, 1).$$

*Hint: First use part (a). To rule out $F(0, 1) = (-1, 0)$, use the fact that $(1, 0)$ and $(0, 1)$ have two common adjacent points, while $(1, 0)$ and $(-1, 0)$ do not.*

**Problem 67** ***(25 points) The normalized transformation fixes part of the grid.***

*Let $F$ be a transformation of $\mathcal{M}_{\mathbb{Z}^2}$. Assume that*

$$F(0,0) = (0,0), \qquad F(1,0) = (1,0), \qquad F(0,1) = (0,1).$$

(a) ***(8 points)*** *Show that $F(1,1) = (1,1)$.*

*Hint: The point $(1,1)$ is adjacent to both $(1,0)$ and $(0,1)$. It is the unique point, different from $(0,0)$, with this property.*

(b) ***(8 points)*** *Show that $F(-1,0) = (-1,0)$ and $F(0,-1) = (0,-1)$.*

*Hint: By Problem 2(a), the remaining neighbors of $(0,0)$ must be sent to the remaining neighbors of $(0,0)$. To rule out switching them, use the fact that $(1,0)$ and $(0,-1)$ have two common adjacent points, while $(1,0)$ and $(-1,0)$ do not.*

(c) ***(9 points)*** *Show, by induction, that*

$$F(k,0) = (k,0)$$

*for every integer $k \geq 0$.*

*Hint: This is similar to the proof for the integer line from lecture, but there is one extra difficulty. Once you know $F(k-1,0) = (k-1,0)$ and $F(k,0) = (k,0)$, use the line $\{(k,0),(k+1,0)\}$ to restrict the possible values of $F(k+1,0)$. Then use injectivity and compare the number of common adjacent points to rule out the wrong possibilities.*

**Remark 9.** The remaining part of the proof, namely showing that $F$ fixes every point of $\mathbb{Z}^2$, will not be graded. However, I highly recommend thinking about it. A scaffolded version appears in the challenge problems.

**Remark 10.** The previous problems, together with Challenge Problem (a), prove the following classification.

Every transformation of $\mathcal{M}_{\mathbb{Z}^2}$ can be written as

$$T \circ r \circ s,$$

where $T$ is a translation, $r \in \{\mathrm{id}, R_x, R_y, R_x \circ R_y\}$, and $s \in \{\mathrm{id}, S\}$.

Here is how the conclusion works. Let $F$ be a transformation of $\mathcal{M}_{\mathbb{Z}^2}$. By Problem 2, we can choose a translation $T$, an element

$$r \in \{\mathrm{id}, R_x, R_y, R_x \circ R_y\},$$

and an element

$$s \in \{\mathrm{id}, S\}$$

such that

$$s \circ r \circ T \circ F$$

sends $(0,0)$ to $(0,0)$, sends $(1,0)$ to $(1,0)$, and sends $(0,1)$ to $(0,1)$.

The challenge problem below completes the proof that any transformation fixing these three points must be the identity. Assuming this result, we get

$$s \circ r \circ T \circ F = \mathrm{id}\,.$$

Now compose on the left by $s^{-1}$, then by $r^{-1}$, then by $T^{-1}$. We obtain

$$F = T^{-1} \circ r^{-1} \circ s^{-1}.$$

By Problem 1, the inverse of a translation is a translation. Also, each element of

$$\{\mathrm{id}, R_x, R_y, R_x \circ R_y\}$$

is its own inverse, and $S^{-1} = S$. Therefore $T^{-1}$ is another translation, $r^{-1}$ is again one of

$$\mathrm{id}, R_x, R_y, R_x \circ R_y,$$

and $s^{-1}$ is again one of

$$\mathrm{id}, S.$$

Thus $F$ can be written in the desired form

$$F = T' \circ r' \circ s',$$

where $T'$ is a translation, $r' \in \{\mathrm{id}, R_x, R_y, R_x \circ R_y\}$, and $s' \in \{\mathrm{id}, S\}$.

In other words, every transformation of the square grid is obtained by combining a translation, possibly a reflection in one or both coordinate axes, and possibly a coordinate swap.

**Problem 68** ***Weekly reflection (2 bonus points).***

*Please answer in a few complete sentences. Your responses will be* ***anonymous****.*

(a) *Have you worked in groups on your homework? Are there any barriers that make it difficult for you to work in groups?*

(b) *Did you enjoy the ungraded quizzes? If so, would you like to see more of them?*

(c) *Would you like to have a recap class, even if it meant that we would cover less new content?*

**Problem 69** ***Challenge problems (not graded).***

(a) ***(Completing the proof for the square grid †)*** *Continue Problem 3 and show that the normalized transformation is the identity.*

*More precisely, let $F$ be a transformation of $\mathcal{M}_{\mathbb{Z}^2}$, and assume that*

$$F(0,0) = (0,0), \qquad F(1,0) = (1,0), \qquad F(0,1) = (0,1).$$

*In Problem 3, you proved some first consequences of these assumptions. The goal is to prove that*

$$F(k,\ell) = (k,\ell)$$

*for every $(k,\ell) \in \mathbb{Z}^2$.*

*(i) Show, using arguments similar to Problem 3(c), that $F$ fixes the negative part of the horizontal axis and the entire vertical axis.*

*(ii) Show that $F(k,\ell) = (k,\ell)$ for every $k,\ell \geq 0$.*

*Suggested strategy: Use induction on $k+\ell$. The induction hypothesis should be: $F(a,b) = (a,b)$ for every point $(a,b)$ with $a,b \geq 0$ and $a+b \leq n$. Then prove the result for a point $(k,\ell)$ with $k,\ell \geq 0$ and $k+\ell = n+1$.*

*Hint: If $k,\ell \geq 1$, then $(k,\ell)$ is the unique point, different from $(k-1,\ell-1)$, adjacent to both $(k-1,\ell)$ and $(k,\ell-1)$.*

*(iii) Adapt the same argument to the other three quadrants.*

*(iv) Conclude that $F = \mathrm{id}$.*

(b) ***(Higher-dimensional square grids †)*** *Consider the model with point set $\mathbb{Z}^n$, where lines are edges between points that differ by 1 in exactly one coordinate. Describe the transformations of this model.*

(c) ***(Lattice lines in $\mathbb{Z}^2$ †)*** *Consider the model with point set $\mathbb{Z}^2$, where lines are subsets of the form*

$$v + \mathbb{Z}u = \{v + ku : k \in \mathbb{Z}\},$$

*where $v \in \mathbb{Z}^2$ and $u = (a,b) \in \mathbb{Z}^2$ satisfies $a\mathbb{Z} + b\mathbb{Z} = \mathbb{Z}$. Equivalently, $u$ is primitive, meaning $\gcd(a,b) = 1$. Can you describe the transformations of this model?*

(d) ***(A model with no nontrivial transformations †)*** *Can you construct a model satisfying axioms **I1** and **I2** whose only transformation is the identity?*

# 34 Homework 8 Solutions

*Solutions to Homework 8 for MATH 128A.*

**MATH 128A: Homework 8 Solutions**

In this homework, we study the following incidence geometry $\mathcal{M}_{\mathbb{Z}^2}$. The points are elements of $\mathbb{Z}^2$, i.e. ordered pairs $(k, \ell)$, where $k, \ell \in \mathbb{Z}$. The lines are the horizontal and vertical edges

$$\{(k,\ell),(k+1,\ell)\} \qquad \text{and} \qquad \{(k,\ell),(k,\ell+1)\},$$

where $k, \ell \in \mathbb{Z}$. Two points are called **adjacent** if they lie on one common line.

**Problem 70** ***(40 points) Examples of transformations of the square grid.***

(a) ***(7 points)*** *For $(a, b) \in \mathbb{Z}^2$, define $T_{(a,b)} : \mathbb{Z}^2 \to \mathbb{Z}^2$ by*

$$T_{(a,b)}(k,\ell) = (k+a, \ell+b).$$

*Show that $T_{(a,b)}$ is a transformation of $\mathcal{M}_{\mathbb{Z}^2}$.*

(b) ***(7 points)*** *Show that the inverse of a translation is a translation. More precisely, show that*

$$T_{(a,b)}^{-1} = T_{(-a,-b)}.$$

(c) ***(7 points)*** *Define*

$$R_x(k,\ell) = (-k,\ell), \qquad R_y(k,\ell) = (k,-\ell).$$

*Show that $R_x$ and $R_y$ are transformations of $\mathcal{M}_{\mathbb{Z}^2}$.*

(d) ***(6 points)*** *Show that each of the transformations*

$$\mathrm{id}, \qquad R_x, \qquad R_y, \qquad R_x \circ R_y$$

*is its own inverse.*

(e) ***(7 points)*** *Define $S(k, \ell) = (\ell, k)$. Show that $S$ is a transformation of $\mathcal{M}_{\mathbb{Z}^2}$.*

(f) ***(6 points)*** *Show that $S$ is its own inverse, i.e.*

$$S^{-1} = S.$$

**Solution:** **(a)** First, $T_{(a,b)}$ is a bijection. Indeed, for every $(k,\ell) \in \mathbb{Z}^2$,

$$T_{(-a,-b)}(T_{(a,b)}(k,\ell)) = T_{(-a,-b)}(k+a, \ell+b) = (k,\ell),$$

and similarly

$$T_{(a,b)}(T_{(-a,-b)}(k,\ell)) = T_{(a,b)}(k-a, \ell-b) = (k,\ell).$$

Therefore $T_{(a,b)}$ is a bijection, with inverse $T_{(-a,-b)}$.

We now show that $T_{(a,b)}$ sends lines to lines. If

$$L = \{(k,\ell), (k+1,\ell)\}$$

is a horizontal line, then

$$T_{(a,b)}(L) = \{(k+a, \ell+b), (k+1+a, \ell+b)\},$$

which is again a horizontal line. If

$$L = \{(k,\ell), (k,\ell+1)\}$$

is a vertical line, then

$$T_{(a,b)}(L) = \{(k+a, \ell+b), (k+a, \ell+1+b)\},$$

which is again a vertical line. Therefore $T_{(a,b)}$ is a transformation.

**(b)** See previous question.

**(c)** We first consider $R_x$. The map $R_x(k,\ell) = (-k,\ell)$ is a bijection. Indeed, for every $(k,\ell) \in \mathbb{Z}^2$,

$$R_x(R_x(k,\ell)) = R_x(-k,\ell) = (k,\ell).$$

Therefore $R_x^{-1} = R_x$, so $R_x$ is a bijection.

If

$$L = \{(k,\ell), (k+1,\ell)\},$$

then

$$R_x(L) = \{(-k,\ell), (-k-1,\ell)\} = \{(-k-1,\ell), (-k,\ell)\},$$

which is a horizontal line. If

$$L = \{(k,\ell), (k,\ell+1)\},$$

then

$$R_x(L) = \{(-k,\ell), (-k,\ell+1)\},$$

which is a vertical line. Therefore $R_x$ is a transformation.

Similarly, $R_y(k,\ell) = (k,-\ell)$ is a bijection. It sends horizontal lines to horizontal lines and vertical lines to vertical lines:

$$\{(k,\ell), (k+1,\ell)\} \longmapsto \{(k,-\ell), (k+1,-\ell)\},$$

and

$$\{(k,\ell),(k,\ell+1)\} \longmapsto \{(k,-\ell),(k,-\ell-1)\}.$$

Therefore $R_y$ is also a transformation.

**(d)** See previous question.

**(e)** The map

$$S(k,\ell) = (\ell,k)$$

is a bijection. Indeed, for every $(k,\ell) \in \mathbb{Z}^2$,

$$S(S(k,\ell)) = S(\ell,k) = (k,\ell).$$

Therefore $S^{-1} = S$, so $S$ is a bijection.

If

$$L = \{(k,\ell),(k+1,\ell)\}$$

is horizontal, then

$$S(L) = \{(\ell,k),(\ell,k+1)\},$$

which is vertical. If

$$L = \{(k,\ell),(k,\ell+1)\}$$

is vertical, then

$$S(L) = \{(\ell,k),(\ell+1,k)\},$$

which is horizontal. Hence $S$ sends lines to lines. Therefore $S$ is a transformation.

**(f)** See previous question.

**Problem 71** ***(35 points) Normalizing a transformation.***

*Let $F$ be a transformation of $\mathcal{M}_{\mathbb{Z}^2}$.*

(a) ***(7 points)*** *Let $p \in \mathbb{Z}^2$. Show that $F$ sends the four points adjacent to $p$ to the four points adjacent to $F(p)$.*

(b) ***(7 points)*** *Show that, by composing $F$ on the left with a translation, we may assume that*

$$F(0,0) = (0,0).$$

(c) ***(7 points)*** *Now assume that $F(0,0) = (0,0)$. Show that $F(1,0)$ must be one of*

$$(1,0), \quad (-1,0), \quad (0,1), \quad (0,-1).$$

(d) ***(7 points)*** *Show that, by composing $F$ on the left with transformations chosen from*

$$\mathrm{id}, \quad R_x, \quad R_y, \quad R_x \circ R_y, \quad S, \quad R_x \circ S, \quad R_y \circ S, \quad R_x \circ R_y \circ S,$$

*we may assume that*

$$F(0,0) = (0,0) \qquad \textit{and} \qquad F(1,0) = (1,0).$$

(e) ***(7 points)*** *Now assume that* $F(0,0) = (0,0)$ *and* $F(1,0) = (1,0)$. *Show that*

$$F(0,1) = (0,1) \qquad \text{or} \qquad F(0,1) = (0,-1).$$

*Then explain why, by composing with* $R_y$ *if necessary, we may assume that*

$$F(0,0) = (0,0), \qquad F(1,0) = (1,0), \qquad F(0,1) = (0,1).$$

**Solution:** **(a)** Let $p \in \mathbb{Z}^2$. If $q$ is adjacent to $p$, then $\{p,q\}$ is a line. Since $F$ is a transformation,

$$F(\{p,q\}) = \{F(p), F(q)\}$$

is a line. Therefore $F(q)$ is adjacent to $F(p)$.

The point $p$ has exactly four adjacent points. Since $F$ is injective, the images of these four points are four distinct adjacent points of $F(p)$. But $F(p)$ also has exactly four adjacent points. Therefore $F$ sends the four points adjacent to $p$ exactly to the four points adjacent to $F(p)$.

We will also use the following consequence. If two points $p, q$ have $r$ common adjacent points, then $F(p)$ and $F(q)$ also have $r$ common adjacent points. Indeed, by the previous paragraph, $F$ sends the set of neighbors of $p$ bijectively to the set of neighbors of $F(p)$, and similarly for $q$.

**(b)** Suppose

$$F(0,0) = (a,b).$$

Define

$$G := T_{(-a,-b)} \circ F.$$

Since translations are transformations and the composition of transformations is a transformation, $G$ is a transformation. Moreover,

$$G(0,0) = T_{(-a,-b)}(F(0,0)) = T_{(-a,-b)}(a,b) = (0,0).$$

Thus, by composing $F$ on the left with a translation, we may assume that

$$F(0,0) = (0,0).$$

**(c)** Assume

$$F(0,0) = (0,0).$$

Since $(1,0)$ is adjacent to $(0,0)$, part (a) implies that $F(1,0)$ is adjacent to $F(0,0) = (0,0)$. The four points adjacent to $(0,0)$ are

$$(1,0), \quad (-1,0), \quad (0,1), \quad (0,-1).$$

Therefore

$$F(1,0) \in \{(1,0), (-1,0), (0,1), (0,-1)\}.$$

**(d)** By part (b), after composing on the left with a translation, we may assume that

$$F(0,0) = (0,0).$$

By part (c),

$$F(1,0) \in \{(1,0), (-1,0), (0,1), (0,-1)\}.$$

We now compose on the left with one of the listed transformations, all of which fix $(0,0)$, to send $F(1,0)$ to $(1,0)$.

If $F(1,0) = (1,0)$, use id. If $F(1,0) = (-1,0)$, use $R_x$. If $F(1,0) = (0,1)$, use $S$. If $F(1,0) = (0,-1)$, use $R_x \circ S$, since

$$(R_x \circ S)(0,-1) = R_x(-1,0) = (1,0).$$

Therefore, after composing $F$ on the left with transformations from the given list, we may assume that

$$F(0,0) = (0,0) \qquad \text{and} \qquad F(1,0) = (1,0).$$

**(e)** Assume

$$F(0,0) = (0,0), \qquad F(1,0) = (1,0).$$

Since $(0,1)$ is adjacent to $(0,0)$, part (a) implies that $F(0,1)$ is adjacent to $(0,0)$. Hence

$$F(0,1) \in \{(1,0), (-1,0), (0,1), (0,-1)\}.$$

Since $F(1,0) = (1,0)$ and $F$ is injective, we cannot have

$$F(0,1) = (1,0).$$

We now rule out $F(0,1) = (-1,0)$. The points $(1,0)$ and $(0,1)$ have two common adjacent points:

$$(0,0) \qquad \text{and} \qquad (1,1).$$

By part (a), $F(1,0)$ and $F(0,1)$ must also have two common adjacent points. But if $F(0,1) = (-1,0)$, then $F(1,0) = (1,0)$, and the points $(1,0)$ and $(-1,0)$ have only one common adjacent point, namely $(0,0)$. This is impossible.

Therefore

$$F(0,1) = (0,1) \qquad \text{or} \qquad F(0,1) = (0,-1).$$

If $F(0,1) = (0,1)$, we do nothing. If $F(0,1) = (0,-1)$, we compose on the left with $R_y$. Since

$$R_y(0,0) = (0,0), \qquad R_y(1,0) = (1,0), \qquad R_y(0,-1) = (0,1),$$

we may assume

$$F(0,0) = (0,0), \qquad F(1,0) = (1,0), \qquad F(0,1) = (0,1).$$

**Problem 72** ***(25 points) The normalized transformation fixes part of the grid.***

*Let $F$ be a transformation of $\mathcal{M}_{\mathbb{Z}^2}$. Assume that*

$$F(0,0) = (0,0), \qquad F(1,0) = (1,0), \qquad F(0,1) = (0,1).$$

(a) ***(8 points)*** *Show that $F(1,1) = (1,1)$.*

(b) ***(8 points)*** *Show that $F(-1,0) = (-1,0)$ and $F(0,-1) = (0,-1)$.*

(c) ***(9 points)*** *Show, by induction, that*

$$F(k,0) = (k,0)$$

*for every integer $k \geq 0$.*

**Solution:** Assume throughout that

$$F(0,0) = (0,0), \qquad F(1,0) = (1,0), \qquad F(0,1) = (0,1).$$

**(a)** The point $(1,1)$ is adjacent to both $(1,0)$ and $(0,1)$. Therefore $F(1,1)$ is adjacent to both

$$F(1,0) = (1,0) \qquad \text{and} \qquad F(0,1) = (0,1).$$

The common adjacent points of $(1,0)$ and $(0,1)$ are exactly

$$(0,0) \qquad \text{and} \qquad (1,1).$$

Hence

$$F(1,1) = (0,0) \qquad \text{or} \qquad F(1,1) = (1,1).$$

Since $F(0,0) = (0,0)$ and $F$ is injective, we cannot have $F(1,1) = (0,0)$. Therefore

$$F(1,1) = (1,1).$$

**(b)** By Problem 2(a), $F$ sends the four neighbors of $(0,0)$ to the four neighbors of $(0,0)$. Since

$$F(1,0) = (1,0) \qquad \text{and} \qquad F(0,1) = (0,1),$$

the remaining two neighbors must be sent to the remaining two neighbors:

$$F(\{(-1,0),(0,-1)\}) = \{(-1,0),(0,-1)\}.$$

Thus either

$$F(-1,0) = (-1,0), \quad F(0,-1) = (0,-1),$$

or

$$F(-1,0) = (0,-1), \quad F(0,-1) = (-1,0).$$

We rule out the second possibility. The points $(1,0)$ and $(0,-1)$ have two common adjacent points:

$$(0,0) \qquad \text{and} \qquad (1,-1).$$

Therefore their images must also have two common adjacent points. If $F(0,-1) = (-1,0)$, then the images would be

$$F(1,0) = (1,0) \qquad \text{and} \qquad F(0,-1) = (-1,0).$$

But $(1,0)$ and $(-1,0)$ have only one common adjacent point, namely $(0,0)$. This is impossible.

Therefore

$$F(-1,0) = (-1,0) \qquad \text{and} \qquad F(0,-1) = (0,-1).$$

**(c)** We prove by induction that

$$F(k,0) = (k,0)$$

for every integer $k \geq 0$.

The base cases $k = 0$ and $k = 1$ hold by assumption.

Assume that $n \geq 1$, and suppose that

$$F(k,0) = (k,0)$$

for every $0 \leq k \leq n$. We prove that

$$F(n+1,0) = (n+1,0).$$

Since $(n+1,0)$ is adjacent to $(n,0)$, its image $F(n+1,0)$ is adjacent to

$$F(n,0) = (n,0).$$

Hence

$$F(n+1,0) \in \{(n-1,0),(n+1,0),(n,1),(n,-1)\}.$$

The first possibility is impossible, because $F(n-1,0) = (n-1,0)$ by the induction hypothesis and $F$ is injective.

We now rule out the possibilities $(n,1)$ and $(n,-1)$. The original points

$$(n-1,0) \qquad \text{and} \qquad (n+1,0)$$

have exactly one common adjacent point, namely $(n,0)$. Therefore their images must also have exactly one common adjacent point.

If

$$F(n+1,0) = (n,1),$$

then the two image points

$$F(n-1,0) = (n-1,0) \qquad \text{and} \qquad F(n+1,0) = (n,1)$$

have two common adjacent points:

$$(n,0) \qquad \text{and} \qquad (n-1,1).$$

This is impossible.

Similarly, if

$$F(n+1,0) = (n,-1),$$

then the two image points

$$(n-1,0) \qquad \text{and} \qquad (n,-1)$$

have two common adjacent points:

$$(n,0) \qquad \text{and} \qquad (n-1,-1).$$

This is also impossible.

Therefore the only remaining possibility is

$$F(n+1,0) = (n+1,0).$$

By induction,

$$F(k,0) = (k,0)$$

for every $k \geq 0$.

# 35 Week 9 Lecture 1

*Finishing the transformations of the integer line, then introducing Pappus' theorem and the reduction from the affine plane to the projective plane.*

## 1. Finishing the integer line

Recall the model $\mathcal{M}_{\mathbb{Z}}$. Its points are the integers,

$$\mathcal{P} = \mathbb{Z},$$

and its lines are the subsets

$$\{k, k+1\}, \qquad k \in \mathbb{Z}.$$

Last time, we introduced the translations

$$T_\ell(k) = k + \ell, \qquad \ell \in \mathbb{Z},$$

and the reflection

$$s(k) = -k.$$

Each $T_\ell$ is a transformation, and $s$ is also a transformation. Since transformations form a group under composition, every map of the form

$$T_\ell \circ S, \qquad \ell \in \mathbb{Z}, \quad S \in \{\mathrm{id}, s\},$$

is a transformation.

The goal is to prove that these are all the transformations of $\mathcal{M}_{\mathbb{Z}}$.

## 2. Normalization

Let

$$f \in \mathrm{Aut}(\mathcal{M}_{\mathbb{Z}}).$$

Set

$$L = f(0).$$

We want to simplify $f$ by composing it with transformations that we already understand. Define

$$g := T_{-L} \circ f.$$

Then

$$g(0) = T_{-L}(f(0)) = T_{-L}(L) = 0.$$

Since $\{0,1\}$ is a line and $g$ is a transformation,

$$g(\{0,1\}) = \{g(0), g(1)\} = \{0, g(1)\}$$

is a line. The only lines containing 0 are $\{-1,0\}$ and $\{0,1\}$. Hence

$$g(1) = 1 \qquad \text{or} \qquad g(1) = -1.$$

Choose $S \in \{\mathrm{id}, s\}$ as follows:

$$S = \begin{cases} \mathrm{id}, & \text{if } g(1) = 1, \\ s, & \text{if } g(1) = -1. \end{cases}$$

Define

$$h := S \circ g = S \circ T_{-L} \circ f.$$

Then

$$h(0) = 0 \qquad \text{and} \qquad h(1) = 1.$$

This is the normalization step: starting from an arbitrary transformation $f$, we constructed a simpler transformation $h$ fixing both 0 and 1.

## 3. The normalized transformation is the identity

We now prove that if $h \in \mathrm{Aut}(\mathcal{M}_{\mathbb{Z}})$, $h(0) = 0$, and $h(1) = 1$, then

$$h = \mathrm{id}\,.$$

From last time, we already proved by strong induction that

$$h(k) = k$$

for every $k \geq 0$. It remains to prove the same statement for negative integers.

First, since $\{-1,0\}$ is a line,

$$h(\{-1,0\}) = \{h(-1), h(0)\} = \{h(-1), 0\}$$

is a line. Thus $h(-1) = 1$ or $h(-1) = -1$. But $h(1) = 1$, and $h$ is injective. Since $-1 \neq 1$, we cannot have $h(-1) = 1$. Therefore

$$h(-1) = -1.$$

We now prove by strong induction that

$$h(-k) = -k$$

for every $k \geq 0$. We already know this for $k = 0$ and $k = 1$.

Assume that $n \geq 1$, and suppose that

$$h(0) = 0, \quad h(-1) = -1, \quad \ldots, \quad h(-n) = -n.$$

We prove that

$$h(-(n+1)) = -(n+1).$$

Since $\{-(n+1), -n\}$ is a line, its image is a line:

$$h(\{-(n+1), -n\}) = \{h(-(n+1)), h(-n)\} = \{h(-(n+1)), -n\}.$$

Thus $h(-(n+1))$ is adjacent to $-n$. Hence

$$h(-(n+1)) = -(n+1) \qquad \text{or} \qquad h(-(n+1)) = -(n-1).$$

But $h(-(n-1)) = -(n-1)$ by the induction hypothesis, and $h$ is injective. Since $-(n+1) \neq -(n-1)$, we cannot have

$$h(-(n+1)) = -(n-1).$$

Therefore

$$h(-(n+1)) = -(n+1).$$

By strong induction,

$$h(-k) = -k$$

for every $k \geq 0$. Combining this with the result for nonnegative integers, we get

$$h(k) = k$$

for every $k \in \mathbb{Z}$. Hence

$$h = \mathrm{id}\,.$$

## 4. Classification of the transformations

Recall that

$$h = S \circ T_{-L} \circ f.$$

Since $h = \mathrm{id}$, we have

$$S \circ T_{-L} \circ f = \mathrm{id}\,.$$

We now solve this equation for $f$. Composing on the left by $S^{-1}$, we get

$$T_{-L} \circ f = S^{-1}.$$

Composing on the left by $T_L$, we get

$$f = T_L \circ S^{-1}.$$

Now $S$ is either id or $s$. In both cases,

$$S^{-1} = S.$$

Indeed, $\mathrm{id}^{-1} = \mathrm{id}$, and

$$s(s(k)) = s(-k) = k,$$

so $s^{-1} = s$. Therefore

$$f = T_L \circ S, \qquad S \in \{\mathrm{id}, s\}.$$

Thus the transformations of $\mathcal{M}_{\mathbb{Z}}$ are exactly the maps

$$k \longmapsto k + L \qquad \text{and} \qquad k \longmapsto -k + L, \qquad L \in \mathbb{Z}.$$

## 5. Pappus' theorem in the affine plane

We now turn to a classical theorem of incidence geometry.
Let $\mathbb{F}$ be $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, or more generally any field. We work in the affine plane

$$\mathbb{A}^2(\mathbb{F}).$$

**Theorem (Affine Pappus).** Let $\ell$ and $m$ be two distinct lines in $\mathbb{A}^2(\mathbb{F})$. Let

$$A, B, C \in \ell$$

be three distinct points, and let

$$A', B', C' \in m$$

be three distinct points. Assume the following non-degeneracy conditions:

- none of the six points $A, B, C, A', B', C'$ lies on $\ell \cap m$;
- the three intersection points

  $$P = AB' \cap A'B, \qquad Q = AC' \cap A'C, \qquad R = BC' \cap B'C$$

  exist in the affine plane.

Then $P, Q, R$ are collinear.

**Question 73** *Construct this configuration in GeoGebra or Cinderella. Move the points $A, B, C, A', B', C'$. What seems to happen to $P, Q, R$?*

The condition that $P, Q, R$ exist is important in the affine plane. For example, it may happen that the lines $AB'$ and $A'B$ are parallel. In that case, the point

$$P = AB' \cap A'B$$

does not exist in the affine plane.

However, if $AB'$ and $A'B$ are parallel, then they meet at a point at infinity in the projective completion. This suggests that the theorem should become cleaner in the projective plane.

## 6. Why pass to the projective plane?

There are two main reasons to prove Pappus' theorem projectively.

First, in the affine plane, there are different cases. The two lines $\ell$ and $m$ may be parallel or they may meet. A direct affine proof has to treat these cases separately.

Second, some of the intersection points $P, Q, R$ may fail to exist in the affine plane because the corresponding lines are parallel. In the projective plane, distinct lines always meet.

Thus the projective plane removes both difficulties: there is no parallel case, and the missing affine intersections become points at infinity.

## 7. From affine to projective

Recall the standard embedding

$$\varphi : \mathbb{A}^2(\mathbb{F}) \longrightarrow \mathbb{P}^2(\mathbb{F}), \qquad \varphi(x, y) = [x : y : 1].$$

This map lets us view the affine plane as the part of the projective plane where the last homogeneous coordinate is nonzero.

A key property is the following.

**Collinearity is preserved.** For points $P, Q, R \in \mathbb{A}^2(\mathbb{F})$,

$$P, Q, R \text{ are collinear in } \mathbb{A}^2(\mathbb{F})$$

if and only if

$$\varphi(P), \varphi(Q), \varphi(R) \text{ are collinear in } \mathbb{P}^2(\mathbb{F}).$$

Indeed, an affine line has an equation

$$ax + by + c = 0,$$

and its projective completion has equation

$$aX + bY + cZ = 0.$$

If $(x, y)$ lies on the affine line, then $[x : y : 1]$ lies on the corresponding projective line. Conversely, a projective line containing points with $Z = 1$ gives an affine line after setting $Z = 1$.

Therefore, to prove affine Pappus, it is enough to prove a projective version of Pappus.

## 8. Pappus' theorem in the projective plane

**Theorem (Projective Pappus).** Let $\ell$ and $m$ be two distinct lines in $\mathbb{P}^2(\mathbb{F})$. Let

$$A, B, C \in \ell$$

be three distinct points, and let

$$A', B', C' \in m$$

be three distinct points. Assume that none of these six points is equal to the point

$$\ell \cap m.$$

Define

$$P = AB' \cap A'B, \qquad Q = AC' \cap A'C, \qquad R = BC' \cap B'C.$$

Then $P, Q, R$ are collinear.

In the projective plane, the points $P, Q, R$ always exist, because any two distinct projective lines meet. Thus the theorem has a cleaner statement than in the affine plane.

# 36 Week 9 Section Worksheet: Transformations of Infinite Incidence Geometries

*The purpose of this worksheet is to classify all transformations of two infinite incidence geometries.*

## Problem 1: The one-sided infinite line

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. Define $\mathcal{M}_{\mathbb{N}} = (\mathcal{P}_{\mathbb{N}}, \mathcal{L}_{\mathbb{N}})$ by

$$\mathcal{P}_{\mathbb{N}} = \mathbb{N}, \qquad \mathcal{L}_{\mathbb{N}} = \big\{\{n, n+1\} : n \in \mathbb{N}\big\}.$$

(a) Draw $\mathcal{M}_{\mathbb{N}}$. What object do you picture?

(b) Guess all transformations of $\mathcal{M}_{\mathbb{N}}$.

(c) Show that 0 is the unique point incident with exactly one line, and that every other point is incident with exactly two lines.

*Hint: Transformations preserve the number of incident lines. Also,* 1 *is the unique point adjacent to* 0*.*

(d) Prove by induction that $F(n) = n$ for all $n \in \mathbb{N}$.

(e) What can you conclude?

## Problem 2: The two-sided ladder

Define $\mathcal{M}_2 = (\mathcal{P}_2, \mathcal{L}_2)$ by

$$\mathcal{P}_2 = \{(0, i) : i \in \mathbb{Z}\} \cup \{(1, i) : i \in \mathbb{Z}\},$$

and

$$\mathcal{L}_2 = \big\{\{(0, i), (0, i+1)\}, \{(1, i), (1, i+1)\}, \{(0, i), (1, i)\} : i \in \mathbb{Z}\big\}.$$

We call $\{(0, i), (0, i+1)\}$ and $\{(1, i), (1, i+1)\}$ *vertical lines*, and $\{(0, i), (1, i)\}$ *rungs*.

(a) Draw the part of $\mathcal{M}_2$ with $-3 \leq i \leq 3$. What object do you picture?

(b) Can you think of some transformations of this model?

(c) Show that the following maps are transformations and determine their inverse.:

$$T_k(\epsilon, i) = (\epsilon, i + k), \qquad S(\epsilon, i) = (1 - \epsilon, i), \qquad R(\epsilon, i) = (\epsilon, -i),$$

where $k \in \mathbb{Z}$, $\epsilon \in \{0, 1\}$, and $i \in \mathbb{Z}$.

(d) A *4-cycle* is a set of four distinct points $\{p_0, p_1, p_2, p_3\}$ which can be ordered so that consecutive points, including $p_3, p_0$, are joined by lines. Show that every set

$$Q_i = \{(0, i), (0, i + 1), (1, i), (1, i + 1)\}$$

is a 4-cycle, and that every 4-cycle is equal to $Q_i$ for a unique $i \in \mathbb{Z}$.

*Hint: Start with one point and list its three neighbours.*

(e) Show that each rung belongs to exactly two 4-cycles, while each vertical line belongs to exactly one 4-cycle. Conclude that every transformation sends rungs to rungs and vertical lines to vertical lines.

(f) Let $F$ be any transformation. Since $\{(0, 0), (0, 1)\}$ is vertical, prove that there exist $\epsilon \in \{0, 1\}$, $k \in \mathbb{Z}$, and $\sigma \in \{1, -1\}$ such that

$$F(0, 0) = (\epsilon, k), \qquad F(0, 1) = (\epsilon, k + \sigma).$$

Using the transformations from (c), precompose $F$ with a suitable transformation $G$ so that

$$(G \circ F)(0, 0) = (0, 0), \qquad (G \circ F)(0, 1) = (0, 1).$$

*Hint: Use, as needed, $S$, a translation, and $R$.*

(g) Prove the normalized case: if $H$ is a transformation with $H(0, 0) = (0, 0)$ and $H(0, 1) = (0, 1)$, then $H$ is the identity.

*Hint: First show $H(1, 0) = (1, 0)$, $H(0, -1) = (0, -1)$, $H(1, 1) = (1, 1)$, and $H(0, 2) = (0, 2)$. Then use induction upward and downward on $i$, using vertical lines and rungs alternately.*

(h) What can you conclude?

## Challenge problems

You are welcome, and even encouraged, to invent your own infinite incidence geometry. For each model, try to draw a picture, guess some transformations, then guess all transformations. Below are two models you can think about.

**C1. A wider ladder.** Fix $r \geq 2$. Define a model with

$$\mathcal{P} = \{0, 1, \ldots, r\} \times \mathbb{Z},$$

where two points $(a, i)$ and $(b, j)$ form a line exactly when

$$|a - b| + |i - j| = 1.$$

What transformations do you expect?

*Hint: You should at least find vertical translations, vertical reflections, and the horizontal reflection $a \mapsto r - a$. Are there any others? Try to distinguish boundary points from interior points by counting incident lines.*

**C2. An infinite tree.** Consider the infinite rooted binary tree: one point is called the root; the root is joined to two new points; each of those is joined to two new points; and so on forever. What transformations can you imagine?

*Hint: This model has many more transformations than the line or the ladder. For example, at each branching point, you may swap the two infinite subtrees growing from that point. Think about why this gives infinitely many independent choices.*

# 37 Week 9 Lecture 2

*Continuing Pappus' theorem: reducing the affine statement to the projective statement, introducing projective transformations, and normalizing the two lines.*

## 1. Plan for the end of the course

Today, our main goal is to finish the proof of Pappus' theorem, one of the classical theorems of incidence geometry. This proof will use several ideas from the last few weeks:

- affine and projective planes;
- homogeneous coordinates;
- transformations;
- normalization.

The final exam will focus only on the material covered after the midterm: incidence geometry, affine and projective planes, finite and infinite models, counting arguments, homogeneous coordinates, projective completion, and transformations.

## 2. Reminder: affine Pappus

Let $\mathbb{F}$ be $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, or more generally any field. We want to prove Pappus' theorem in the affine plane

$$\mathbb{A}^2(\mathbb{F}).$$

Recall the statement. Let $\ell$ and $m$ be two distinct lines in $\mathbb{A}^2(\mathbb{F})$. Let

$$A, B, C \in \ell$$

be three distinct points, and let

$$A', B', C' \in m$$

be three distinct points. Assume that none of these six points lies on $\ell \cap m$. Assume also that the three intersection points

$$P = AB' \cap A'B, \qquad Q = AC' \cap A'C, \qquad R = BC' \cap B'C$$

exist in the affine plane. Then

$$P, Q, R$$

are collinear.

A direct affine proof is lengthy because there are cases to distinguish. The lines $\ell$ and $m$ may be parallel, or they may intersect. The point of passing to the projective plane is to remove this distinction.

## 3. From the affine plane to the projective plane

Recall the map

$$\varphi : \mathbb{A}^2(\mathbb{F}) \longrightarrow \mathbb{P}^2(\mathbb{F}), \qquad \varphi(x, y) = [x : y : 1].$$

This map has two important properties:

- it sends affine lines to projective lines;
- points $P, Q, R \in \mathbb{A}^2(\mathbb{F})$ are collinear if and only if $\varphi(P), \varphi(Q), \varphi(R)$ are collinear in $\mathbb{P}^2(\mathbb{F})$.

Thus, starting with a Pappus configuration in the affine plane, we may apply $\varphi$ to everything. The affine lines $\ell$ and $m$ become projective lines

$$\widetilde{\ell} = \varphi(\ell), \qquad \widetilde{m} = \varphi(m),$$

and the points $A, B, C, A', B', C'$ become points

$$\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{A}', \widetilde{B}', \widetilde{C}'$$

on $\widetilde{\ell}$ and $\widetilde{m}$.

If we can prove that

$$\varphi(P), \varphi(Q), \varphi(R)$$

are collinear in $\mathbb{P}^2(\mathbb{F})$, then $P, Q, R$ are collinear in $\mathbb{A}^2(\mathbb{F})$.

Let us justify this last implication directly. Suppose

$$\varphi(P) = [x_P : y_P : 1], \qquad \varphi(Q) = [x_Q : y_Q : 1], \qquad \varphi(R) = [x_R : y_R : 1]$$

belong to a projective line

$$\widetilde{n} = \{[X : Y : Z] \in \mathbb{P}^2(\mathbb{F}) : aX + bY + cZ = 0\}.$$

Then

$$ax_P + by_P + c = 0, \qquad ax_Q + by_Q + c = 0, \qquad ax_R + by_R + c = 0.$$

Therefore the affine points

$$P = (x_P, y_P), \qquad Q = (x_Q, y_Q), \qquad R = (x_R, y_R)$$

all lie on the affine line

$$n = \{(x, y) \in \mathbb{A}^2(\mathbb{F}) : ax + by + c = 0\}.$$

Hence $P, Q, R$ are collinear.

# 4. Projective Pappus

We now prove the projective version.

**Theorem (Projective Pappus).** Let $\ell$ and $m$ be distinct lines in $\mathbb{P}^2(\mathbb{F})$. Let

$$A, B, C \in \ell$$

be three distinct points, and let

$$A', B', C' \in m$$

be three distinct points. Assume that none of these six points is equal to

$$O = \ell \cap m.$$

Define

$$P = AB' \cap A'B, \qquad Q = AC' \cap A'C, \qquad R = BC' \cap B'C.$$

Then $P, Q, R$ are collinear.

Notice that in the projective plane, there is no need to assume that $P, Q, R$ exist. Indeed, any two distinct projective lines meet.

There is also no need to distinguish the cases $\ell \parallel m$ and $\ell \cap m \neq \emptyset$, because in a projective plane any two distinct lines meet.

# 5. Why transformations help

The proof of Projective Pappus has two main steps.

(a) Use a projective transformation to send the two lines $\ell$ and $m$ to the simple coordinate lines

$$\{X = 0\} \qquad \text{and} \qquad \{Y = 0\}.$$

(b) Prove Pappus' theorem in this normalized configuration.

This is a powerful use of transformations. Instead of proving the theorem for two completely arbitrary lines, we transform the whole configuration to a simpler one. Since transformations send lines to lines and preserve collinearity, proving the theorem after transformation proves it before transformation.

More precisely, suppose $F$ is a projective transformation. If

$$F(P), F(Q), F(R)$$

are collinear, then $P, Q, R$ are collinear, because applying $F^{-1}$ sends the line through $F(P), F(Q), F(R)$ to a line through $P, Q, R$.

## 6. Projective transformations from matrices

Let $A \in GL_3(\mathbb{F})$ be an invertible $3 \times 3$ matrix. It defines a map

$$F_A : \mathbb{P}^2(\mathbb{F}) \longrightarrow \mathbb{P}^2(\mathbb{F})$$

by

$$F_A([X : Y : Z]) = [A(X, Y, Z)^T].$$

This is well-defined because if two nonzero vectors are proportional, then their images under $A$ are still proportional.

Since $A$ is invertible, $F_A$ is bijective. Also, $F_A$ sends projective lines to projective lines. Indeed, projective lines in $\mathbb{P}^2(\mathbb{F})$ correspond to planes in $\mathbb{F}^3$ passing through the origin, and an invertible linear map sends planes through the origin to planes through the origin.

Thus $F_A$ is a transformation of $\mathbb{P}^2(\mathbb{F})$.

**Example.** Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$A \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} X + Y + Z \\ Y + Z \\ Z \end{pmatrix}.$$

So

$$F_A([X : Y : Z]) = [X + Y + Z : Y + Z : Z].$$

## 7. Why translations do not work directly

In the affine plane, translations are natural transformations. In the projective plane, the situation is different.

Recall that points of $\mathbb{P}^2(\mathbb{F})$ correspond to lines in $\mathbb{F}^3$ passing through the origin. If we translate all vectors in $\mathbb{F}^3$ by a fixed nonzero vector, a line through the origin need not be sent to a line through the origin.

For example, let

$$v = (1, 0, 0).$$

Consider the translation

$$T_v(X, Y, Z) = (X, Y, Z) + v.$$

The line

$$\{(0, \lambda, 0) : \lambda \in \mathbb{F}\}$$

passes through the origin. Its image under $T_v$ is

$$\{(1, \lambda, 0) : \lambda \in \mathbb{F}\},$$

which does not pass through the origin. Therefore this translation does not define a transformation of $\mathbb{P}^2(\mathbb{F})$.

## 8. The normalization step

We now prove the normalization step. Let $\ell$ and $m$ be distinct lines in $\mathbb{P}^2(\mathbb{F})$. Write

$$\ell = \{[X : Y : Z] \in \mathbb{P}^2(\mathbb{F}) : aX + bY + cZ = 0\},$$

and

$$m = \{[X : Y : Z] \in \mathbb{P}^2(\mathbb{F}) : dX + eY + fZ = 0\}.$$

Let

$$L = (a, b, c), \qquad M = (d, e, f).$$

Since $\ell$ and $m$ are distinct lines, the vectors $L$ and $M$ are linearly independent.

Choose a vector

$$N = (g, h, i)$$

such that $L, M, N$ form a basis of $\mathbb{F}^3$. Define

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$

Then $A$ is invertible.

We claim that the projective transformation $F_A$ sends $\ell$ to the line $\{X = 0\}$ and $m$ to the line $\{Y = 0\}$.

Indeed, if $[X : Y : Z] \in \ell$, then

$$aX + bY + cZ = 0.$$

The first coordinate of

$$A \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

is $aX + bY + cZ$, so the first coordinate of $F_A([X : Y : Z])$ is zero. Hence

$$F_A(\ell) \subseteq \{X = 0\}.$$

Conversely, since $A$ is invertible, every point on $\{X = 0\}$ has a preimage, and the same computation applied to $A^{-1}$ shows that this preimage lies on $\ell$. Therefore

$$F_A(\ell) = \{X = 0\}.$$

Similarly,

$$F_A(m) = \{Y = 0\}.$$

Thus it is enough to prove Projective Pappus in the normalized case

$$\ell = \{X = 0\}, \qquad m = \{Y = 0\}.$$

## 9. Beginning the normalized coordinate proof

Assume from now on that

$$\ell = \{X = 0\}, \qquad m = \{Y = 0\}.$$

First, we find the intersection point

$$O = \ell \cap m.$$

Let

$$O = [X : Y : Z].$$

Since $O \in \ell$, we have $X = 0$. Since $O \in m$, we have $Y = 0$. Since $[X : Y : Z]$ is a projective point, the vector $(X, Y, Z)$ is not zero, so $Z \neq 0$. Therefore

$$O = [0 : 0 : Z] = [0 : 0 : 1].$$

Now let $A' \in m$. Write

$$A' = [X : Y : Z].$$

Since $A' \in m = \{Y = 0\}$, we have $Y = 0$. We claim that $X \neq 0$. If $X = 0$, then since $(X, Y, Z) \neq (0, 0, 0)$, we would have $Z \neq 0$, and hence

$$A' = [0 : 0 : Z] = [0 : 0 : 1] = O.$$

But by hypothesis $A' \neq O$. Therefore $X \neq 0$. Dividing by $X$, we can write

$$A' = [1 : 0 : a']$$

for some $a' \in \mathbb{F}$.

The same argument gives

$$B' = [1 : 0 : b'], \qquad C' = [1 : 0 : c']$$

for some $b', c' \in \mathbb{F}$. Since $A', B', C'$ are distinct, the scalars

$$a', b', c'$$

are distinct.

Similarly, because $A, B, C \in \ell = \{X = 0\}$ and none of them is equal to $O$, we can write

$$A = [0 : 1 : a], \qquad B = [0 : 1 : b], \qquad C = [0 : 1 : c],$$

where $a, b, c \in \mathbb{F}$ are distinct.

This is the normalized coordinate setup for the rest of the proof.

# 38 Homework 9

*Homework 9 for MATH 128A.*

**MATH 128A: Homework 9** **Due date: Thursday, June 4**

**Instructions.** You will submit your solutions on Gradescope, which you can access through our Canvas course page by clicking on the assignment.

You are welcome to submit your homework either handwritten or in LaTeX. If you choose to submit handwritten work, please use pen rather than pencil, and make sure that both your handwriting and your scans are as readable as possible in order to facilitate grading.

I encourage you to attend office hours if you would like hints on the homework or if you would like to discuss more advanced material. Also, you are highly encouraged to work in groups!

### Problem 74 *(20 points) Calculations in the projective plane.*

*In this problem, we work in the real projective plane* $\mathbb{P}^2(\mathbb{R})$*. Recall that a point has homogeneous coordinates*

$$[x : y : z],$$

*where* $(x, y, z) \neq (0, 0, 0)$ *and*

$$[x : y : z] = [\lambda x : \lambda y : \lambda z]$$

*for every nonzero real number* $\lambda$*.*

*A projective line is given by an equation*

$$ax + by + cz = 0,$$

*where* $a, b, c \in \mathbb{R}$ *are not all zero.*

*One way to find the line through two distinct projective points*

$$P = [p_1 : p_2 : p_3] \qquad \text{and} \qquad Q = [q_1 : q_2 : q_3]$$

*is to use the determinant formula*

$$\det \begin{pmatrix} x & y & z \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{pmatrix} = 0.$$

*This works because a point* $[x : y : z]$ *lies on the projective line through* $P$ *and* $Q$ *exactly when the three vectors*

$$(x, y, z), \qquad (p_1, p_2, p_3), \qquad (q_1, q_2, q_3)$$

*are linearly dependent.*

(a) ***(5 points)*** *Find an equation for the projective line passing through*

$$A = [1 : 0 : 2] \qquad \text{and} \qquad B = [0 : 1 : 3].$$

(b) ***(5 points)*** *Find an equation for the projective line passing through*

$$C = [1 : 1 : 1] \qquad \text{and} \qquad D = [2 : 1 : 0].$$

(c) ***(5 points)*** *Find the intersection point of the two projective lines*

$$\ell : \ x + 2y - z = 0 \qquad \text{and} \qquad m : \ 2x - y + z = 0.$$

(d) ***(5 points)*** *Find the intersection point of the two projective lines*

$$\ell : \ x + y - z = 0 \qquad \text{and} \qquad m : \ x + y + z = 0.$$

**Problem 75** ***(80 points) Reviewing, correcting, and completing past homework.***

*This problem is graded for completion. The purpose of this question is to push you to review your past homework and submit regrade requests where appropriate. This should do three useful things at once:*

- *improve your understanding of the material;*
- *help you prepare for the final exam;*
- *and, when appropriate, improve your final grade.*

*Choose one question from a previous homework assignment on which you struggled, lost points, or felt uncertain. Then write a short reflection answering the following questions.*

(a) *Which homework problem did you choose?*

(b) *What was the main mathematical idea of the problem?*

(c) *What did you do wrong, or what did you find confusing?*

(d) *After reading the solution, do you now understand the correct argument? Explain the key step in your own words.*

(e) *Correct your mistake. This can mean rewriting the full solution, rewriting the part where your argument went wrong, or clearly explaining how your original solution should be fixed.*

*Once you are done, submit your corrected exercise to Mike and I by email, and make a regrade request for the corresponding question.*

*If you earned full credit on every previous homework assignment, challenge yourself and attempt one of the challenge problems from a past homework assignment instead.*

**Problem 76** ***Weekly reflection (2 bonus points).***

*Please fill in the SETS survey using this link:*

*blue.ucsc.edu*.

*You will receive* ***2 bonus points*** *if the class response rate reaches at least* ***80%***.

*Your feedback will be very beneficial to me. It will help me understand what worked well, what did not work well, and what I should change next time I teach this class. Thank you in advance!*

# 39 Week 10 Lecture 1

*Finishing the proof of Pappus' theorem in the normalized case, then concluding the course by discussing Hilbert planes and Euclidean planes.*

## 1. Finishing Pappus' theorem

Last time, we reduced Pappus' theorem to the following normalized projective situation. We work in $\mathbb{P}^2(\mathbb{F})$, and we assume

$$\ell = \{Y = 0\}, \qquad m = \{X = 0\}.$$

Then

$$O = \ell \cap m = [0 : 0 : 1].$$

Since $A, B, C \in \ell$ and none of them is equal to $O$, we may write

$$A = [1 : 0 : a], \qquad B = [1 : 0 : b], \qquad C = [1 : 0 : c],$$

where $a, b, c \in \mathbb{F}$ are distinct. Similarly, since $A', B', C' \in m$ and none of them is equal to $O$, we may write

$$A' = [0 : 1 : a'], \qquad B' = [0 : 1 : b'], \qquad C' = [0 : 1 : c'],$$

where $a', b', c' \in \mathbb{F}$ are distinct.

We want to show that

$$P = AB' \cap A'B, \qquad Q = AC' \cap A'C, \qquad R = BC' \cap B'C$$

are collinear.

## 2. Equation of the line $A'B$

The line $A'B$ is the unique projective line through

$$A' = [0 : 1 : a'] \qquad \text{and} \qquad B = [1 : 0 : b].$$

Recall that lines in $\mathbb{P}^2(\mathbb{F})$ correspond to planes in $\mathbb{F}^3$ passing through the origin. Thus $A'B$ corresponds to the plane through $(0, 0, 0)$ containing the vectors

$$(0, 1, a') \qquad \text{and} \qquad (1, 0, b).$$

A point $[X : Y : Z]$ lies on this line if and only if the vectors

$$(0, 1, a'), \qquad (1, 0, b), \qquad (X, Y, Z)$$

are linearly dependent. Equivalently,

$$\det \begin{pmatrix} 0 & 1 & X \\ 1 & 0 & Y \\ a' & b & Z \end{pmatrix} = 0.$$

This determinant is

$$a'Y + bX - Z.$$

Therefore

$$A'B = \{[X : Y : Z] \in \mathbb{P}^2(\mathbb{F}) : a'Y + bX - Z = 0\}.$$

Similarly,

$$AB' = \{[X : Y : Z] \in \mathbb{P}^2(\mathbb{F}) : b'Y + aX - Z = 0\}.$$

## 3. Expression for $P = AB' \cap A'B$

Let

$$P = [X : Y : Z] \in A'B \cap AB'.$$

Then

$$a'Y + bX - Z = 0$$

and

$$b'Y + aX - Z = 0.$$

Subtracting the two equations gives

$$(a' - b')Y + (b - a)X = 0.$$

Since $a' \neq b'$, we obtain

$$Y = \frac{b - a}{b' - a'}X.$$

Then

$$Z = a'Y + bX = a'\frac{b - a}{b' - a'}X + bX = \frac{bb' - aa'}{b' - a'}X.$$

Thus the solution space is one-dimensional:

$$\left\{ \left( X, \frac{b - a}{b' - a'}X, \frac{bb' - aa'}{b' - a'}X \right) : X \in \mathbb{F} \right\}.$$

Therefore

$$P = [b' - a' : b - a : bb' - aa'].$$

## 4. The other two intersection points

Proceeding in exactly the same way, we obtain

$$Q = AC' \cap A'C = [c' - a' : c - a : cc' - aa']$$

and

$$R = BC' \cap B'C = [c' - b' : c - b : cc' - bb'].$$

Now choose vector representatives

$$p = (b' - a', b - a, bb' - aa'),$$

$$q = (c' - a', c - a, cc' - aa'),$$

and

$$r = (c' - b', c - b, cc' - bb').$$

Then

$$q - p = (c' - b', c - b, cc' - bb') = r.$$

Hence

$$q - p - r = 0.$$

Therefore the three vectors $p, q, r$ are linearly dependent. Since collinearity in $\mathbb{P}^2(\mathbb{F})$ is equivalent to linear dependence of vector representatives, the points

$$P, Q, R$$

are collinear.

This proves Pappus' theorem in the normalized projective case. Since last time we showed that every projective Pappus configuration can be normalized by a projective transformation, this proves projective Pappus. Since affine Pappus follows from projective Pappus by the embedding

$$\mathbb{A}^2(\mathbb{F}) \longrightarrow \mathbb{P}^2(\mathbb{F}), \qquad (x, y) \longmapsto [x : y : 1],$$

we have also proved affine Pappus.

## 5. Course conclusion

We began this course with Euclid's *Elements*. This helped us:

- learn how to read mathematical proofs;
- learn compass-and-straightedge constructions;
- understand what an axiomatic theory is;

- notice gaps in some of Euclid's arguments;
- practice proving statements rigorously using results such as SAS, SSS, and ASA.

Then we moved toward a more rigorous treatment of geometry, especially through Hilbert's axioms. This led us to incidence geometry and to non-Euclidean geometries, including:

- finite geometries with different parallel properties;
- infinite geometries such as $\mathbb{A}^2(\mathbb{F})$, $\mathbb{P}^2(\mathbb{F})$, $\mathcal{M}_{\mathbb{Z}}$, and $\mathcal{M}_{\mathbb{Z}^2}$;
- transformations and symmetry groups;
- applications of transformations, such as the normalization proof of Pappus' theorem.

Along the way, we also practiced several proof strategies: direct proof, proof by contradiction, proof by induction, and proving that maps are injective, surjective, or bijective.

The last question is: what must we add to our incidence axioms if we want to recover Euclid's geometry more fully?

The incidence axioms **I1**–**I3** are not enough. They know about points and lines, but they do not know about segments, rays, angles, lengths, or angle measurements. To recover Euclid's geometry, Hilbert added two more families of axioms:

- betweenness axioms;
- congruence axioms.

# 6. Hilbert planes

A **Hilbert plane** is a model $\mathcal{M} = (\mathcal{P}, \mathcal{L}, \mathcal{R})$ satisfying the incidence axioms **I1-3** together with:

- a betweenness relation satisfying axioms **B1**–**B4**;
- congruence relations for segments and angles satisfying axioms **C1**–**C6**.

# 7. Betweenness

Betweenness is a relation on triples of points. We write

$$A * B * C$$

to mean that $B$ is between $A$ and $C$.

The betweenness axioms include the following ideas.

**B1.** If

$$A * B * C,$$

then $A, B, C$ are distinct and collinear.

**B2.** If $A \neq B$, then there exists a point $C$ such that

$$A * B * C.$$

This is a rigorous version of the idea that a segment can be extended beyond one of its endpoints. In particular, lines must contain infinitely many points.

**B3.** If $A, B, C$ are distinct and collinear, then exactly one of the three points lies between the other two.

**B4 (Pasch's axiom).** Let $A, B, C$ be three non-collinear points, and let $\ell$ be a line not incident with any of $A, B, C$. Suppose that $D \in \ell$ and

$$A * D * B.$$

Then either there exists $E \in \ell$ such that

$$B * E * C,$$

or there exists $E \in \ell$ such that

$$A * E * C,$$

but not both.

Geometrically, Pasch's axiom says that if a line enters a triangle through one side and does not pass through a vertex, then it must leave through exactly one of the other two sides.

## 8. Segments, rays, and angles

Once we have betweenness, we can define segments, rays, and angles.

If $A \neq B$, we write

$$(AB)$$

for the unique line through $A$ and $B$.

The segment from $A$ to $B$ is

$$[AB] = \{A, B\} \cup \{C \in \mathcal{P} : A * C * B\}.$$

The ray starting at $A$ and passing through $B$ is

$$[AB) = [AB] \cup \{C \in \mathcal{P} : A * B * C\}.$$

If $A, B, C$ are non-collinear, the angle $\angle BAC$ is the union of two rays:

$$\angle BAC = [AB) \cup [AC).$$

## 9. Congruence

Hilbert also introduces congruence relations:

- a congruence relation on line segments, written

$$[AB] \sim [CD];$$

- a congruence relation on angles, written

$$\angle BAC \sim \angle EDF.$$

The congruence axioms include the following ideas.

**C1.** Given a segment $[AB]$ and a ray $r$ starting at $C$, there exists a point $D \in r$ such that

$$[AB] \sim [CD].$$

This is the rigorous version of copying a segment onto a given ray.

**C2.** Segment congruence is an equivalence relation.

**C3.** If

$$A * B * C \qquad \text{and} \qquad D * E * F,$$

and

$$[AB] \sim [DE], \qquad [BC] \sim [EF],$$

then

$$[AC] \sim [DF].$$

This is an addition rule for segment lengths.

**C4.** Given an angle $\angle BAC$ and a ray $[DE)$, there exists a unique ray $[DF)$ such that

$$\angle BAC \sim \angle EDF.$$

This is the rigorous version of copying an angle onto a given ray. It corresponds to Euclid I.23.

**C5.** Angle congruence is an equivalence relation.

**C6 (SAS).** Let $ABC$ and $DEF$ be triangles. If

$$[AB] \sim [DE], \qquad [AC] \sim [DF], \qquad \angle BAC \sim \angle EDF,$$

then

$$[BC] \sim [EF],$$

and the remaining corresponding angles are congruent:

$$\angle ABC \sim \angle DEF, \qquad \angle ACB \sim \angle DFE.$$

This is the Hilbert-plane version of the SAS theorem.

## 10. Examples of Hilbert planes

The standard example is

$$\mathbb{A}^2(\mathbb{R}).$$

This is the usual Euclidean plane, with the usual notions of betweenness, segment congruence, and angle congruence.

Another example is

$$\mathbb{A}^2(\mathbb{K}),$$

where $\mathbb{K} \subset \mathbb{R}$ is the field of constructible numbers. A real number $k$ is constructible if, using straightedge and compass starting from a unit segment, we can construct a segment of length $|k|$. For example,

$$\sqrt{2} \in \mathbb{K},$$

because it is the diagonal length of a unit square.

Another example is

$$\mathbb{A}^2(\overline{\mathbb{Q}} \cap \mathbb{R}),$$

where $\overline{\mathbb{Q}} \cap \mathbb{R}$ is the field of real algebraic numbers. These are the real numbers which solve a polynomial equation

$$P(x) = 0$$

with rational coefficients. For example,

$$\sqrt[5]{2} \in \overline{\mathbb{Q}} \cap \mathbb{R},$$

because it is a solution of

$$x^5 - 2 = 0.$$

There are also hyperbolic Hilbert planes. These satisfy the Hilbert plane axioms, but they do not satisfy the Euclidean parallel property.

Another important example is the Moulton plane. Its point set is $\mathbb{R}^2$, but the lines are modified: usual lines are kept, except that lines of negative slope bend when they cross the $y$-axis. This gives a nonstandard geometry where many Euclidean-looking axioms still hold, but some familiar theorems such as Desargues' theorem fail.

## 11. Non-examples of Hilbert planes

The rational affine plane

$$\mathbb{A}^2(\mathbb{Q})$$

is not a Hilbert plane with the usual Euclidean congruence. The issue is segment-copying.

Consider the segment

$$[(0,0),(1,1)].$$

Its length is $\sqrt{2}$. If we try to copy this segment onto the ray from $(0,0)$ through $(1,0)$, the endpoint would have to be

$$(\sqrt{2},0).$$

But

$$(\sqrt{2},0) \notin \mathbb{Q}^2.$$

Thus axiom **C1** fails.

The complex affine plane

$$\mathbb{A}^2(\mathbb{C})$$

is also not a Hilbert plane in the usual sense, because there is no good order relation on $\mathbb{C}$ compatible with the geometry. In particular, complex lines do not behave like ordered real lines, so there is no suitable Hilbert-style betweenness relation.

# 12. Euclidean planes

A **Euclidean plane** is a Hilbert plane satisfying additional axioms. The most important additional axioms for us are:

- the Euclidean parallel property;
- the circle-circle intersection property;
- further axioms related to area and congruence of rectilinear figures.

The Euclidean parallel property says that, given a line $\ell$ and a point $p$ not incident with $\ell$, there is a unique line through $p$ parallel to $\ell$.

The circle-circle intersection property says, informally, that if one circle contains a point inside another circle and also a point outside that circle, then the two circles meet.

More precisely, fix two circles

$$C_1 = \{P \in \mathcal{P} : [A_1P] \sim [A_1B_1]\}$$

and

$$C_2 = \{P \in \mathcal{P} : [A_2P] \sim [A_2B_2]\}.$$

If $C_2$ contains a point $P$ inside $C_1$ and a point $Q$ outside $C_1$, then

$$C_1 \cap C_2 \neq \emptyset.$$

Here $P$ being inside $C_1$ means that

$$[A_1P] < [A_1B_1],$$

and $Q$ being outside $C_1$ means that

$$[A_1Q] > [A_1B_1].$$

This circle-circle intersection property is what guarantees constructions such as Euclid I.1, the construction of an equilateral triangle on a given segment.

# 13. Euclid's *Elements* inside Euclidean planes

A major theorem is:

**Theorem.** Euclid's *Elements*, Books I–IV, are valid in any Euclidean plane.

More precisely, different parts of the *Elements* require different amounts of structure.

For Book I:

| Propositions | Axioms needed |
|---|---|
| 1.1–1.28, except 1.1 and 1.22 | Hilbert plane |
| 1.1 and 1.22 | Hilbert plane + circle-circle intersection |
| 1.29–1.34 | Hilbert plane + Euclidean parallel property |
| 1.35–1.48 | Hilbert plane + parallel property + area theory |

For Book II:

| Propositions | Axioms needed |
|---|---|
| 2.1–2.13 | Hilbert plane + parallel property + area theory |
| 2.14 | Euclidean plane |

For Book III:

| Propositions | Axioms needed |
|---|---|
| 3.1–3.19, except 3.1 and 3.17 | Hilbert plane |
| 3.1 and 3.17 | Hilbert plane + circle-circle intersection |
| 3.20–3.34 | Hilbert plane + parallel property + circle-circle intersection |
| 3.35–3.37 | Euclidean plane + area theory |

For Book IV:

| Propositions | Axioms needed |
|---|---|
| 4.1–4.16 | Euclidean plane |

Thus, to recover the full Euclidean geometry of the *Elements*, incidence axioms are not enough. We need betweenness, congruence, a parallel axiom, circle intersection principles, and eventually area theory.

# 14. Closing

The purpose of this course was not only to learn a list of classical theorems. It was also to see how geometry can be built from axioms, how changing axioms changes the geometry, and how transformations can simplify difficult configurations.

I hope this class has sparked some interest in geometry. Hartshorne's book is a good reference for going further, and my door is always open if you would like to discuss more! Here is my personal email, for when I lose my UCSC email address:

ike693@gmail.com

# 40 Week 10 Section Worksheet: Transformations, Ceva, and Conics

*This worksheet explores how transformations can simplify geometric proofs. We prove Ceva's theorem by affine normalization and prove that five points determine a conic by projective normalization.*

## 1. Transformations as a proof strategy

A recurring idea in this course is that transformations allow us to replace a complicated configuration by a simpler one.

For example, in the proof of Pappus' theorem, we may send two projective lines to convenient coordinate axes.

The general strategy is:

(a) Normalize the configuration.

(b) Prove the theorem in the normalized case.

(c) Transform back to the original configuration.

## 2. Ceva's theorem by affine normalization

We begin with a classical theorem about concurrence in a triangle.

**Theorem (Ceva's theorem).** Let $ABC$ be a triangle, and let

$$D \in BC, \qquad E \in CA, \qquad F \in AB$$

be points on the three sides, none of them equal to a vertex. Then the three lines

$$AD, \qquad BE, \qquad CF$$

are concurrent if and only if

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$

In this section, we prove Ceva's theorem using transformations. The key idea is that affine transformations preserve:

- collinearity;
- concurrency of lines;
- ratios along a line.

Therefore, instead of proving Ceva's theorem in an arbitrary triangle, we may use an affine transformation to send the triangle to the standard triangle

$$(0,0), \qquad (1,0), \qquad (0,1).$$

The proof then becomes a coordinate calculation.

**Problem 77** ***Affine normalization of a triangle.***

*Let $A, B, C \in \mathbb{R}^2$ be three non-collinear points.*

(a) *Explain why the vectors $B - A$ and $C - A$ form a basis of $\mathbb{R}^2$.*

(b) *Define*

$$\Phi(u, v) = A + u(B - A) + v(C - A).$$

*Show that $\Phi$ is an affine transformation.*

(c) *Compute*

$$\Phi(0,0), \qquad \Phi(1,0), \qquad \Phi(0,1).$$

*Conclude that any nondegenerate triangle can be sent by an affine transformation to the standard triangle*

$$(0,0), \qquad (1,0), \qquad (0,1).$$

**Problem 78** ***Ceva's theorem in the standard triangle.***

*We now prove Ceva's theorem for the standard triangle. Let*

$$A = (0,0), \qquad B = (1,0), \qquad C = (0,1).$$

*Let $D \in BC$, $E \in CA$, and $F \in AB$, with none of $D, E, F$ equal to a vertex. Write*

$$D = (1 - t, t), \qquad E = (0, u), \qquad F = (v, 0),$$

*where $0 < t, u, v < 1$.*

(a) *Show that*

$$\frac{BD}{DC} = \frac{t}{1-t}, \qquad \frac{CE}{EA} = \frac{1-u}{u}, \qquad \frac{AF}{FB} = \frac{v}{1-v}.$$

(b) *Show that the equation of the line $AD$ is*

$$tx - (1-t)y = 0.$$

(c) *Show that the equations of the lines $BE$ and $CF$ are*

$$ux + y - u = 0 \qquad \text{and} \qquad x + vy - v = 0.$$

(d) *Explain why the three lines $AD$, $BE$, and $CF$ are concurrent exactly when the system*

$$\begin{cases} tx - (1-t)y = 0, \\ ux + y - u = 0, \\ x + vy - v = 0 \end{cases}$$

*has a common solution $(x, y)$.*

(e) *Rewrite this system as*

$$\begin{pmatrix} t & -(1-t) & 0 \\ u & 1 & -u \\ 1 & v & -v \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

*Deduce that concurrency implies*

$$\det \begin{pmatrix} t & -(1-t) & 0 \\ u & 1 & -u \\ 1 & v & -v \end{pmatrix} = 0.$$

(f) *Compute this determinant and show that the condition above is equivalent to*

$$u(1-t)(1-v) = tv(1-u).$$

(g) *Conclude that $AD$, $BE$, and $CF$ are concurrent if and only if*

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$

**Problem 79** ***Ceva's theorem in an arbitrary triangle.***

*Explain why the result for the standard triangle implies Ceva's theorem for any nondegenerate triangle.*

*Your answer should mention:*

- *affine transformations;*
- *preservation of collinearity;*
- *preservation of concurrency;*
- *preservation of ratios along a line.*

## 3. Five points determine a conic

We now move from affine normalization to projective normalization.

Recall that a conic in $\mathbb{P}^2(\mathbb{F})$ is the zero set of a nonzero homogeneous quadratic equation

$$aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ = 0.$$

Two equations which differ by a nonzero scalar define the same conic.

**Theorem.** Let

$$P_1, P_2, P_3, P_4, P_5 \in \mathbb{P}^2(\mathbb{F})$$

be five points such that no three are collinear. Then there is a unique conic passing through them.

We will prove this theorem by using a projective transformation to simplify four of the points.

**Problem 80** ***Normalizing four points.***

*Let $P_1, P_2, P_3, P_4 \in \mathbb{P}^2(\mathbb{F})$ be four points such that no three of them are collinear.*

(a) *Choose vector representatives*

$$P_i = [v_i], \qquad v_i \in \mathbb{F}^3 \setminus \{0\}.$$

*Explain why $v_1, v_2, v_3$ form a basis of $\mathbb{F}^3$.*

(b) *Write*

$$v_4 = \alpha v_1 + \beta v_2 + \gamma v_3.$$

*Explain why*

$$\alpha, \beta, \gamma \neq 0.$$

(c) *Define a linear map $A : \mathbb{F}^3 \to \mathbb{F}^3$ by*

$$A(\alpha v_1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad A(\beta v_2) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad A(\gamma v_3) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

*Explain why $A$ is invertible.*

(d) *Let $[A]$ be the induced projective transformation of $\mathbb{P}^2(\mathbb{F})$. Show that*

$$[A](P_1) = [1 : 0 : 0], \qquad [A](P_2) = [0 : 1 : 0], \qquad [A](P_3) = [0 : 0 : 1],$$

*and*

$$[A](P_4) = [1 : 1 : 1].$$

**Problem 81** ***Conics through four normalized points.***

*Consider the four points*

$$[1:0:0], \qquad [0:1:0], \qquad [0:0:1], \qquad [1:1:1]$$

*in* $\mathbb{P}^2(\mathbb{F})$.

(a) *Let*

$$aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ = 0$$

*be a conic passing through*

$$[1:0:0], \qquad [0:1:0], \qquad [0:0:1].$$

*Show that*

$$a = b = c = 0.$$

(b) *Deduce that the equation has the form*

$$dXY + eXZ + fYZ = 0.$$

(c) *Now impose that the conic passes through* $[1:1:1]$. *Show that*

$$d + e + f = 0.$$

**Problem 82** ***Adding the fifth point.***

*Let*

$$P = [u:v:w] \in \mathbb{P}^2(\mathbb{F})$$

*be a fifth point, and suppose that the five points*

$$[1:0:0], \qquad [0:1:0], \qquad [0:0:1], \qquad [1:1:1], \qquad [u:v:w]$$

*are distinct and no three are collinear.*

(a) *Explain why*

$$u, v, w \neq 0.$$

(b) *Show that a conic passing through the five points must have equation*

$$dXY + eXZ + fYZ = 0,$$

*where*

$$d + e + f = 0$$

*and*

$$duv + euw + fvw = 0.$$

(c) *Explain why the two equations*

$$d + e + f = 0 \qquad \text{and} \qquad duv + euw + fvw = 0$$

*are independent.*

*Hint: If they were dependent, then* $(uv, uw, vw)$ *would be proportional to* $(1, 1, 1)$.

(d) *Conclude that the space of triples* $(d, e, f)$ *satisfying these equations is one-dimensional.*

(e) *Explain why this means that there is a unique conic passing through the five points.*

**Problem 83** ***The five-point theorem for conics.***

*Prove the following theorem.*

***Theorem.*** *Let*

$$P_1, P_2, P_3, P_4, P_5 \in \mathbb{P}^2(\mathbb{F})$$

*be five points such that no three are collinear. Then there is a unique conic passing through them.*

*Your proof should use:*

- *the projective normalization of four points;*
- *the normalized computation above;*
- *the fact that projective transformations send conics to conics.*